Math776: Graph Theory (I)
Fall, 2017
Homework 1 solution

1. **[page 30, #2]** Determine the average degree, number of edges, diameter, girth, and circumference of the hypercube graph Q_d.

Solution:

- It is a d-regular graph so the average degree is d.
- The number of edges is $2^{d-1} \times d$.
- The diameter is d.
- The girth is 4 for $d \geq 2$; and is ∞ for $d = 1$.
- The circumference is 2^d since Q_d is Hamiltonian.

2. **[page 30, #3]** Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

Solution: Let $C = (V_c, E_c)$ denote the cycle in G, and let $P = (V_p, E_p)$ denote the path of length at least k in G. Let $|V_p \cap V_c| = s$. If $s \geq \sqrt{k}$, then C is is the cycle that we want. Thus, we can assume $2 \leq s < \sqrt{k}$.

The set $V_p \cap V_c$ divides P into s segment. There is a segment P_i with at least

$$\frac{k}{s-1} \geq \frac{k}{\sqrt{k}-1} > \sqrt{k}$$

edges.

Taking this segment P_i together with edges on C connecting two ends of P_i results with a cycle of length at least $\sqrt{k}$.

3. **[page 30, #8]** Show that every connected graph G contains a path of length at least $\min\{2\delta(G), |G| - 1\}$.

Solution: Let $P = x_0 x_1 \ldots x_m$ be a path in graph G of maximal length.
We will denote the length of P by l.

If $l \geq |G| - 1$, we are done. Otherwise, the set $O = V(G) \setminus V(P)$ is nonempty, and since the graph is connected, there exists a $V(P) - O$ path, $P' = y_0 y_1 \ldots y_k$ that is non-trivial.

Claim: If $l < 2\delta(G)$ then there is a cycle spanning $V(P)$.

Observe that $N(x_0) \subset P$ and $N(x_m) \subset P$, because if either endpoint of P is adjacent to a vertex outside of P, then the path can be extended and is not maximal. If $x_0 x_{i+1}$ and $x_m x_i$ are both edges in P, then there is a cycle $C = x_0 \ldots x_i x_m \ldots x_{i+1} x_0$.

The occurrence of these two edges can be shown by the Pigeonhole Principle.

A special case occurs where there is an edge $x_0 x_m$, since the other edge is given in the path.

The vertex x_0 has at least $\delta(G) - 1$ neighbors out of $\{x_2 \ldots x_{m-1}\}$ because it is adjacent to x_1 and not adjacent to x_m(would create an obvious cycle). For each neighbor x_i there is a corresponding vertex x_{i-1} to which x_m is not adjacent. So, x_m must have at least $\delta(G) - 1$ neighbors out of $\{x_1 \ldots x_{m-2}\}$, and of those $\delta(G) - 1$ are forbidden. Since $m < 2\delta(G)$, there are fewer than $2\delta(G) - 2$ possible neighbors. So by the Pigeonhole Principle one of the neighbors is forbidden, so there is a cycle.

By deleting an edge of the cycle spanning $V(P)$ incident with y_0 you can extend the remaining path with P', forming a path longer than P, which is a contradiction.

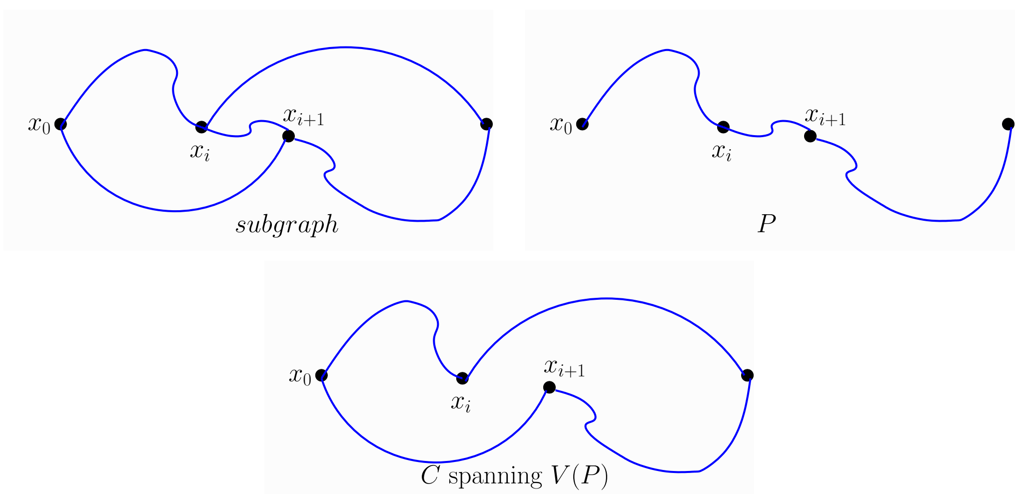

Figure 1: Visualization of proof

4. **[page 30, #9**] Show that a connected graph of diameter k and minimum degree d has at least about $kd/3$ vertices but need not have substantially more.

Solution: Let x_0 and x_k be vertices such that the shortest path, P, between the two has length k, the diameter. Let v be a vertex not on P that is adjacent to a vertex on P. Let i be the smallest integer such that x_i is adjacent to v. If $j > i + 2$ then x_j cannot be adjacent to v or $x_0 P x_i v x_j P x_k$ would be a path from x_0 to x_k of length less than k, a contradiction. Thus any vertex off of P can only be adjacent to at most 3 vertices on P. Now we consider the number of edges leaving P. Two ends of P contribute $(d-1)$ each. Every internal vertex of P contributes $(d-2)$ edges. Thus there are

$$2(d-1) + (k-1)(d-2) = kd - 2k$$

edges leaving P.

Since any vertex off of P can only be adjacent to at most 3 vertices on P, The number of neighbors of P is at least

$$\frac{kd - 2k}{3}.$$

Thus the total number of vertices is at least

$$\frac{kd-2k}{3} + k + 1 = \frac{k(d+1)}{3} + 1 > \frac{kd}{3}.$$

On the other hand, we can reverse-engineer the proof to construct a graph G from a path P by adding vertices to connect each three consecutive vertices of P properly. This graph will not substantially more than $\frac{kd}{3}$ vertices.

5. [page 30, #12] Determine $\kappa(G)$ and $\lambda(G)$ for $G = P_m, C_n, K_n, K_{m,n}$, and Q_d; $d, m, n \geq 3$.

Proof:

$\kappa(P_m) = \lambda(P_m) = 1.$

$\kappa(C_n) = \lambda(C_n) = 2.$

$\kappa(K_n) = \lambda(K_n) = n - 1.$

$\kappa(K_{m,n}) = \lambda(K_{m,n}) = \min\{m, n\}.$

$\kappa(Q_d) = \lambda(Q_d) = d.$

6. [page 31, #18] Show that a tree without a vertex of degree 2 has more leaves than other vertices. Can you find a very short proof that does not use induction?

Proof: Let G be a tree with no vertex of degree 2. Let L be the set of leaves in G and let O be the set of vertices which are not leaves in G. Note that the minimum degree of an element of O is 3 because no vertex has degree 2. So,

$$2(|V| - 1) = \sum_{v \in V} d(v) \geq |L| + 3|O|.$$

Since $|V| = |L| + |O|$, we get

$$|L| \geq |O| + 2.$$