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Graphs with three distinct eigenvalues and largest eigenvalue less than 8

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ABSTRACT

In this paper we consider graphs with three distinct eigenvalues, and we characterize those with the largest eigenvalue less than 8. We also prove a simple result which gives an upper bound on the number of vertices of graphs with a given number of distinct eigenvalues in terms of the largest eigenvalue.

1. Introduction

Let G be an undirected finite simple graph with n vertices and the adjacency matrix $A(G)$. Since $A(G)$ is a real symmetric matrix, its eigenvalues are real numbers. So we can assume that $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{n-1}$ are the adjacency eigenvalues of G. The multiset of the eigenvalues of $A(G)$ is called the *adjacency spectrum*. The maximum eigenvalue of $A(G)$ is called the *index* of G. A graph G is called an *integral graph* if its adjacency eigenvalues are integers.

Graphs with few distinct eigenvalues form an interesting class of graphs. Clearly if all the eigenvalues of a graph coincide, then we have a trivial graph (a graph without edges). Connected graphs with only two distinct eigenvalues are easily proven to be complete graphs. The first nontrivial graphs with three distinct eigenvalues are the strongly regular graphs. Graphs with exactly three distinct

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eigenvalues are generalizations of strongly regular graphs by dropping regularity. A large family of (in general) non-regular examples is given by the complete bipartite graphs $K_{m,n}$ with the spectrum $\{[\sqrt{mn}]^1, [0]^{m+n-2}, [-\sqrt{mn}]^1\}$. Other examples were found by Bridges and Mena [1] and Muzychuk and Klin [9], most of them being cones. A cone over a graph H is obtained by adding a vertex to H that is adjacent to all vertices of H. Those with the least eigenvalue -2 have been characterized by Van Dam (see [6]). Further results on graphs with few different eigenvalues can be found in [4–9]. In this paper we characterize graphs with three distinct eigenvalues and index less than 8. Moreover we show that the number of connected graphs with fix number of distinct eigenvalues and with largest eigenvalue not exceeding the given number are finite. The main result is the following theorem (the definitions of the following graphs are given in the next section).

Theorem 1. *If G is a connected graph with three distinct eigenvalues and index less than 8, then G is one of the following graphs: the complete bipartite graph, the strongly regular graphs with parameters srg-(5, 2, 0, 1), srg-(13, 6, 2, 3) and srg-(9, 6, 3, 6), the lattice graphs $L_2(3), L_2(4)$, the triangular graphs $T(4)$, $T(5)$, the cocktail party graph $CP(4)$, the Shrikhande graph, the cone over the Petersen graph, the Petersen graph, the Hoffman-Singleton graph and the Clebsch graph.*

2. Some definitions and preliminaries

In this section we express some useful results. First we give some definitions that will be used in the sequel. A t-(v, k, λ) *design* is a set of v points and a set of k-subsets of points, called *blocks*, such that any t-subset of points is contained in precisely λ blocks. The point x and the block b are called *incident* if $x \in b$. The *incidence graph* of a design is the bipartite graph with vertices the points and blocks of the design, where a point and a block are adjacent if and only if they are incident. A *projective plane of order n* is a 2-$(n^2 + n + 1, n + 1, 1)$ design. The *Fano plane* is the projective plane of order 2. The unique strongly regular graphs with parameters srg-(50, 7, 0, 1) and srg-(27, 10, 1, 5) are called the *Hoffman-Singleton graph* and the *Schläfli graph*, respectively. The *Shrikhande graph* is a strongly regular graph with parameters srg-(16, 6, 2, 2). A *cocktail party graph* $CP(n)$ is the complement of the disjoint union of n edges. The *Clebsch graph* is the unique strongly regular graph with parameters srg-(16, 5, 0, 2). The *triangular graph* $T(n)$ is the line graph of the complete graph K_n. The *lattice graph* $L_2(n)$ is the line graph of the complete bipartite graph $K_{n,n}$.

In this paper we assume that G be a simple connected graph with three distinct eigenvalues $\lambda_0 > \lambda_1 > \lambda_2$. Moreover let $N_G(C_3)$ denote the number of triangles of G and suppose $N_G^i(C_3)$ is the number of triangles containing v_i.

Lemma 1 [7]. *Suppose A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{n-1}$ and suppose s is the sum of the entries of A. Then $\lambda_0 \geqslant s/n \geqslant \lambda_{n-1}$ and equality on either side implies that every row sum of A equals s/n.*

An important property of connected graphs with three eigenvalues is that $(A - \lambda_1 I)(A - \lambda_2 I)$ is a rank one matrix. It follows that for some Perron–Frobenius eigenvector α corresponding to the eigenvalue λ_0 of G, we have $(A - \lambda_1 I)(A - \lambda_2 I) = \alpha\alpha^t$.

