

Math 776 Graph Theory Lecture Note 1

Basic concepts

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Graph theory was founded by the great Swiss mathematician Leonhard Euler (1707-1783) after he solved the Königsberg Bridge problem: Is it possible to cross seven bridges exactly once?

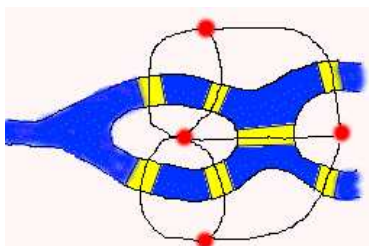


Figure 1: The map of Königsberg

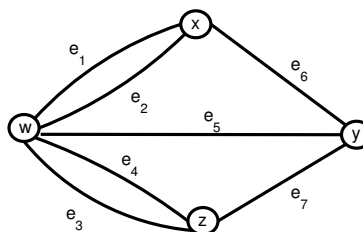


Figure 2: The graph associated with the Königsberg Bridges

Using graph terminology, Euler gave a complete solution for any similar problem with any number of bridges connecting any number of landmasses! We will show and prove his result later.

Definition 1 A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its endpoints.

Example 1 For the problem of the Königsberg Bridges, one can define the graph as follows. Vertices are landmasses while edges are bridges. We have

$$\begin{aligned} V(G) &= \{w, x, y, z\} \\ E(G) &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \\ e_1 &\leftrightarrow \{w, x\} \\ e_2 &\leftrightarrow \{w, x\} \\ e_3 &\leftrightarrow \{w, z\} \\ e_4 &\leftrightarrow \{w, z\} \\ e_5 &\leftrightarrow \{w, y\} \\ e_6 &\leftrightarrow \{x, y\} \\ e_7 &\leftrightarrow \{y, z\} \end{aligned}$$

Definition 2 A loop is an edge whose endpoints are the same. Multiple edges are edges that have the same pair of endpoints. A simple graph is a graph without loops or multiple edges.

For a simple graph, an edge e is uniquely represented by its endpoints u and v . In this case, we write $e = uv$ (or $e = vu$), and we say u and v are *adjacent*. u is a *neighbor* of v , and vice versa. An edge e is incident to u (and v).

A simple graph G on n vertices can have at most $\binom{n}{2}$ edges. The simple graph with n vertices and $\binom{n}{2}$ edges is called the *complete* graph, denoted by K_n . The simple graph G on n vertices with 0 edge is called the *empty* graph. The graph with 0 vertices and 0 edges is called the *null* graph.

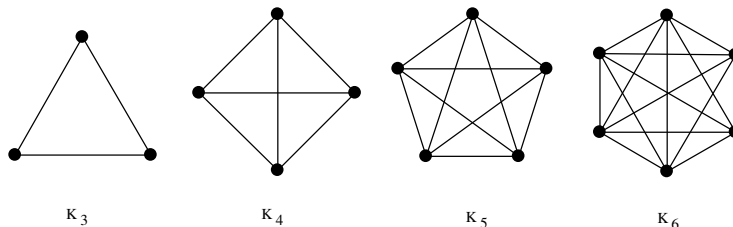


Figure 3: Complete graphs K_3 , K_4 , K_5 , and K_6 .

A cycle on n vertices, written C_n , is a graph with vertex set $V(G) = [n] = \{1, 2, 3, \dots, n\}$ and edge set $E(G) = \{12, 23, 34, \dots, (n-1)n, n1\}$.

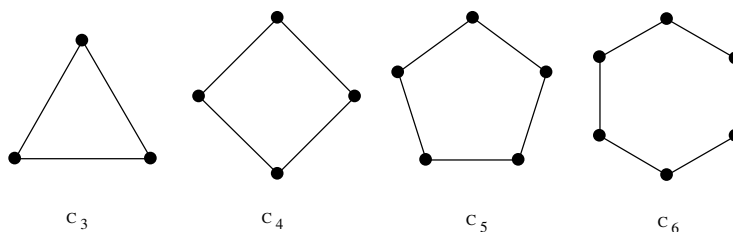


Figure 4: Cycles C_3 , C_4 , C_5 , and C_6 .

A path on n vertices, written P_n , is a graph with vertex set $V(G) = [n] = \{1, 2, 3, \dots, n\}$ and edge set $E(G) = \{12, 23, 34, \dots, (n-1)n\}$.

Definition 3 The complement \bar{G} of a simple graph G is the simple graph with vertex set $V(\bar{G}) = V(G)$ and edge set $E(\bar{G})$ defined by $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$. A clique in a graph is a set of pairwise adjacent vertices. An independent set in a graph is a set of pairwise non-adjacent vertices.

