Math 776 Graph Theory Lecture Notes 10
Graph Coloring

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Definition: A *proper k-coloring* of a graph, G, is a labeling $f:V(G)\to S$ such that:

$|S| = k$

$f(u) \neq f(v)$ if $uv \in E(G)$

Definition: The *chromatic number* of a graph, G, is the minimum k such that G is k-colorable and is noted: X(G)

For any non-trivial bipartite graph, H, X(H)=2

cycle, $X(C_{2k+1})=3$

complete graph, $X(K_n)=n$

Definition: The *clique number* of a graph, G, is te maximum size of a set of pairwise adjacent vertices in G and is noted: $\omega(G)$

Examples of Lower Bounds:

1. $X(G) \geq X(H)$ for any subgraph H of G
2. $X(G) \geq \omega(G)$
3. $X(G) \geq \dfrac{n(G)}{\alpha(G}$

If G is k-colorable, then $V(G) = I_1 \cup \ldots \cup I_k$ where the I's are independent sets. Then

$$|I_i| \leq \alpha(G)$$

$$n(G) = \sum_{i=1}^{k} |I_i| \leq k\alpha(G)$$

$$k \geq \frac{n(G)}{\alpha(G}$$

$$X(G) \geq \frac{n(G)}{\alpha(G}$$

Examples of New Operators:

1. Disjoint union G + H
   V(G+H) = V(G) +V(H)
   E(G+H) = E(G) + E(H)
   X(G+H) = max{X(G), X(H)}
2. Join $G \vee H = \overline{\overline{G} + \overline{H}}$
   $V(G \vee H) = V(G) + V(H)$
   $E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$
   $X(G \vee H) = X(G) + X(H)$
3. Cartesian product $G \square H$
   $V(G \square H) = \{(u,v) \mid u \in V(G), v \in V(H)\}$
   $E(G \square H) = \{(u_1v_1)(u_2v_2) \mid u_1 = u_2, v_1v_2 \in E(H) \text{ or } v_1 = v_2, u_1u_2 \in E(G)\}$
   $X(G \square H) = \max\{X(G), X(H)\}$

Proof that $X(G \square H) = \max\{X(G),X(H)\}$: (Vizing, 1963, Aberth 1964)

1. $X(G \square H) \geq X(G)$ $\quad X(G \square H) \geq X(H)$
   $X(G \square H) = \max\{X(G),X(H)\}$
2. Suppose G is k_1-colorable and H is k_2-colorable with $k_1 \leq k_2$
   $f1: V(G) \rightarrow [k_1]$
   $f2: V(H) \rightarrow [k_2]$
   $f: V(G \square H) \rightarrow k_2$
   $f(u,v) = f_1(u)+f_2(v) \bmod k_2$
   If (u_1,v_1) is adjacent to (u_2,v_2) then either:
   1. $u_1 = u_2$ and $v_1v_2 \in E(H)$ or
   2. $v_1 = v_2$ and $u_1u_2 \in E(G)$
      In case (1.) consider $f(u_1,v_1) - f(u_2,v_2) = f_2(v_1) - f_2(v_2) \neq 0 \bmod k$
      In case (2.) consider $f(u_1,v_1) - f(u_2,v_2) = f_1(u_1) - f_1(u_2) \neq 0 \bmod k$

   So $G \square H$ is k_2-colorable
   So $X(G \square H) \leq \max\{X(G),X(H)\}$

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Example of an Upper Bound:

1. $X(G) \leq \Delta(G) + 1$ ($\Delta(G)$ is the maximum degree of G)
   Proof: (Greedy Coloring)
   Order V(G) into any order $v_1,\ldots v_n$.
   Suppose we color $v_1,\ldots v_k$
   Since v_{k+1} has at most $\Delta(G)$ lower-index neighbors, we can always pick up a color which is not used in its neighbors and we color v_{k+1} by this color.

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Furthermore, this upper bound is reachable for $G = K_n$ and $G = C_{2k+1}$.

