# Math 776 Graph Theory Lecture Note 3
# Eulerian circuits and directed graphs

**Lectured by Lincoln Lu**
**Transcribed by Lincoln Lu**

## 1 Eulerian circuits for undirected graphs

An Eulerian circuit/trail in a graph $G$ is a circuit containing all the edges. A graph is Eulerian if it has an Eulerian circuit. We first prove the following lemma.

**Lemma 1** *If every vertex of a (finite) graph $G$ has degree at least 2, then $G$ contains a cycle.*

**Proof:** Let $P$ be a maximal path in $G$, and let $u$ be an endpoint of $P$. On one hand, $P$ can not be extended, every neighbor of $u$ is in $V(P)$. On the other hand, $u$ has a neighbor $v$ via an edge not in $P$. This edge $uv$ and the path from $v$ to $u$ form a cycle.

**Theorem 1** *A graph $G$ is Eulerian if and only if $G$ has at most one nontrivial component and its vertices all have even degrees.*

**Proof:**

**Necessity:** Suppose $G$ is Eulerian. All edges are on a Eulerian cycle. Therefore, all edges are in one component. Other components have no edge. Thus, they are isolated vertices. Let us fix an orientation of the Eulerian circuit. For any vertex $v$ in the nontrivial component, the number of edges leaving $v$ is equal to the number of edges entering $v$. The degree $d_v$ is the sum of edges which are either leaving or entering $v$. Thus, $d_v$ is even.

**Sufficiency:** We will prove it by induction on the number $m$ of edges.

If $m = 0$, the Eulerian cycle is empty. It holds.

Suppose that the statement holds for any graph with at most $m$ edges. In another words, if a graph $G$ with at most $m$ edges has at most one nontrivial component and its vertices all have even degrees, then $G$ is Eulerian.

Now we consider a graph with $m+1$ edges, which has at most one nontrivial component $H$ and whose vertices all have even degrees. By Lemma 2, it contains a cycle $C$. Deleting all edges on $C$ from $G$, $H$ might be breaking into several components, say $H_1, H_2, \ldots, H_r$. The degree of a vertex $v$ either decreases by 2 if $v \in C$, or remains the same if $v \not inC$. All degrees remain even after deleting the edges of $C$.

Each component $H_i$ has at most $m$ edges. By inductive hypothesis, There is an Eulerian circuit $C_i$ for each component $H_i$. Since $G$ has only one non-trivial component, the cycle $C$ must intersect with every component $H_i$. Pick one vertex $v_i \in V(C) \cap V(H_i)$. The vertices $v_1, v_2, \ldots, v_r$ break the cycle $C$ into $r$ paths, say $v_1 P_1 v_2, v_2 P_2 v_3, \ldots, v_r P_r v_1$. Arrange Eulerian circuit $C_i$ so that the starting vertex and end vertex is $v_i$. Now we construct an Eulerian circuit as follows.

$$C_1 P_1 C_2 P_2 \dots C_r P_r v_1$$

It contains all edges of $G$. $\square$

## 2 Directed Graphs

**Definition 1** *A* directed graph $G$ *(or digraph, for short) is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge an ordered pair of vertices called the head and the tail.*

**Definition 2** *A* loop *is an edge whose head and tail are the same vertex.* Multiple *edges are edges that have the same pair of the head and the tail. A* simple *digraph is a digraph without loops or multiple edges.*

For a simple digraph, an edge $e$ is uniquely represented by its head $u$ and its tail $v$. In this case, we write $e = uv$, and say $u$ is an *predecessor* of $v$ and $v$ is an *successor* of $u$. For any vertex $v$, the out-degree $d^+(v)$ is the number of successors of $v$; the in-degree $d^-(v)$ is the number of predecessors of $v$.

For a digraph $G$ on the vertex set $\{v_1, v_2, \dots, v_n\}$, the adjacency matrix $A = (a_{ij})_{n\times n}$ of $G$ is define to be

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

The out-degree $d^+(v_i)$ is the sum of entries in $i$-th row of the adjacency matrix $A$. The in-degree $d^-(v_j)$ is the sum of entries in $j$-column row of the adjacency matrix $A$. The total number of edges is

$$\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i).$$

The following concepts are similar to those for undirected graphs.

A *walk* (on a digraph $G$) is a list $v_0, e_1, v_1, e_1, \dots, e_k, v_k$, satisfying $e_i = v_{i-1}v_i$ is an edge for all $i = 1, 2, \dots, k$. $k$ is called the length of the walk.

A $u, v$-*walk* is a walk with $v_0 = u$ and $v_k = v$.

A *trail* is a walk with no repeated edge.

A *path* is a walk with no repeated vertices.

A *closed walk* is a walk with the same endpoints, i.e., $v_0 = v_k$.

A *cycle* is a closed walk with no repeated vertices except for the endpoints.

An Eulerian circuit/trail of a digraph $G$ is a circuit containing all the edges. A digraph is Eulerian if it has an Eulerian circuit. We first prove the following lemma.

