Math 776 Graph Theory Lecture Notes 6
Matchings

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Definition 1 *A matching in a graph G is a set of non-loop edges with no shared endpoints.*

Definition 2 *The vertices incident to the edges of a matching M are saturated by M. Note: All other vertices are said to be unsaturated.*

Definition 3 *A perfect matching in a graph G is a matching that saturates every vertex.*

Definition 4 *An M-alternating path is a path that alternates between edges that are in M and edges that are not in M.*

Definition 5 *An M-augmenting path is a M-alternating path where the endpoints of the path are unsaturated by M.*

Definition 6 *A maximal matching M is a matching such that M united with any other edge is not a matching. Also, M is a maximal matching if $|M| \geq |M'|$ for any matching M'.*

Lemma 1 *Every component of the symmetric difference of two matchings is a path or is an even cycle.*

Definition 7 *The symmetric difference of two sets A and B, denoted $A \Delta B$ is*

$$A \Delta B = (A \backslash B) \cup (B \backslash A)$$

Proof: Let M_1 and M_2 be two matchings. Also let

$$F = M_1 \Delta M_2.$$

Considering F as a subgraph we can see that $d_F(v) \leq 2$ for any vertex v. So, F is a graph that is the disjoint union of cycles and paths. But those cycles can not be odd cycles since odd cycles are not a union of two matchings. Therefore each component of F is either a path or an even cycle. $\square$.

Theorem 1 (Berge 1957) *A matching M in a graph G is a maximal matching on G if and only if G has no M-augmenting paths.*

Proof: $\Rightarrow$ Proof by Contraposition:
We need to show:
G has an M-augmented path $\Longrightarrow$ M is not a maximal matching. Let P be an

M-augmented path in G (written so that the first and final edges in the path are not in M). Also, let

$$M' = M \Delta P.$$

Then M' is a matching.

$$|M'| = |M| + 1.$$

Therefore M is not a maximal matching.

($\Leftarrow$) Proof by Contraposition We need to show: M is not a maximal matching $\Longrightarrow \exists$ an M-augmented path Let M' be a maximal matching. Considering $M \Delta M'$, we see that $M \Delta M'$ is the disjoint union of paths and even cycles. **Case 1 :** If $\exists$ a path in $M \Delta M'$.

1. $\mid P \mid$ is even. Then P's contribution to

$$\mid M' \mid - \mid M \mid = 0.$$

2. $\mid P \mid$ is odd and the first edge of the path is an element of M. Then P's contribution to

$$\mid M' \mid - \mid M \mid = -1.$$

3. $\mid P \mid$ is odd and the first edge of the path is an element of M'. Then P's contribution to

$$\mid M' \mid - \mid M \mid = 1.$$

Case 2 : If $\exists$ an even cycle in $M \Delta M'$. Then the cycle's contribution to

$$\mid M' \mid - \mid M \mid = 0.$$

If we sum up all contributions then we get that

$$\mid M' \mid - \mid M \mid = 1.$$

Therefore we know that part three of Case one must occur. So then $\exists$ a path P where the first edge of the path is an element of M'. P is then an M-augmented path. Therefore $\exists$ an M-augmented path. $\square$.

Hall's Matching Condition

Theorem 2 (Hall 1935) *A $X - Y$ bipartite graph G has a matching that saturates X if and only if*

$$\mid N(S) \mid \leq \mid S \mid \forall S \subseteq X.$$

Proof: ($\Rightarrow$) Let M be a matching in G that saturates X. Then M defined a map f such that

$$f : X \longrightarrow Y$$

is a one-to-one mapping (since M saturates X). Then we know that

$$N(S) \supseteq f(S).$$

$$| N(S) | \geq | f(S) | = | S |$$

(since f is one-to-one).

($\Leftarrow$) Proof by Contraposition We need to show: If M is a maximal matching and M does not saturate $X \Rightarrow \exists$ a set S such that $| N(S) | \leq | S |$. Let x be a vertex in X which is not saturated by M. Also let

$$S = \{s | s can be reached from x using an M - alternating path and s \in X\}.$$

Also let

$$T = \{t | t can be reached from x using an M - alternating path and s \in Y\}.$$

⧫ Claim: M matches T with $S \backslash \{x\}$. **Proof:** $\forall\, s \in S \backslash \{x\}\ \exists$ an M-alternating path from X to S. The length of this path is even and the matching M defines a map f from $S \backslash \{x\} \longrightarrow T$ where $f(S)=$ the vertex directly before s in the M-augmented path.

Then f is well-defined, one-to-one, and onto. (Note: f is onto since the matching M is maximal).

Now we know that

$$| T | = | S \backslash \{x\} | = | S | - 1$$

We also know that

$$N(s) = T$$

(Trivially) Therefore we can conclude using these two equations that...

$$| N(s) | = | T | = | S | - 1$$

$\square$.