Brook Theorem (1941)

If G is a connected graph other than a completed graph or odd cycle, then $X(G) \leq \Delta(G)$.

Proof: Let $k = \Delta(G)$.

Case 1: $k = 1$ $G = K_2$

Case 2: $k = 2$ G a path or cycle

Since G is not an odd cycle (by hypothesis) G is either a path or an even cycle. In both cases, $X(G) = 2 = \Delta(G)$.

Case 3: $k \geq 3$

subcase a: G not a k-regular graph

Pick up a vertex, vn, with degree $\leq k - 1$.
Let T be any spanning tree of G.
Orient T such that T is an in-tree rooted at v_n.
We extend this partial ordering to a total ordering on V(G):

$$v_1, v_2, \ldots, v_n$$

Color the vertices greedily according to this order.
The number of lower-indexed neighbors of any v_i is at most k-1.
Hence $X(G) \leq k$.

subcase b: G is a k-regular graph

If G has a cut-vertex, x, then by subcase a:

$$X(G_1 \cup \{x\}) \leq k$$
$$X(G_2 \cup \{x\}) \leq k$$

Without loss of generality, suppose $f_1(x) = f_2(x)$.
We define coloring of G by union of f_1 and f_2.
If $K(G) \geq 2$ (where K(G) is the connectivity of G) find

$$v_1v_2 \notin E(G)$$
$$v1v_n \in E(G)$$
$$v_2v_n \in E(G)$$
$$G - \{v_1,v_2\} \text{ is connected}$$

Find a spanning tree of $G - \{v_1,v_2\}$.
Orient it and extend it to a total ordering of $v_2, v_4, \ldots, v_n$.
v_n has k neighbors. v_1 and v_2 have same color.
So v_n has k–1 different-colored neighbors.
(i.e. greedy coloring gives a proper k-coloring)
It is sufficient to find v_1, v_2, v_n satisfying

$$v_1v_2 \notin E(G)$$
$$v_1v_n \in E(G)$$
$$v_2v_n \in E(G)$$

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Borodinad and Kostocha (1977) conjectured that:

$$\omega(G) < \Delta(G) \Rightarrow X(G) < \Delta(G) \text{ if } \Delta(G) > 9$$

Reed (1998) proved this conjecture for $\Delta(G) \geq 10^{10}$.

Reed's conjecture, still open:

$$X(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil$$

Definition: A graph, G, is *color-critical* if $X(H) < X(G)$ for any proper subgraph H of G.
examples: C_5, K_5
(if G is color-critical and X(G)=k, then G is *k-critical*)

Theorem: (Szekeres-wilf 1968)
If G is a graph, then $X(G) \leq 1 + \max_{H \subseteq G} \delta(H)$

Lemma: If H is k-critical, then $\delta(H) \geq k - 1$

Proof: Assume there is a vertex, v, with degree $dv < k - 1$
Show that $H \setminus \{v\}$ is not k-critical.
By definition of k-critical, we have $X(H \setminus \{v\}) < X(H) = k$.

So we can properly color H \ {v} using k – 1 colors.
Since dv $\leq$ k – 2, only at most k – 2 colors are used for the neighbors of v.
Coloring v with a different color results in using k – 1 colors.
Then X(H) $\leq$ k – 1, which is a contradiction. ▪

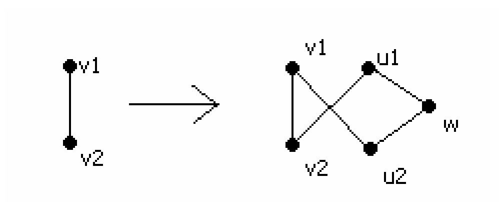
Exercise: prove that a k-critical graph is k – 1 edge-connected.