Lemma 2 [6]. *Let α be the Perron–Frobenius eigenvector corresponding to the eigenvalue λ_0 of G with three distinct eigenvalues such that $(A - \lambda_1 I)(A - \lambda_2 I) = \alpha\alpha^t$. Then*

(i) $d_i = -\lambda_1\lambda_2 + \alpha_i^2$ *is the degree of vertex v_i,*
(ii) $\lambda_{ij} = \lambda_1 + \lambda_2 + \alpha_i\alpha_j$ *is the number of common neighbors of v_i and v_j, if they are adjacent,*
(iii) $\mu_{ij} = \alpha_i\alpha_j$ *is the number of common neighbors of v_i and v_j, if they are not adjacent.*

Corollary 1. *Let G be an integral connected graph with three distinct eigenvalues. Then for each v_i and $v_j \in V(G)$, $\alpha_i\alpha_j = \sqrt{(d_i + \lambda_1\lambda_2)(d_j + \lambda_1\lambda_2)}$ is integer.*

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By Lemma 2, we can see that if G is a regular graph, then G is a strongly regular graph. Let A be the adjacency matrix of a graph on n vertices, m edges and let $\{[\lambda_0]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}\}$ for $\lambda_0 > \lambda_1 > \lambda_2$ be its adjacency spectrum (m_i is the multiplicity of the eigenvalue λ_i for $i = 1, 2$). We know that $\mathrm{tr}(A^i)$ gives the total number of closed walks of length i. On the other hand $\mathrm{tr}(A^i)$ equals to the sum of the ith powers of the adjacency eigenvalues. So if $N_G(H)$ is the number of subgraphs of type H of G, then we have

$$m_1 + m_2 = n - 1, \tag{1}$$

$$\lambda_1 m_1 + \lambda_2 m_2 + \lambda_0 = 0, \tag{2}$$

$$\lambda_1^2 m_1 + \lambda_2^2 m_2 + \lambda_0^2 = 2m, \tag{3}$$

$$\lambda_1^3 m_1 + \lambda_2^3 m_2 + \lambda_0^3 = 6N_G(C_3). \tag{4}$$

Using (1)–(4) we obtain the following equalities:

$$\begin{aligned} m_2 &= \frac{\lambda_1(n-1)+\lambda_0}{\lambda_1-\lambda_2} = \frac{2m+\lambda_0\lambda_1-\lambda_0^2}{\lambda_2(\lambda_2-\lambda_1)} \\ &= \frac{6N_G(C_3)+\lambda_1^2\lambda_0-\lambda_0^3}{\lambda_2^3-\lambda_1^2\lambda_2} = \frac{6N_G(C_3)-2m\lambda_1+\lambda_0^2\lambda_1-\lambda_0^3}{\lambda_2^3-\lambda_2^2\lambda_1}. \end{aligned} \tag{5}$$

Chang showed that up to isomorphism there are four strongly regular graphs with parameters srg-(28, 12, 6, 4), namely $T(8)$ and three other graphs, known as the Chang graphs. *Switching* with respect to some subset of the vertices means that we interchange the edges and the non-edges between the subset and its complement. Muzychuk and Klin found parametric conditions for switching in a strongly regular graph to obtain a non-regular graph with three eigenvalues. Moreover, they proved that the only such graph that can be obtained by switching in a triangular graph is the one obtained by switching in $T(9)$ with respect to an 8-clique. This gives a graph with spectrum $\{[21]^1, [5]^7, [-2]^{28}\}$. We have the following graph with spectrum $\{[11]^1, [3]^7, [-2]^{16}]\}$, which is related to the strongly regular lattice graph $L_2(5)$. For a vertex x in $L_2(5)$, the set of its neighbors can be partitioned into two 4-sets, each inducing a 4-clique. Now delete x and (switch) interchange edges and non-edges between one of the 4-sets and the set of non-neighbors of x (see [6]). In the next theorem all connected graphs with three distinct eigenvalues, each at least -2, are characterized.

Theorem 2. *Let G be a connected graph with three distinct eigenvalues.*

(i) [6] *If each eigenvalue of G is greater than* -2, *then G is either* $K_{1,2}$, $K_{1,3}$, *or* C_5,
(ii) [2] *If G is a strongly regular graph with the least eigenvalue* -2, *then G is one of* $L(K_n)$, $L(K_{n,n})$, $CP(n)$, *complement of the Schläfli graph, the Shrikhande graph, complement of the Clebsch graph, the Petersen graph and the three Chang graphs.*
(iii) [6] *If each eigenvalue of G is at least* -2 *and G is not a strongly regular graph or a complete bipartite graph, then G is one of the following graphs: the cone over the Petersen graph, the graph derived from the complement of the Fano plane, the cone over the Shrikhande graph, the cone over the lattice graph* $L_2(4)$, *the graph on the points and planes of* $AG(3, 2)$, *the graph related to the lattice graph* $L_2(5)$ *(see above), the cones over the Chang graphs, the cone over the triangular graph* $T(8)$ *and the graph obtained by switching in* $T(9)$ *with respect to an 8-clique.*

Lemma 3 [6]. *The only connected non-regular graphs with three distinct eigenvalues and at most twenty vertices, which are not complete bipartite are the cone over the Petersen graph, the graph derived from the complement of the Fano plane, the cone over the Shrikhande graph and the cone over the lattice graph* $L_2(4)$.

Now we will determine all connected integral graphs with three distinct eigenvalues each at least -2 and index less than 8.