Definition 4 A graph G is bipartite if $V(G)$ is the union of two disjoint independent sets called partite sets of G .

Definition 5 A graph is k -partite if $V(G)$ can be expressed as the union of k independent sets.

Definition 6 The chromatic number of a graph G , written $\chi(G)$, is the minimum number k such that G is k -partite.

A *subgraph* of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We say G contains H . A *induced subgraph* of a graph G is a subgraph H satisfying

$$E(H) = \{uv \mid u, v \in V(H), uv \in E(G)\}.$$

Example 2 Let G_1 be the simple graph defined by (see Figure 5)

$$\begin{aligned} V(G_1) &= \{1, 2, 3, 4, 5\} \\ E(G_1) &= \{12, 23, 13, 24, 34, 45\}. \end{aligned}$$

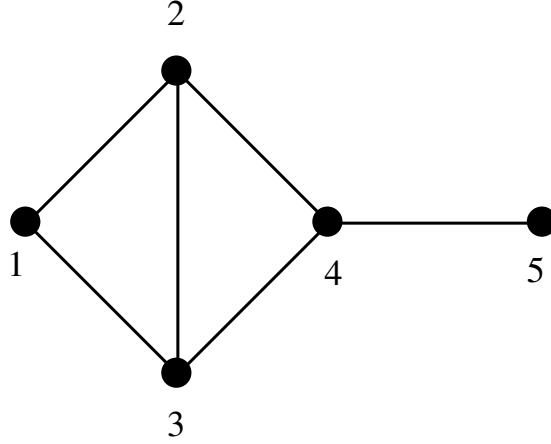


Figure 5: A simple graph G_1 .

$\{1, 2, 3\}$ is a clique. $\{1, 4\}$ is an independent set. G_1 is not a bipartite graph. The chromatic number is $\chi(G_1) = 3$.

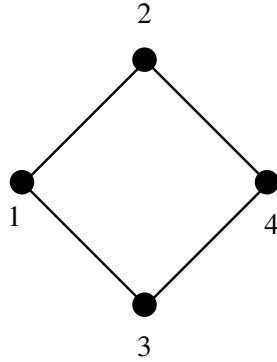


Figure 6: A subgraph of G_1 .

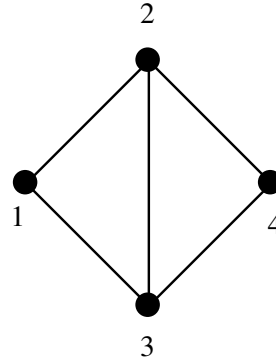


Figure 7: A induced subgraph of G_1 .

An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \rightarrow V(H)$ satisfying

$$uv \in E(G) \text{ iff } f(u)f(v) \in E(H).$$

We say G is isomorphic to H , denoted $G \cong H$.

The isomorphism relation is an equivalence relation. I.e., it is

1. reflexive: $A \cong A$.
2. symmetric: if $A \cong B$, then $B \cong A$.
3. transitive: if $A \cong B$ and $B \cong C$, then $A \cong C$.

An isomorphism class of graphs is an equivalence class of graphs under the isomorphism relation. It is also known as an “unlabeled graph”.

Definition 7 The adjacency matrix of a graph G , written $A(G)$, is the n -by- n matrix in which entry a_{ij} is the number of edges with endpoints $\{v_i, v_j\}$ in G . Here $V(G) = v_1, v_2, \dots, v_n$ is the vertex set of G .

For a simple graph G , we have

$$A(G) = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

$A(G)$ is a symmetric 0-1 matrix with 0s on the diagonal. For a vertex v of the graph G , the degree d_v is the number of edges which are incident to v . If $v = v_i$, then d_{v_i} is the i -th row/column sum of $A(G)$.

Example 3 For graph G_1 (Figure 5), the adjacency matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $[n] = \{1, 2, \dots, n\}$. The Petersen Graph G is defined as follows:

$$V(G) = \{\text{all 2-subsets of } [5]\}.$$

$$E(G) = \{ST | S, T \in V(G); S \cap T = \emptyset\}.$$

The Petersen Graph has the following properties.

1. The automorphism group $\text{Aut}(G)$ is S_5 .
2. It is a 3-regular graph.
3. The girth is 5. So, G contains no cycles C_3 or C_4 .

4. G has 10 vertices and 15 edges.

Example 4 Prove that the Petersen Graph G contains no cycle of length 10.

