Math 778S Spectral Graph Theory
Handout #3: Eigenvalues of Adjacency Matrix

The *Cartesian product* (denoted by $G \,\square\, H$) of two simple graphs G and H has the vertex-set $V(G) \times V(H)$. For any $u, v \in V(G)$ and $x, y \in V(H)$, (u, x) is adjacent to (v, y) if either "$u = v$ and $xy \in E(H)$" or "$uv \in E(G)$ and $x = y$".

Lemma 1 *Suppose $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the adjacency matrix of a graph G and $\mu_1, \ldots, \mu_m$ are eigenvalues of the adjacency matrix of a graph H. Then the eigenvalues of the adjacency matrix of the Cartesian product $G \,\square\, H$ are $\lambda_i + \mu_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.*

Proof: Let A (or B) be the adjacency matrix of G (or H) respectively. For any eigenvalue λ of A and any eigenvalue μ of B, we would like to show $\lambda + \mu$ is an eigenvalue of $G \,\square\, H$. Let α be the eigenvector of A corresponding to λ and β be the eigenvector of B corresponding to μ. We have

$$
\begin{aligned}
A\alpha &= \lambda\alpha && (1)\\
B\beta &= \mu\beta. && (2)
\end{aligned}
$$

Equivalently, for any $u \in V(G)$,

$$\sum_{v \sim u} \alpha_v = \lambda \alpha_u;$$

for any $x \in V(H)$,

$$\sum_{y \sim x} \beta_y = \mu \beta_x.$$

Let $\alpha \otimes \beta$ be the $n \times m$ column vector defined by entries

$$(\alpha \otimes \beta)_{u,x} = \alpha_u \beta_x.$$

Let C be the adjacency matrix of $G \,\square\, H$. We would like to show $\alpha \otimes \beta$ is an eigenvector of C. We have, for any $(u, x) \in V(G \,\square\, H)$,

$$
\begin{aligned}
\sum_{(v,y)\sim(u,x)} (\alpha \otimes \beta)_{v,y} &= \sum_{(v,y)\sim(u,x)} \alpha_v \beta_y \\
&= \sum_{(u,y)\sim(u,x)} \alpha_u \beta_y + \sum_{(v,x)\sim(u,x)} \alpha_v \beta_x \\
&= \sum_{y \sim x} \alpha_u \beta_y + \sum_{v \sim u} \alpha_v \beta_x \\
&= \alpha_u \sum_{y \sim x} \beta_y + \beta_x \sum_{v \sim u} \alpha_v \\
&= \alpha_u \mu \beta_x + \beta_x \lambda \alpha_u \\
&= (\lambda + \mu)(\alpha \otimes \beta)_{u,x}.
\end{aligned}
$$

This is equivalent to

$$C(\alpha \times \beta) = (\lambda + \mu)(\alpha \times \beta).$$

Thus, $\lambda + \mu$ is an eigenvalue of $G \,\square\, H$.

For $1 \leq i \leq n$ and $1 \leq j \leq m$, $\lambda_i + \mu_j$ are eigenvalues of $G \,\square\, H$. Since $G \,\square\, H$ has nm vertices, these eigenvalues (with multiplicity) are all eigenvalues of $G \,\square\, H$. $\square$

Remark: The adjacency matrix of $G \,\square\, H$ can be written as $A \otimes I_m + I_n \otimes B$. Here $\otimes$ is tensor product of matrices.

Hypercube Q_n: The vertices of Q_n are points in n-dimensional space over the field of two elements $F_2 = \{0, 1\}$. Two points are adjacent in Q_n if and only if they differ by exactly one coordinate.

We have $Q_1 = P_2$, $Q_2 = C_4$, and Q_3 is the cube in 3-dimensional space. We have $Q_{n+1} = Q_1 \,\square\, Q_n$. The eigenvalues of Q_n can be determined from the eigenvalues of Q_1 and the above lemma.

$Q_1 = P_2$ has eigenvalues ± 1. Q_n has eigenvalues $n - 2i$ with multiplicity $\binom{n}{i}$ for $0 \leq i \leq n$.