Theorem 3. *All connected integral graphs with three distinct eigenvalues, index less than 8 and the least eigenvalue at least -2 are $K_{1,4}, K_{2,2}$, the cone over the Petersen, the Shrikhande graph, the Petersen graph, the lattice graphs $L_2(3), L_2(4)$, the triangular graphs $T(4), T(5)$, the cocktail party graph $CP(4)$.*

Proof. Using Theorem 2, the result follows. □

3. Bound on the number of vertices

In this section we obtain an upper bound on the number of vertices of graphs with a given number of distinct eigenvalues in terms of the largest eigenvalue. Therefore we conclude that the number of connected graphs with fix number of distinct eigenvalues and with largest eigenvalue not exceeding the given number is finite.

Lemma 4 [3]. *Let G be a connected graph and suppose H is a proper subgraph of G. Then $\lambda_0(H) < \lambda_0(G)$ where $\lambda_0(G)$ is the largest eigenvalue of G.*

It is well-known that the number of distinct eigenvalues of G is at least $d + 1$ where d is the diameter of G (see [3]). The following theorem gives an upper bound on the number of vertices in terms of the largest eigenvalue.

Theorem 4. *Let G be a connected graph with r distinct eigenvalues on n vertices and the largest eigenvalue λ_0. Then $n \leqslant 1 + \lambda_0^2 \frac{(\lambda_0^2-1)^{r-1}-1}{\lambda_0^2-2}$.*

Proof. Let Δ be the maximum degree of G. Since $d + 1 \leqslant r$ we have

$$\begin{aligned} n &\leqslant 1 + \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 + \cdots \Delta(\Delta-1)^{(r-2)} \\ &= 1 + \Delta\frac{(\Delta-1)^{r-1}-1}{\Delta-2}. \end{aligned}$$

On the other hand $K_{1,\Delta}$ is the subgraph of G and so by Lemma 4, we have $\sqrt{\Delta} \leqslant \lambda_0$. Therefore by the above inequality we have

$$n \leqslant 1 + \lambda_0^2 \frac{(\lambda_0^2-1)^{r-1}-1}{\lambda_0^2-2}. \quad \square$$

Since each complete graph has two distinct eigenvalues the diameter of each graph with three distinct eigenvalues is 2 and so we have the following result.

Lemma 5. *Let G be a connected graph with three distinct eigenvalues on n vertices and the largest eigenvalue λ_0. Then $n \leqslant 1 + \Delta^2 \leqslant 1 + \lambda_0^4$.*

4. Non-integral graphs

In this section we consider non-integral graphs with three distinct eigenvalues and index less than 8. Using the following facts, we will show that each of these graphs is either a complete bipartite graph or a strongly regular graph.

Lemma 6 [6]. *Let G be a non-integral connected graph with three distinct eigenvalues on n vertices and suppose G is not a complete bipartite graph. Then n is odd and $\lambda_0 = (n-1)/2$, $(\lambda_1, \lambda_2) = (-1+\sqrt{b}/2, -1-$*

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$\sqrt{b}/2$), for some $b \equiv 1 (mod\, 4)$ and $b \geqslant n$, with equality if and only if G is a strongly regular graph. Moreover, if $n \equiv 1 (mod\, 4)$, then all vertex degrees are even and if $n \equiv 3 (mod\, 4)$, then $b \equiv 1 (mod\, 8)$.

Lemma 7 [6]. *If the largest eigenvalue of a connected graph G with three distinct eigenvalues is not an integer, then G is a complete bipartite graph.*

Theorem 5. *Let G be a non-integral connected graph with three distinct eigenvalues and index less than 8. Then G is one of the following graphs: the complete bipartite graph, the strongly regular graphs with parameters srg-$(5, 2, 0, 1)$ and srg-$(13, 6, 2, 3)$.*

Proof. Suppose that G has distinct eigenvalues $\lambda_0 > \lambda_1 > \lambda_2$. If G is a complete bipartite graph, then $G = K_{p,q}$ where $pq < 64$ is non-square. Now let G be not a complete bipartite graph. By Lemma 7, λ_0 is integer. By applying Lemma 6, we can see that $n \leqslant 15$ is an odd number. If G is a regular graph, then G is a strongly regular graph with $k = \lambda_0 \leqslant 7$ and so G is a non-integral strongly regular graph with parameters srg-(v, k, λ, μ) where $(v, k, \lambda, \mu) \in \{(5, 2, 0, 1), (13, 6, 2, 3)\}$. Finally from Lemma 3, we can see that each non-regular non-integral connected graph G with $n \leqslant 15$ is complete bipartite. □

5. Integral graphs

In this section we will characterize all integral graphs with three distinct eigenvalues and index less than 8. First we give some useful lemmas.

The minimal polynomial $P_A(\lambda)$ of the adjacency matrix $A(G)$ of a graph G is the unique monic polynomial of minimal degree such that $P_A(A) = 0$. If $\{\lambda_0, \lambda_1, \ldots, \lambda_r\}$ is the set of distinct eigenvalues of a graph G, then (see [3])

$$P_A(\lambda) = \prod_{i=0}^{r}(\lambda - \lambda_i).$$

Lemma 8. *Let G be an integral connected graph with three distinct eigenvalues and suppose $\lambda_1 \leqslant 0$ and $\lambda_0 \leqslant 7$. Then G is either the complete bipartite graph $K_{p,q}$ for square $1 < pq \leqslant 49$, $CP(4)$, $T(4)$ or the strongly regular graph with parameters srg-$(9, 6, 3, 6)$.*