Corollary: $\forall\, k > 0$ every k-regular bipartite graph has a perfect matching. **Proof:** Let X and Y be the two partitions of a k-regular bipartite graph. Then we know that

$$k* | X | = k* | Y |$$

so

$$| X | = | Y | .$$

Therefore, any matching that saturates X also saturates Y and vice versa. Then to prove the corollary, it is sufficient to show that Hall's Matching Condition holds here. Therefore we must show that... $\forall\, S \subseteq X$, $| N(S) | \geq | S |$

So, the number of edges between S and

$$N(S) = k* | S |$$

which is less than the number or edges leaving $N(S)$. Therefore, $k* | S | \leq k* | N(S) | \Rightarrow | S | \leq | N(S) |$. $\square$.

Definition 8 *A vertex cover of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge.*

Definition 9 *An edge cover of a graph G is a subset L of the edges of G such that every vertex of $V(G)$ is incident to some edge of L.*

Definition 10 *$\alpha(G)$ is the maximum size of an independent set in a graph G.*

Definition 11 *$\alpha'(G)$ is the minimum size of all of the matchings of a graph G.*

Definition 12 *$\beta(G)$ is the minimum size of all vertex covers of a graph G.*

Definition 13 *$\beta'(G)$ is the minimum size of all the edge covers of a graph G.*

Theorem 3 (König and Egervary 1931) *If a graph G is a bipartite graph then the maximum size of a matching in G equals the minimum size of a vertex cover. So,*

$$\alpha'(G) = \beta(G).$$

Example 1 *This theorem does not hold in general if the graph is not bipartite. Let*

$$G = C_5.$$

Then

$$\alpha'(G) = 2 \neq 3 = \beta(G).$$

Proof: First observe that for any matching M and any vertex cover Q we have $\mid Q \mid \geq \mid M \mid$. Suppose that Q is a minimal vertex cover. (We aim to construct a matching of size $\mid Q \mid$). Let

$$R = Q \cap X$$

and let

$$T = Q \cap Y$$

where X and Y are the two partitions of the bipartite graph G. Now construct a matching from R to $Y \backslash T$. This matching saturates R. Now it is sufficient to show that Hall's Condition on the induced subgraph H on $R \cup Y \backslash T$ holds.
Reminder: Hall's Conditions says that for any subset $S \subseteq R$, $\mid N_H(S) \mid \geq \mid S \mid$.

⧫ Assume Hall's condition does not hold.
If $\exists$ an S such that $\mid N_H(S) \mid < \mid S \mid$ replace S by $N_H(S)$ in Q. Now $Q \cup (N(S)) \backslash S$ is a vertex cover. But the size of this vertex cover is less than the size of Q. This is a contradiction to the minimality of Q. Therefore Hall's Condition is verified. Now we can say that there is a matching from R to $Y - T$ that saturates R. Similarly there is a matching from T to $X - R$ that saturates T. Let M be the union of these two matchings. Then we know that

$$\mid M \mid = \mid R \mid + \mid T \mid = \mid Q \mid$$

and we have successfully constructed a matching of size $\mid Q \mid$. $\square$.

Proposition:

$$\alpha(G) + \beta(G) = n(G)$$

OR A set S is an independent set of $G \iff$ the complement of S is a vertex cover of G.

Theorem 4 (Gallai 1959) *If G is a graph without isolated vertices then*

$$\alpha'(G) + \beta'(G) = n(G).$$

Proof: Show:

$$\beta'(G) \leq n(G) - \alpha'(G)$$

Let M be a maximum matching, then $\mid M \mid = \alpha'(G)$. We need to construct an edge cover L of size $n(G) - \mid M \mid$.
For any unsaturated vertex v pick an edge e_x which is incident to x.
Then $L = M \cup \{e_x\}$ {where x is an unsaturated vertex}. Then $\mid L \mid = \mid M \mid + \mid \{unsaturatedvertices\} \mid = \mid M \mid + n(G) - 2 \mid M \mid = n(G) - \mid M \mid$. Therefor L is a vertex cover.
For the second part Show:

$$\alpha'(G) \geq n(G) - \beta'(G).$$

Assume that L is a minimal edge cover,

$$\mid L \mid = \beta'(G).$$

Our goal is to construct a matching M with size equal to $n(G) - \mid L \mid$. (Note: L contains no P_4). Based on this we know that L is the disjoint union of k stars. Then

$$\mid L \mid = n(G) - k.$$

So,

$$k = n(G) - \mid L \mid.$$

Pick one edge in each star and let M equal that set. So,

$$\mid M \mid = k = n(G) - \mid L \mid.$$

Now, M is a matching with the appropriate size.
Combining parts one and two yields...

$$\alpha'(G) + \beta'(G) = n(G).$$

□.

Theorem 5 (König) *If G is a bipartite graph with no isolated vertices then*

$$\alpha(G) = \beta'(G).$$

The proof begins the next section.