1 Eulerian circuits for undirected graphs

An Eulerian circuit/trail in a graph $G$ is a circuit containing all the edges. A
graph is Eulerian if it has an Eulerian circuit. We first prove the following
lemma.

**Lemma 1** If every vertex of a (finite) graph $G$ has degree at least 2, then $G$
contains a cycle.

**Proof:** Let $P$ be a maximal path in $G$, and let $u$ be an endpoint of $P$. On one
hand, $P$ can not be extended, every neighbor of $u$ is in $V(P)$. On the other
hand, $u$ has a neighbor $v$ via an edge not in $P$. This edge $uv$ and the path from
$v$ to $u$ form a cycle.

**Theorem 1** A graph $G$ is Eulerian if and only if $G$ has at most one nontrivial
component and its vertices all have even degrees.

**Proof:**

**Necessity:** Suppose $G$ is Eulerian. All edges are on a Eulerian cycle. There-
fore, all edges are in one component. Other components have no edge. Thus,
they are isolated vertices. Let us fix an orientation of the Eulerian circuit. For
any vertex $v$ in the nontrivial component, the number of edges leaving $v$ is equal
to the number of edges entering $v$. The degree $d_v$ is the sum of edges which are
either leaving or entering $v$. Thus, $d_v$ is even.

**Sufficiency:** We will prove it by induction on the number $m$ of edges.
If $m = 0$, the Eulerian cycle is empty. It holds.
Suppose that the statement holds for any graph with at most $m$ edges. In
another words, if a graph $G$ with at most $m$ edges has at most one nontrivial
component and its vertices all have even degrees, then $G$ is Eulerian.
Now we consider a graph with $m + 1$ edges, which has at most one nontrivial
component $H$ and whose vertices all have even degrees. By Lemma 2, it contains
a cycle $C$. Deleting all edges on $C$ from $G$, $H$ might be breaking into several
components, say $H_1, H_2, \ldots, H_r$. The degree of a vertex $v$ either decreases by 2
if $v \in C$, or remains the same if $v \not\in C$. All degrees remain even after deleting
the edges of $C$.

Each component $H_i$ has at most $m$ edges. By inductive hypothesis, There is
an Eulerian circuit $C_i$ for each component $H_i$. Since $G$ has only one non-trivial
component, the cycle $C$ must intersect with every component $H_i$. Pick one
vertex $v_i \in V(C) \cap V(H_i)$. The vertices $v_1, v_2, \ldots, v_r$ break the cycle $C$ into $r$
paths, say $v_1P_1v_2, v_2P_2v_3, \ldots, v_rP_rv_1$. Arrange Eulerian circuit $C_i$ so that the
starting vertex and end vertex is \( v_i \). Now we construct an Eulerian circuit as follows.

\[
C_1 P_1 C_2 P_2 \ldots C_r P_r v_1
\]

It contains all edges of \( G \).

## 2 Directed Graphs

**Definition 1** A directed graph \( G \) (or digraph, for short) is a triple consisting of a vertex set \( V(G) \), an edge set \( E(G) \), and a relation that associates with each edge an ordered pair of vertices called the head and the tail.

**Definition 2** A loop is an edge whose head and tail are the same vertex. Multiple edges are edges that have the same pair of the head and the tail. A simple digraph is a digraph without loops or multiple edges.

For a simple digraph, an edge \( e \) is uniquely represented by its head \( u \) and its tail \( v \). In this case, we write \( e = uv \), and say \( u \) is a predecessor of \( v \) and \( v \) is a successor of \( u \). For any vertex \( v \), the out-degree \( d^+(v) \) is the number of successors of \( v \); the in-degree \( d^-(v) \) is the number of predecessors of \( v \).

For a digraph \( G \) on the vertex set \( \{v_1, v_2, \ldots, v_n\} \), the adjacency matrix \( A = (a_{ij})_{n \times n} \) of \( G \) is define to be

\[
a_{ij} = \begin{cases} 
1 & \text{if } v_i v_j \in E(G) \\
0 & \text{otherwise.}
\end{cases}
\]

The out-degree \( d^+(v_i) \) is the sum of entries in \( i \)-th row of the adjacency matrix \( A \). The in-degree \( d^-(v_j) \) is the sum of entries in \( j \)-column row of the adjacency matrix \( A \). The total number of edges is

\[
\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i).
\]

The following concepts are similar to those for undirected graphs.

A walk (on a digraph \( G \)) is a list \( v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k \), satisfying \( e_i = v_{i-1} v_i \) is an edge for all \( i = 1, 2, \ldots, k \). \( k \) is called the length of the walk.

A \( u, v \)-walk is a walk with \( v_0 = u \) and \( v_k = v \).