Proof. Since $\lambda_1 \leqslant 0$, G is a complete multipartite graph (see [3]). On the other hand G is not a complete graph. So G has a color class with more than one vertex. Since the rows of the adjacency matrix of G corresponding to the vertices of a given color class are equal, the rank of the adjacency matrix is less than n and so $\lambda_1 = 0$. It is clear that if G is a complete bipartite graph, then G is $K_{p,q}$ for square $1 < pq \leqslant 49$. Now assume that G is not a complete bipartite graph. By equality (2), it is clear that $\lambda_2 \mid \lambda_0$. From Theorem 2, we can see that there is no integral graph with three distinct eigenvalues and $\lambda_2 > -2$. It is clear that for each prime $\lambda_0 \leqslant 7$, we have $\lambda_2 = -\lambda_0$. Therefore G is complete bipartite. So the possible spectra are $S_1 = \{[4]^1, [0]^{m_1}, [-2]^2\}$, $S_2 = \{[6]^1, [0]^{m_1}, [-2]^3\}$ and $S_3 = \{[6]^1, [0]^{m_1}, [-3]^2\}$. By Theorem 3, we know that the only graph with spectrum S_1 is $T(4)$ and the graph with spectrum S_2 is $CP(4)$. Now let the spectrum of G be S_3. If G is regular, then by Theorem 2, G is the strongly regular graph with parameters srg-$(9, 6, 3, 6)$. So suppose that G is non-regular. The minimal polynomial of $A(G)$ is $P_A(\lambda) = \lambda^3 - 3\lambda^2 - 18\lambda$. So for $v_i \in V(G)$ we have $2N_G^i(C_3) = 3d_i$. On the other hand from the equalities (3), (4), $N_G(C_3) = m = 27$. Therefore

$$(N_G^i(C_3), d_i) \in \{(3, 2), (6, 4), (9, 6), (12, 8), (15, 10), (18, 12), (21, 14), (24, 16), (27, 18)\}.$$

But it is clear that $(N_G^i(C_3), d_i) \neq (3, 2)$. Since from Corollary 1, $d_i d_j$ must be a square, so the only possible degree sequences are $\{8, 18\}$ and $\{4, 16\}$. Let $x_k = |\{v_i \in V(G) | d_i = k\}|$. First let possible degree of each vertex be 8 or 18. Then we have $8x_8 + 18x_{18} = 54$. Since $N_G(C_3) = 27$ and each vertex of degree 18 lies on 27 triangles, $x_{18} = 1$. Hence $8x_8 = 36$ which is not true. Now suppose each possible

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degree of each vertices is 4 or 16. Then we have $4x_4 + 16x_{16} = 54$. Since $4 \nmid 54$ and the left side of the equality is a multiple of 4, this case is impossible.

Lemma 9. *Let G be a connected graph with three distinct eigenvalues. Suppose G is not a complete bipartite graph. Then*

(i) *If G is a non-regular graph, then* $N_G(C_3) > 0$.
(ii) *If G has a vertex* v_i *with* $N^i_G(C_3) = 0$, *then the degree of* v_i *is* $d_i = -\lambda_0\lambda_1\lambda_2/(\lambda_0 + \lambda_1 + \lambda_2)$.

Proof. (i) Since G has three distinct eigenvalues, the degree of the minimal polynomial of $A(G)$ is 3 and we have $A^3 = (\lambda_0 + \lambda_1 + \lambda_2)A^2 - (\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2)A + (\lambda_0\lambda_1\lambda_2)I$. So we have $2N^i_G(C_3) = (\lambda_0 + \lambda_1 + \lambda_2)d_i + (\lambda_0\lambda_1\lambda_2)$. If $(\lambda_0 + \lambda_1 + \lambda_2) = 0$, then $\lambda_1 \leqslant 0$ and so G is complete multipartite. On the other hand G is not complete bipartite and so $N_G(C_3) > 0$. Now let $(\lambda_0 + \lambda_1 + \lambda_2) \neq 0$. Since G is non-regular it is clear that there is at least one vertex of G with $N^i_G(C_3) > 0$. Consequently we have $N_G(C_3) > 0$,

(ii) If $(\lambda_0 + \lambda_1 + \lambda_2) = 0$, then $\lambda_1 \leqslant 0$ and so G is complete multipartite. Since $N^i_G(C_3) = 0$, G is complete bipartite, which is impossible. Now let $(\lambda_0 + \lambda_1 + \lambda_2) \neq 0$. Since $N^i_G(C_3) = 0$, from the equality $2N^i_G(C_3) = (\lambda_0 + \lambda_1 + \lambda_2)d_i + (\lambda_0\lambda_1\lambda_2)$ the result follows.

Lemma 10. *Let G be a connected graph with three distinct eigenvalues* $\lambda_0 > \lambda_1 > \lambda_2$. *Then the inequality* $-\lambda_2\lambda_1 < \lambda_0$ *holds.*

Proof. Suppose that G has n vertices and m edges. Let the spectrum of G be $\{[\lambda_0]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}\}$. Now from Eqs. (1) and (2) we get $(\lambda_1 - \lambda_2)m_2 = n\lambda_1 + (\lambda_0 - \lambda_1)$. By Eqs. (2) and (3) we obtain $\lambda_2(\lambda_2 - \lambda_1)m_2 = 2m - \lambda_0(\lambda_0 - \lambda_1)$. By comparing these equations we have $-\lambda_2(n\lambda_1 + \lambda_0 - \lambda_1) = 2m - \lambda_0(\lambda_0 - \lambda_1)$, and so $-\lambda_2\lambda_1 n + (\lambda_0 - \lambda_2)(\lambda_0 - \lambda_1) = 2m$.