Regular graphs: The degree of a vertex v in G is the number of edges incident to v. If all degrees are equal to d, then G is called a d-regular graph. Let $\mathbf{1}$ be the column vector of all entries equal to 1. If G is a regular graph, then $A\mathbf{1} = d\mathbf{1}$. Hence, $\mathbf{1}$ is an eigenvector for the eigenvalue d.

Eigenvalues of K_n: Let $J = \mathbf{1}'\mathbf{1}$ be the $n \times n$-matrix with all entries 1. Since J is a rank 1 matrix, J has eigenvalues 0 with multiplicity $n - 1$. It is easy to see that the nonzero eigenvalue of J is n. The complete graph K_n has the adjacency matrix $J - I$. Thus, K_n has an eigenvalue $n - 1$ of multiplicity 1 and -1 of multiplicity $n - 1$.

Eigenvalues of C_n: Let $Q = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$.

(Q can be viewed as the adjacency matrix of the directed cycle.) We have $A = Q + Q'$. Note that $Q^n = I$. Let λ be the eigenvalue of Q. We have $\lambda^n = 1$. The eigenvalues of Q are precisely n-th root of 1:

$$\rho^k = \cos(\frac{2k\pi}{n}) + \sqrt{-1}\sin(\frac{2k\pi}{n}), \quad \text{for } 0 \leq k \leq n-1.$$

Note $Q' = Q^{n-1}$. Thus, $A = Q + Q'$ has eigenvalues

$$\rho^k + \rho^{k(n-1} = 2\Re(\rho^k) = 2\cos(\frac{2k\pi}{n})$$

for $k = 0, 1, 2, \ldots, n-1$.

Let $\mu_1 \geq \mu_2 \geq \ldots \mu_n$ be the eigenvalues of the adjacency matrix of a graph G. We refer $\mu_1 = \mu_{\max}$ and $\mu_n = \mu_{\min}$. We have

$$\begin{aligned}
\mu_{\max} &= \sup_{\|x\|=1} x'Ax \\
\mu_{\min} &= \inf_{\|x\|=1} x'Ax
\end{aligned}$$

Suppose $f(x) = x'Ax$ reaches the maximum at α on the unit sphere. Then all coordinates of α are non-negative.

Lemma 2 *If H is a subgraph of G, then we have*

$$\mu_{\max}(G) \geq \mu_{\max}(H).$$

Proof: Without loss of generality, we assume $V(H) = V(G)$. (Otherwise, we add some isolated vertices to H. It doesn't change the maximum eigenvalue of H.)

Let α be the eigenvector A_H corresponding to $\mu_{\max}(H)$. We have

$$\begin{aligned}
\mu_{\max}(H) &= \alpha' A_H \alpha \\
&= 2 \sum_{ij \in E(H)} \alpha_i \alpha_j \\
&\leq 2 \sum_{ij \in E(G)} \alpha_i \alpha_j \\
&= \alpha' A_G \alpha \\
&\leq \sup_{\|x\|=1} x' A_G x \\
&= \mu_{\max}(G).
\end{aligned}$$

□

Let δ be the minimum degree and Δ be the maximum degree of G. We have the following bound on $\mu_{\max}$.

Lemma 3 *For every graph G, we have*

$$\delta(G) \leq \mu_{\max}(G) \leq \Delta(G).$$

Proof: Let α be an eigenvector for eigenvalue $\mu = \mu_{\max}(G)$. Since $\alpha \neq 0$, we can assume α has at least one positive coordinate. (If all coordinates are none-positive, we consider $-\alpha$ instead.)

Let $\alpha_k = \max_i \alpha_i$ be the largest coordinate of α. Since $A\alpha = \mu\alpha$, we have

$$\mu \alpha_k = (A\alpha)_k = \sum_{i \sim k} \alpha_i \leq \Delta \alpha_k.$$

Thus, $\mu \leq \Delta$.