A trail is a walk with no repeated edge.

A path is a walk with no repeated vertices.

A closed walk is a walk with the same endpoints, i.e., \( v_0 = v_k \).

A cycle is a closed walk with no repeated vertices except for the endpoints.

An Eulerian circuit/trail of a digraph \( G \) is a circuit containing all the edges. A digraph is Eulerian if it has an Eulerian circuit. We first prove the following lemma.

**Lemma 2** If every vertex of a (finite) graph \( G \) has out-degree (or in-degree) at least 1, then \( G \) contains a cycle.
Proof: Let $P$ be a maximal path in $G$, and $u$ be the last vertex on $P$. Since $P$ can not be extended, every successor of $u$ is in $V(P)$. There is at least one successor of $u$, say $v$. This edge $uv$ and the path from $v$ to $u$ form a cycle. □

**Theorem 2** A digraph $G$ is Eulerian if and only if $G$ has at most one nontrivial component and $d^+(v) = d^-(v)$ for each vertex $v$.

**Proof:**

**Necessity:** Suppose $G$ is Eulerian. All edges are on an Eulerian cycle. Therefore, all edges are in one component. Other components have no edges. Thus, they are isolated vertices. For any vertex $v$ in the nontrivial component, the number of edges leaving $v$ is equal to the number of edges entering $v$. Thus, $d^+(v) = d^-(v)$.

**Sufficiency:** We will prove it by induction on the number $m$ of edges.

If $m = 0$, the Eulerian cycle is empty. It holds.

Suppose that the statement holds for any graph with at most $m$ edges. In another words, if a graph $G$ with at most $m$ edges has at most one nontrivial component and its vertices all have even degrees, then $G$ is Eulerian.

Now we consider a graph with $m + 1$ edges, which has at most one nontrivial component $H$ and $d^+(v) = d^-(v) \geq 1$ for all $v \in V(H)$. By Lemma 2, it contains a cycle $C$. Deleting all edges on $C$ from $G$, $H$ might be breaking into several components, say $H_1, H_2, \ldots, H_r$. It is clear that $d^+(v) = d^-(v)$ still holds for every vertex $v$.

Each component $H_i$ has at most $m$ edges. By inductive hypothesis, There is an Eulerian circuit $C_i$ for each component $H_i$. Since $G$ has only one non-trivial component, the cycle $C$ must intersect with every component $H_i$. Pick one vertex $v_i \in V(C) \cap V(H_i)$. The vertices $v_1, v_2, \ldots, v_r$ break the cycle $C$ into $r$ paths, say $v_1P_1v_2, v_2P_2v_3, \ldots, v_rP_rv_1$. Arrange Eulerian circuit $C_i$ so that the starting vertex and end vertex is $v_i$. Now we construct an Eulerian circuit as follows.

$$C_1P_1C_2P_2 \ldots C_rP_r, v_1$$

It contains all edges of $G$. □

3 Applications

A de Bruijn sequence of window-size $n$ is a circular binary string of length $2^n$ such that every substring of consecutive $n$-bits are distinct. For example, for $n = 4$, 00001111101100101 is a de Bruijn sequence.

A de Bruijin digraph $D_n$ is a digraph $(V, E)$ satisfying

1. The vertex set $V = \{\text{all binary strings of length } n - 1\}$.

2. The edge set $E = \{uv \mid \text{the last } n - 2 \text{ bits of } u \text{ agree with the first } n - 2 \text{ bits of } v.\}$. Each edge can be labeled by the last bit of its head.
There is a bijection between the set of de Bruijn sequences and Eulerian circuits of de Bruijn digraphs. The bijection is obtained by collecting the labels of edges on an Eulerian circuit.

**Remark:** The de Bruijn sequences and de Bruijn digraphs can be defined over any alphabet.

### 4 Orientations and tournaments

**Definition 3** An orientation of a graph $G$ is a digraph $D$ obtained from $G$ by choosing an orientation $x \rightarrow y$ or $y \rightarrow x$ for every edge $xy \in E(G)$. A tournament is an orientation of a complete graph.

In a digraph, a **king** is a vertex from which every vertex is reachable by a path of length at most 2.

**Theorem 3 (Landau 1953)** Every tournament has a king.

**Proof:** Let $x$ be a vertex with maximum out-degree in a tournament $T$. We claim $x$ is a king. We will prove this claim by contradiction.

Otherwise, there is a vertex $y$ can not reached by $x$ in at most 2 steps. So $y$ must reach $x$. If $z$ can be reached by $x$, $y$ must reach $z$ as well. In particular, we have $d^+(y) > d^+(x)$. Contradiction to the choice of $x$. 

\[ \square \]