Using Lemma 1, we obtain

$$-\lambda_2\lambda_1 + \frac{(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_1)}{n} \leqslant \lambda_0.$$

Since $\lambda_0 > \lambda_1 > \lambda_2$, we deduce that $-\lambda_2\lambda_1 < \lambda_0$. □

Lemma 11. *Let G be a connected graph with spectrum* $\{[5]^1, [1]^{m_1}, [-3]^{m_2}\}$ *for some positive integers* m_1 *and* m_2. *Then G is the strongly regular graph with parameters srg-*$(16, 5, 0, 2)$ (*the Clebsch graph*).

Proof. From (5), we have

$$m_2 = \frac{n+4}{4} = \frac{2m-20}{12} = \frac{-6N_G(C_3)+120}{24}.$$

First let $N_G(C_3) > 0$. Since $m_2 = (-6N_G(C_3) + 120)/24$ is integer, $N_G(C_3)$ is a multiple of 4. Moreover we have $2m = -3N_G(C_3) + 80$ and so $m \leqslant 34$. Since $m_2 = (2m - 20)/12$, m is even. On the other hand, $3n = 2m - 32 > 3$. It is clear that there is no even $18 \leqslant m \leqslant 34$ such that $2m - 32$ be a multiple of 3 and $m \leqslant n(n-1)/2$ unless $m = 28, 34$. If $m = 28$, then $G = K_8$ and so $\lambda_0 = 7$, which is not true. For $m = 34$ we have $n = 12$ and $N_G(C_3) = 4$. The minimal polynomial of $A(G)$ is $P_G(\lambda) = \lambda^3 - 3\lambda^2 - 13\lambda + 15$. So we have $2N^i_G(C_3) = 3d_i - 15$. On the other hand $N_G(C_3) = 4$ and so we get

$$(N^i_G(C_3), d_i) \in \{(0,5), (3,7)\}.$$

If G is a 5-regular graph (respectively, 7-regular graph), then $N_G(C_3) = 0$ (respectively, $m = 42$). Which is impossible. So there are vertices v_i and v_j of G with $d_i = 5$ and $d_j = 7$ and so by Lemma 2, we get $\alpha_i\alpha_j = \sqrt{8}$, which contradicts corollary 1.

Now let $N_G(C_3) = 0$. Since G is not complete bipartite by Lemma 9, we have G is a 5-regular graph. Using Lemma 2 and the above equalities, it follows that G is the strongly regular graph with parameters srg-$(16, 5, 0, 2)$, which is the Clebsch graph. □

In the next theorem we determine all graphs with three distinct eigenvalues and index at most 5.

Theorem 6. *The connected integral graphs with three distinct eigenvalues and index at most* 5 *are the triangular graph* $T(4)$, *the lattice graph* $L_2(3)$, $K_{p,q}$ *where* $1 < pq \leqslant 25$ *is square, the cone over the Petersen graph, the Clebsch graph and the Petersen graph.*

Proof. Suppose that G is an integral graph with index at most 5, n vertices, m edges and spectrum $\{[\lambda_0]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}\}$.

If $\lambda_2 = -\lambda_0$, then G is a complete bipartite graph, and so $G = K_{p,q}$ where $1 < pq \leqslant 25$ is square. If $\lambda_2 \geqslant -2$ then by Theorem 3, G is one of the triangular graph $T(4)$, $K_{1,4}$, $L_2(2) = K_{2,2} = CP(2)$, $L_2(3)$, the cone over the Petersen graph and the Petersen graph.

From Lemma 8, if $\lambda_1 \leqslant 0$, then G is a complete bipartite graph or $T(4)$. Therefore we may assume that $-\lambda_0 < \lambda_2 < -2$ and $\lambda_1 > 0$. Hence we consider the following two cases:

Case 1. Let $\lambda_0 = 4$. Since $-4 < \lambda_2 < -2$ and $0 < \lambda_1 < 4$, we have $\lambda_2 = -3$ and $\lambda_1 \in \{1, 2, 3\}$. By Lemma 10, we can see that $\lambda_1 \neq 2, 3$. So we have $\lambda_1 = 1$.

Using (5), we obtain

$$m_2 = \frac{n+3}{4} = \frac{2m-12}{12} = \frac{-6N_G(C_3)+60}{24}.$$

First suppose that $N_G(C_3) > 0$. From the above equalities, we can see that $2m - 12 = -3N_G(C_3) + 30$ and $2 \mid N_G(C_3)$ and so $m \leqslant 18$. On the other hand we have $3n = 2m - 21$. Since there is no integral graph on $n \leqslant 5$ vertices with three distinct eigenvalues and $3n = 2m - 21$, we have $m > 18$, which is impossible. Now let $N_G(C_3) = 0$. Since G is not complete bipartite by Lemma 9, G is a 6-regular graph. This means that $\lambda_0 = 6$, which is impossible.

Case 2. Let $\lambda_0 = 5$. Since $-5 < \lambda_2 < -2$ and $0 < \lambda_1 < 5$ we have $\lambda_2 \in \{-3, -4\}$ and $\lambda_1 \in \{1, 2, 3, 4\}$.