**Lemma 2** *If every vertex of a (finite) graph $G$ has out-degree (or in-degree) at least 1, then $G$ contains a cycle.*

**Proof:** Let $P$ be a maximal path in $G$, and $u$ be the last vertex on $P$. Since $P$ can not be extended, every successor of $u$ is in $V(P)$. There is at least one successor of $u$, say $v$. This edge $uv$ and the path from $v$ to $u$ form a cycle. $\square$

**Theorem 2** *A digraph $G$ is Eulerian if and only if $G$ has at most one nontrivial component and $d^+(v) = d^-(v)$ for each vertex $v$.*

**Proof:**

**Necessity:** Suppose $G$ is Eulerian. All edges are on a Eulerian cycle. Therefore, all edges are in one component. Other components have no edges. Thus, they are isolated vertices. For any vertex $v$ in the nontrivial component, the number of edges leaving $v$ is equal to the number of edges entering $v$. Thus, $d^+(v) = d^-(v)$.

**Sufficiency:** We will prove it by induction on the number $m$ of edges.

If $m = 0$, the Eulerian cycle is empty. It holds.

Suppose that the statement holds for any graph with at most $m$ edges. In another words, if a graph $G$ with at most $m$ edges has at most one nontrivial component and its vertices all have even degrees, then $G$ is Eulerian.

Now we consider a graph with $m+1$ edges, which has at most one nontrivial component $H$ and $d^+(v) = d^-(v) \geq 1$ for all $v \in V(H)$. By Lemma 2, it contains a cycle $C$. Deleting all edges on $C$ from $G$, $H$ might be breaking into several components, say $H_1, H_2, \ldots, H_r$. It is clear that $d^+(v) = d^-(v)$ still holds for every vertex $v$.

Each component $H_i$ has at most $m$ edges. By inductive hypothesis, There is an Eulerian circuit $C_i$ for each component $H_i$. Since $G$ has only one non-trivial component, the cycle $C$ must intersect with every component $H_i$. Pick one vertex $v_i \in V(C) \cap V(H_i)$. The vertices $v_1, v_2, \ldots, v_r$ break the cycle $C$ into $r$ paths, say $v_1P_1v_2, v_2P_2v_3, \ldots, v_rP_rv_1$. Arrange Eulerian circuit $C_i$ so that the starting vertex and end vertex is $v_i$. Now we construct an Eulerian circuit as follows.

$$C_1P_1C_2P_2 \ldots C_rP_rv_1$$

It contains all edges of $G$. $\square$

## 3 Applications

A de Bruijn sequence of window-size $n$ is a circular binary string of length $2^n$ such that every substring of consecutive $n$-bits are distinct. For example, for $n = 4$, 0000111101100101 is a de Bruijn sequence.

A de Bruijn digraph $D_n$ is a digraph $(V, E)$ satisfying

1. The vertex set $V = \{\text{all binary strings of length } n-1\}$.
2. The edge set $E = \{uv \mid \text{the last } n-2 \text{ bits of } u \text{ agree with the first } n-2 \text{ bits of } v.\}$. Each edge can be labeled by the last bit of its head.

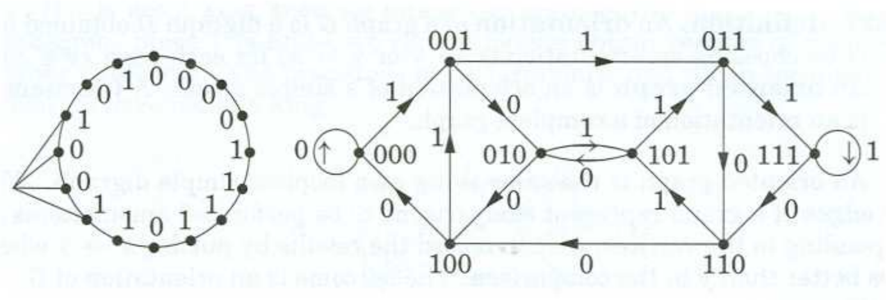

Figure 1: De Bruijn sequence and De Bruijn digraph

There is a bijection between the set of de Bruijn sequences and Eulerian circuits of de Bruijn digraphs. The bijection is obtained by collecting the labels of edges on an Eulerian circuit.

**Remark:** The de Bruijn sequences and de Bruijn digraphs can be defined over any alphabet.

## 4 Orientations and tournaments

**Definition 3** *An* **orientation** *of a graph $G$ is a digraph $D$ obtained from $G$ by choosing an orientation $x \to y$ or $y \to x$ for every edge $xy \in E(G)$. A* **tournament** *is an orientation of a complete graph.*

In a digraph, a **king** is a vertex from which every vertex is reachable by a path of length at most 2.

**Theorem 3 (Landau 1953)** *Every tournament has a king.*

**Proof:** Let $x$ be a vertex with maximum out-degree in a tournament $T$. We claim $x$ is a king. We will prove this claim by contradiction.

Otherwise, there is a vertex $y$ can not reached by $x$ in at most 2 steps. So $y$ must reach $x$. If $z$ can be reached by $x$, $y$ must reach $z$ as well. In particular, we have $d^+(y) > d^+(x)$. Contradiction to the choice of $x$. $\square$