Now we show $\mu_{\max}(G) \geq \delta(G)$.

$$\begin{aligned}
\mu_{\max} &= \sup_{\|x\|=1} x' A_G x \\
&\geq \frac{1}{\sqrt{n}} \mathbf{1}' A_G \frac{1}{\sqrt{n}} \mathbf{1} \\
&= \frac{1}{n} 2 \sum_{i \sim j} a_{ij} \\
&= \frac{2|E(G)|}{n} \\
&\geq \delta(G).
\end{aligned}$$

$\square$

A k-coloring of a graph G is a map $c\colon V(G) \to [k] = \{1, 2, \ldots, k\}$. A k-coloring is said to be *proper* if the end vertices of any edge in G receive different colors. I.e.,

$$c(u) \neq c(v) \text{ for any } u \sim v.$$

In this case, we say G is k-colorable.

The chromatic number denoted by $\chi(G)$ is the minimum integer k such that G is k-colorable. For example, $\chi(K_n) = n$. $\chi(G) = 2$ if and only if G is a nonempty bipartite graph.

There is a simple bound on $\chi(G)$.

Theorem 1 *For every G, $\chi(G) \leq 1 + \Delta(G)$.*

Proof: Given any order $v_1, v_2, \ldots, v_n$, we color vertices one by one using $\Delta + 1$ colors. At time i, we assume $v_1, \ldots, v_{i-1}$ has been colored properly. Note that v_i has at most Δ neighbors in $v_1, \ldots, v_{i-1}$. We can pickup a distinct color for v_i other than those neighbors received. The resulted coloring is a proper coloring. $\square$.

Theorem 2 (Wilf 1967) *For every G, $\chi(G) \leq 1 + \lambda_{\max}(G)$.*

Proof: In the proof of the previous lemma, the graph G is k-colorable if v_i has at most $k-1$ neighbors in the induced subgraph on $v_1, v_2, \ldots, v_i$ for all $i = 1, 2, \ldots, n$.

Since the order of the vertices can be arbitrary, we choose v_n to be the vertex having the minimum degree. For $i = n, n-1, \ldots, 1$, let v_i be the vertex having minimum degree in the induced subgraph G_i on $v_1, v_2, \ldots, v_i$. Note

$$\begin{aligned}
\delta(G_i) &\leq \mu_{\max}(G_i) \\
&\leq \mu_{\max}(G).
\end{aligned}$$

Thus, under this order, the previous greedy algorithm results a proper k-coloring for any $k \leq 1 + \mu_{\max}(G)$. $\square$

Remark: Brook's theorem states that if G is a simple connected graph other than the complete graph and odd cycles then

$$\chi(G) \leq \Delta(G).$$

It is unknown whether similar result can be proved using $\mu_{\max}(G)$ instead.

Assume $\mu_1 > \mu_2 > \ldots > mu_k$ are distinct eigenvalues of A. The $\phi(x) = \prod_{i=1}^{k}(x - \mu_k)$ is called the minimal polynomial of A. We have

$$\phi(A) = 0.$$

Any polynomial $f(x)$ with $f(A) = 0$ is divisible by $\phi(x)$.

For any pair of vertices u, v, the distance $d(u, v)$ is the shortest length of any uv-path. The diameter of graph G is the maximum distance among all pairs of vertices which belongs to the same connected component.

Theorem 3 *The diameter of a graph is less than its number of distinct eigenvalues.*

Proof: Without loss of generality, we can assume G is connected. Let k be the number of distinct eigenvalues. The minimum polynomial $\phi(x)$ has degree k. Since $\phi(A) = 0$, A^k can be expressed as a linear combination of $I, A, \ldots, A^{k-1}$. Suppose the diameter of G is greater than or equal to k. There exists a pair of vertices u and v satisfying $d(u, v) = k$. We have $(A^k)_{uv} \geq 1$ and $(A^i)_{uv} = 0$ for $i = 0, 1, 2, \ldots, A^{k-1}$. This is a contradiction to the fact A^k is a linear combination of $I, A, \ldots, A^{k-1}$. $\square$

This result is tight for the hypercube Q_n.