First let $\lambda_2 = -4$. Then by Lemma 10, we can see that $\lambda_1 \neq 2, 3, 4$. So let $\lambda_1 = 1$. From (5), we get

$$m_2 = \frac{n+4}{5} = \frac{2m-20}{20} = \frac{-6N_G(C_3)+120}{60}.$$

By the equality $(2m-20)/20 = (-6N_G(C_3)+120)/60$ we have $m = -N_G(C_3) + 30$ and so $m \leqslant 30$. Since $(n+4)/5$ is integer, $n - 1 \geqslant 5$ is a multiple of 5. Moreover by the equality $(n+4)/5 = (2m-20)/20$ and the fact that $m \leqslant n(n-1)/2$ we get $n > 6$ and so $m > 30$, which is impossible.

Now let $\lambda_2 = -3$. Then from Lemma 10, we get $\lambda_1 \neq 2, 3, 4$ and so $\lambda_1 = 1$. Therefore by Lemma 11, G is the strongly regular graph with parameters srg-$(16, 5, 0, 2)$, which is the Clebsch graph.

Lemma 12. *Let G be a connected graph with spectrum* $\{[6]^1, [1]^{m_1}, [-3]^{m_2}\}$ *for some positive integers* m_1 *and* m_2. *Then G is the strongly regular graph with parameters srg-*$(15, 6, 1, 3)$ (*the* $(6, 2)$*-Kneser graph*).

Proof. From 5, we obtain

$$m_2 = \frac{n+5}{4} = \frac{2m-30}{12} = \frac{2m-6N_G(C_3)+180}{36}.$$

Since $4m = -6N_G(C_3) + 270$, we have $m \leqslant 67$. On the other hand we have $2m = 3n + 45$, by applying Lemma 1, we get $n \geqslant 15$. So we have $m \geqslant 45$. Moreover $m_2 = (2m - 30)/12$ is integer and so m is a multiple of 3. Therefore $m \in \{45, 48, 51, 54, 57, 60, 63, 66\}$. It follows that $n \in \{15, 17, 19, 21, 23, 25, 27, 29\}$. Again $m_2 = (n+5)/4$ is integer and so $n \neq 17, 21, 25, 29$. First let $n \in \{15, 19\}$. Using Lemma 3, there is no non-regular graph on n vertices with spectrum $\{[6]^1, [1]^{m_1}, [-3]^{m_2}\}$. If G is regular, then G is a 6-regular graph on 15 vertices and so G is the strongly regular graph with parameters srg-$(15, 6, 1, 3)$. Note that there is no strongly regular graph on 19 vertices. Now we assume that $n \in \{23, 27\}$. The minimal polynomial of $A(G)$ is $P_A(\lambda) = \lambda^3 - 4\lambda^2 - 15\lambda + 18$. So for each $v_i \in V(G)$ we have

$$2N_G^i(C_3) = 4d_i - 18.$$

Now assume that $n = 23$, then we get $N_G(C_3) = 7$. It follows that for at most 21 vertices $N_G^i(C_3) \neq 0$. This means that for at least 2 vertices we have $N_G^i(C_3) = 0$ and so $d_i = 18/4$, which is impossible.

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For $n = 27$ we have $N_G(C_3) = 3$. By a similar argument, we can see that $n = 27$ is impossible too. So $n \neq 23, 27$. $\square$

In the next theorem integral graphs with three distinct eigenvalues and index 6 are identified.

Theorem 7. *The only connected integral graphs with three distinct eigenvalues and index* 6 *are the Shrikhande graph, the strongly regular graphs with parameters srg-*$(15, 6, 1, 3)$, *srg-*$(9, 6, 3, 6)$, *the lattice graph* $L_2(4), CP(4), T(5), K_{1,36}, K_{2,18}, K_{3,12}, K_{4,9}$ *and* $K_{6,6}$.

Proof. Suppose that G is an integral graph with index 6, n vertices, m edges and spectrum $\{[6]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}\}$. If $\lambda_2 = -6$, then G is a complete bipartite graph and so G is one of $K_{1,36}, K_{2,18}, K_{3,12}, K_{4,9}$ and $K_{6,6}$. If $\lambda_2 \geqslant -2$ then by Theorem 3, G is one of the Shrikhande graph, the lattice graph $L_2(4)$, $CP(4)$ and $T(5)$. From Lemma 8, if $\lambda_1 \leqslant 0$, then G is a complete bipartite graph, $CP(4)$ or the strongly regular graph with parameters srg-$(9, 6, 3, 6)$. Therefore we can assume that $-6 < \lambda_2 < -2$ and $\lambda_1 > 0$. Since $-6 < \lambda_2 < -2$, we have $\lambda_2 \in \{-5, -4, -3\}$ and so by Lemma 10, $\lambda_1 = 1$. Therefore we have the following three cases.

Case 1. Assume that $\lambda_2 = -5$. Using (5), we get

$$m_2 = \frac{n+5}{6} = \frac{2m-30}{30} = \frac{2m - 6N_G(C_3) + 180}{150}.$$

Since $8m = -6N_G(C_3) + 330$, we have $m \leqslant 41$. Moreover $m_2 = (2m - 30)/30$ is integer and so $m \in \{15, 30\}$. On the other hand $5n = 2m - 55$. It follows that $n = 1$, which is false.