<u>Proof</u> (of theorem)

Let H be a k-critical subgraph of G where k = X(G).

$$\begin{aligned} X(G) &= X(H) \\ &\leq \delta(H) + 1 \\ &\leq \max_{H \subseteq G} \delta(H) + 1 \end{aligned}$$

<u>Mycielski's construction</u>



Construct G' from G.
Suppose V(G) = $\{v_1,\ldots,v_n\}$
Then

$$V(G') = \{v_1,\ldots,v_n\} \cup \{u_1\ldots u_n\} \cup \{w\}$$
$$E(G') = \bigcup_{v_iu_j \in E(G)} \{v_iu_j,\ u_iv_j,\ u_jv_i,\ u_iw,\ u_jw\}.$$
$$|V(G')| = 2\,|V(G)| + 1$$
$$|E(G')| = 3\,|E(G)| + |V(G)|$$

Grotzsch graph: (smallest 4-critical graph)

<u>Mycielski's Theorem</u> (1955)
From a k-chromatic triangle-free graph, G, Mycielski's construction produces a k+1-chromatic triangle-free graph, G'.
<u>Proof:</u>

Suppose G' has a triangle, T. Since u1,…,un form an independent set,

$$w \notin V(T)$$

case 1: V(T) = $\{u_i, u_j, v_k\}$

if k $\neq$ i, j
$\{v_i, v_j, v_k\}$ forms a triangle in G, which is a contradiction.
if k = i
v_iu_i would be an edge in G', which doesn't happen
if k $\neq$ i
by construction we'd have an edge from v_i to v_k and one from v_j to v_k

so there'd be a triangle in G, which is a contradiction.

case 2: $V(T) = \{v_i, u_j, u_k\}$

Suppose $X(G) = k$

Then $X(G') \leq X(G) + 1$

Now it is sufficient to prove $X(G') \geq X(G) + 1$.

Suppose G' has a proper k-coloring; $X(G') = k$

$$f: V(G) \rightarrow [k]$$

Without loss of generality, assume

$$f(w) = k$$
$$f(u_i) \neq k$$
$$f(u_i) \in [k-1]$$

If f(vi) = k, we recolor it to $f(u_i)$

For any vertex $v_j \neq v_i$, if $v_iv_j \in E(G)$,

$$f(v_j) \neq k$$
$$f(v_j) \neq f(u_i)$$

After recoloring $v_1,\ldots,v_n$ it is still a proper coloring.

Hence, $X(G) \leq k + 1$

So $X(G) = X(G') + 1$.

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<u>Theorem:</u>

If G is k-critical, then G' from Mycielski's construction is k+1-critical.

<u>Proof:</u>

$X(G) = k$, $X(G') = k+1$, $X(G \setminus \{e\}) \leq k - 1$ for any e

We will show $X(G \setminus \{e\}) \leq k$.

case 1: $e' = v_iv_j$

Since $X(G \setminus \{e'\}) \leq k - 1$ we have a proper coloring

$$f: V(G) \rightarrow [k-1]$$

Extend $f: V(G) \rightarrow [k]$ by assigning

$$f(v_i) = f(u_i)$$
$$f(w)=k$$

So true for case 1.

case 2: $e' = wu_i$

$X(G \setminus \{v_i\}) \leq k-1$

There exists a coloring:

$$f: V(G \setminus \{v_i\}) \rightarrow [k-1]$$

Extend $f: V(G \setminus \{v_i\}) \rightarrow [k]$ by assigning:

$$f(u_j) = f(v_j) \text{ for any vj} \neq \text{vi}$$
$$f(u_i) = f(v_i) = f(w) = k$$

Then f is a proper k-coloring

$$X(G' \setminus \{e'\}) \leq k$$

case 3: $e' = u_iv_j$

Let $e = v_iv_j \in E(G)$

$X(G \setminus \{e\}) \leq k-1$

There exists a coloring:

$$f: V(G \setminus \{e\}) \rightarrow [k-1]$$

Extend $f: V(G \setminus \{e\}) \rightarrow [k]$ by assigning:

$$f(u_k) = f(v_k) \text{ if } k \neq i$$
$$f(u_i) = k = f(v_i)$$
$$f(w) = f(v_j)$$

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