Case 2. Let $\lambda_2 = -4$. From (5) we have

$$m_2 = \frac{n+5}{5} = \frac{2m-30}{20} = \frac{2m - 6N_G(C_3) + 180}{80}.$$

From the equality $(2m - 30)/20 = (2m - 6N_G(C_3) + 180)/80$ we can see that, $m \leqslant 50$. Moreover it is clear that m is a multiple of 5 and $4n = 2m - 50$. It follows that $m \in \{35, 45\}$. Since $m \leqslant n(n-1)/2$ we have $n = 10$ and $m = 45$ and so $G = K_{10}$. But the maximum eigenvalue of K_{10} is 9.

Case 3. Assume that $\lambda_2 = -3$. By Lemma 12, G is the strongly regular graph with parameters srg-$(15, 6, 1, 3)$. $\square$

Lemma 13. *Let G be a connected graph with spectrum* $\{[7]^1, [2]^{m_1}, [-3]^{m_2}\}$ *for some positive integers m_1 and m_2. Then G is the Hoffman-Singleton graph.*

Proof. From (5), we have

$$m_2 = \frac{2n+5}{5} = \frac{2m-35}{15} = \frac{4m - 6N_G(C_3) + 245}{45}.$$

First let $N_G(C_3) > 0$. By the third equality we deduce that $2m \leqslant 344$. Since by the second equality we have $2m = 6n + 50$, so $n \leqslant 49$. On the other hand by Lemma 1, $\lambda_0 = 7 \geqslant 2m/n = (6n + 50)/n$, so $n \geqslant 50$, which is a contradiction. Now let $N_G(C_3) = 0$. Since G is not complete bipartite by Lemma 9, G is a 7-regular graph. Using the above equalities and Lemma 2, we can see that G is the strongly regular graph with parameters srg-$(50, 7, 0, 1)$, that is, the Hoffman-Singleton graph. $\square$

Lemma 14. *Let m_1 and m_2 be positive integers. There is no connected graph with spectrum* $\{[7]^1, [1]^{m_1}, [-3]^{m_2}\}$.

Proof. Let G be a connected graph with parameters $\{[7]^1, [1]^{m_1}, [-3]^{m_2}\}$. Using (5), we obtain

$$m_2 = \frac{n+6}{4} = \frac{2m-42}{12} = \frac{2m - 6N_G(C_3) + 294}{36}.$$

Since G is not complete bipartite if $N_G(C_3) = 0$, then by Lemma 9, for each vertex v_i we have $d_i = 21/5$, a contradiction. So we have $N_G(C_3) > 0$. From the above equalities, we can see that $N_G(C_3)$ is even.

So by the third equality we get $m \leqslant 102$. Since $(n+6)/4 = (2m-42)/12$, we have $n \leqslant 48$. Moreover by Lemma 1, we get $\lambda_0 = 7 \geqslant 2m/n = (3n+60)/n$ and so $n \geqslant 15$. Since we have $4 \mid n+6$, then $n \in \{18, 22, 26, 30, 34, 38, 42, 46\}$. By Lemma 3 and the fact that there is no strongly regular graph on 18 vertices with index 7 we have $n \neq 18$.

The minimal polynomial of $A(G)$ is $P_A(\lambda) = \lambda^3 - 5\lambda^2 - 17\lambda + 21$. So for $v_i \in V(G)$ we have $2N_G^i(C_3) = 5d_i - 21$. If there is some vertex, like v_i, so that $N_G^i(C_3) = 0$, then $d_i = 21/5$, that is impossible. Hence we can assume that for each $v_i \in V(G)$, $N_G^i(C_3) > 0$. Let $n = 46$. Then $N_G(C_3) = 4$ and so for at most 12 vertices we have $N_G^i(C_3) \neq 0$, which is false. For $n = 42$, $N_G(C_3) = 8$ and for $n = 38$, $N_G(C_3) = 12$. By a similar discussion we can see that these cases are impossible.

For $n \in \{22, 26, 30, 34\}$ we have $(N_G(C_3), n) \in \{(28, 22), (24, 26), (20, 30), (16, 34)\}$, so by the equality $2N_G^i(C_3) = 5d_i - 21$ we get $(N_G^i(C_3), d_i) \in \{(2, 5), (7, 7), (12, 9), (17, 11), (22, 13), (27, 15)\}$. But if the case $(27, 15)$ happens, then $n = 22$ and we can see that there are some vertices with $N_G^i(C_3) = 0$, contradicting to our assumption. It is clear that

$$3N_G(C_3) = \sum_{v_i \in V(G)} N_G^i(C_3).$$

Since $N_G^i(C_3) \geqslant 2$ for each $v_i \in V(G)$ and for $n = 34$ we have $N_G(C_3) = 16$, by the above equation we get $48 = \sum_{i=1}^{34} N_G^i(C_3) \geqslant 68$. Which is a contradiction. Let $n = 30$. Then $N_G(C_3) = 20$ and so we have $60 = \sum_{i=1}^{30} N_G^i(C_3)$. This means that for each $v_i \in V(G)$, $d_i = 5$. So G is a 5-regular graph and so $\lambda_0 = 5$ which is not true.

Now we define $x_k = |\{v_i \in V(G) | N_G^i(C_3) = k\}|$. If we consider the case $n = 26$, then we have $m = 69$ and $N_G(C_3) = 24$. With the above assumption we obtain $x_2 + x_7 + x_{12} + x_{17} + x_{22} = 26$, and $5x_2 + 7x_7 + 9x_{12} + 11x_{17} + 13x_{22} = 138$. From these equations we get

$$2x_7 + 4x_{12} + 6x_{17} + 8x_{22} = 8. \tag{6}$$

Doing the same for the case $n = 22$, we obtain

$$2x_7 + 4x_{12} + 6x_{17} + 8x_{22} = 16. \tag{7}$$

Now by Lemma 2, we compute for each vertex v_i its corresponding α_i and d_i

$$(\alpha_i, d_i) \in \{(\sqrt{2}, 5), (2, 7), (\sqrt{6}, 9), (\sqrt{8}, 11), (\sqrt{10}, 13)\}.$$

Since by Corollary 1, $\alpha_i\alpha_j$ for all i, j is integer, then we have only vertices of degree 5 and 11. It follows that $x_7 = x_{12} = x_{22} = 0$. So for $n = 26$ by (6), we get $6x_{17} = 8$ and for $n = 22$ by (7), we have $6x_{17} = 16$, which are impossible. □

Finally in the next theorem we find all integral graphs with three distinct eigenvalues and index 7.

Theorem 8. *All connected integral graphs with three distinct eigenvalues and index* 7 *are precisely* $K_{1,49}$, $K_{7,7}$ *and the Hoffman-Singleton graph.*

Proof. Suppose that G is an integral graph with index 7, n vertices, m edges and with spectrum $\{[7]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}\}$ where $\lambda_0 > \lambda_1 > \lambda_2$.

If $\lambda_2 = -7$, then G is a complete bipartite graph and so G is either $K_{1,49}$ or $K_{7,7}$. By Theorem 3, there is no G with index 7 and $\lambda_2 \geqslant -2$. From Lemma 8, if $\lambda_1 \leqslant 0$, then G is a complete bipartite graph. Therefore we may assume that $-7 < \lambda_2 < -2$ and $\lambda_1 > 0$.
By Lemma 10, for $\lambda_2 \in \{-6, -5, -4\}$ we have $\lambda_1 = 1$ and for $\lambda_2 = -3$ we have $\lambda_1 \in \{1, 2\}$. So we consider the following four cases.

Case 1. Let $\lambda_2 = -6$. From (5), we get

$$m_2 = \frac{n+6}{7} = \frac{2m-42}{42} = \frac{2m - 6N_G(C_3) + 294}{252}.$$

By the above equalities, we get $m \leqslant 54$ and m is a multiple of 21. Therefore $m \in \{21, 42\}$. Moreover $6n = 2m - 78$ and so $n = -6, 1$, which is not true.

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Case 2. Let $\lambda_2 = -5$. From (5), we obtain

$$m_2 = \frac{n+6}{6} = \frac{2m-42}{30} = \frac{2m-6N_G(C_3)+294}{150}.$$

By the third equality we can see that, $m \leqslant 63$. Since n is a multiple of 6, from the second equality we get $m \geqslant 51$. On the other hand $m_2 = (2m-42)/30$ is integer. Hence m is a multiple of 3 and so $m \in \{51, 54, 57, 60, 63\}$. Using the facts $m \leqslant n(n-1)/2$ and $5 \mid 2m-42$, we get $m \neq 51, 54, 57, 60, 63$.

Case 3. Let $\lambda_2 = -4$. Using (5), we have

$$m_2 = \frac{n+6}{5} = \frac{2m-42}{20} = \frac{2m-6N_G(C_3)+294}{80}.$$

From the third equality we get $m \leqslant 77$. It follows that $n \leqslant 22$. It is clear that $n+6$ is a multiple of 5, so $n \in \{4, 9, 14, 19\}$. Since there is no strongly regular graph on $n \in \{4, 9, 14, 19\}$ vertices with $\lambda_2 = -4$ and $\lambda_0 = 7$, by Lemma 3, there is no graph with $n = 4, 9, 14, 19$.

Case 4. Let $\lambda_2 = -3$. So $\lambda_1 \in \{1, 2\}$. But from Lemmas 13 and 14, we know that the only graph is Hoffman-Singleton. □

In the next theorem integral graphs with three distinct eigenvalues and index less than 8 are identified. Since all non-integral graphs with three distinct eigenvalues and index less than 8 are characterized in Theorem 5, we have the complete characterization of graphs with three distinct eigenvalues and index less than 8.

Theorem 9. *If G is a connected integral graph with three distinct eigenvalues and index less than* 8, *then G is one of the following graphs: the lattice graphs* $L_2(3), L_2(4)$, *the triangular graphs* $T(4), T(5)$, *the cocktail party graph* $CP(4)$, $K_{p,q}$ *where* $1 < pq \leqslant 49$ *is a square, the strongly regular graph with parameters srg*-(9, 6, 3, 6), *the Shrikhande graph, the cone over the Petersen graph, the Petersen graph, the Hoffman-Singleton graph and the Clebsch graph.*

Proof. Summing up the results of Theorems 6, 7 and 8, the result follows. □

Finally using Theorems 5 and 9, Theorem 1 can be proved.

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