Proof: We will prove the statement by contradiction. Suppose G contains C_{10} . We can label the vertices by $1, 2, \dots, 10$ according to the order they appear in C_{10} . Every vertex has one more edge leaving since its degree is 3. These edges are called inner edges. Since G contains no C_3 or C_4 , inner edges must be a diagonal or off-diagonal. (Edge ij is diagonal if $i - j - 5 \cong 0 \pmod{10}$; ij is off-diagonal if $i - j - 5 \pmod{\pm 1 \pmod{10}}$.)

Case 1: There exists an off-diagonal edge. Without loss of generality, we can assume it is 17. Since the edge leaving vertex 2 can only go to a vertex v , which is one of vertices 6, 7, 8. For each case, we get a cycle C_3 or C_4 by restricting to $1, 2, 7, v$. (If $v = 7$, we get a triangle.)

Case 2: All inner edges are diagonal. In this case 1276 is a C_4 .

Contradiction! The Petersen Graph contains no C_{10} . \square

A graph G on n vertices is called Hamiltonian if G contains a spanning cycle C_n . The Petersen Graph is not a Hamiltonian graph. In general, to test whether a graph is Hamiltonian is an NP-Complete problem. We will talk about the complexity in lecture 2.

1 Paths, cycles, and trails

Definition 8 A walk (on a graph G) is a list $v_0, e_1, v_1, e_1, \dots, e_k, v_k$, satisfying $e_i = v_{i-1}v_i$ is an edge for all $i = 1, 2, \dots, k$. k is called the length of the walk.

A u, v -walk is a walk with $v_0 = u$ and $v_k = v$.

A trail is a walk with no repeated edge.

A path is a walk with no repeated vertices.

A closed walk is a walk with the same endpoints, i.e., $v_0 = v_k$.

A cycle is a closed walk with no repeated vertices except for the endpoints.

Lemma 1 Every u, v -walk contains a u, v -path.

Proof: We prove it by induction on the length k of the walk.

When $k = 1$, a u, v -walk is a u, v -path.

We assume that every u, v -walk with length at most k contains a u, v -path. Now let us consider a u, v -walk $u = v_0, e_1, v_1, e_1, \dots, e_{k+1}, v_{k+1} = v$ of length $k + 1$. If the walk has no repeated vertices, it is a u, v -path by definition. Otherwise, say $v_i = v_j$ for $0 \leq i < j \leq k + 1$. By deleting all vertices and edges between v_i and v_j , we get a new walk: $v_0, \dots, v_i = v_j, \dots, v_{k+1}$. This walk has length at most k . By the inductive hypothesis, it contains a u, v -path. \square

Definition 9 A graph G is connected if it has a u, v -path for any $u, v \in V(G)$.

For any u, v , we define a connected relation $u \sim v$ over $V(G)$ if there is a u, v -path in G . The connected relation is an equivalence relation. The equivalence classes for this relation are called *connected components* of G .

A component is *trivial* if it contains only one vertex. A trivial component is also called an *isolated* vertex.

Lemma 2 *Every closed odd walk contains an odd cycle.*

Proof: We prove it by induction on the length k of the closed walk.

When $k = 1$, the walk of length 1 is a loop. Thus, it is an odd cycle.

We assume that every odd walk with length at most $k = 2r - 1$ contains a u, v -path. Now let us consider a closed walk $u = v_0, e_1, v_1, e_2, \dots, e_{2r+1}, v_{2r+1} = v$ of length $2r + 1$. If the walk has no repeated vertices, it is an odd cycle by definition. Otherwise, say $v_i = v_j$ for $0 \leq i < j \leq 2r + 1$. The walk can be split into two closed walks:

$$v_0, \dots, v_i = v_j, \dots, v_{k+1}$$

$$v_i, \dots, e_{i+1}, \dots, v_j.$$

One of them must have odd length of at most $2r - 1$. By the inductive hypothesis, it contains an odd cycle.

The proof is finished. \square .

Theorem 1 (König 1936) *A graph is bipartite if and only if it has no odd cycle.*

Proof: It is sufficient to prove this for any connected graph.

Necessity: Let G be bipartite graph. Every walk of G alternates between the two sets of a bipartition. The lengths of any closed walks are even. Therefore, it has no odd cycle.

Sufficiency: Let G be a graph with no odd cycle. We will construct a bipartition as follows. Choose any vertex u . We define a partition $V(G) = X \cup Y$ as follows.

$$X = \{v \in V(G) \mid \text{there is a } u, v\text{-path of odd length}\}.$$

$$Y = \{v \in V(G) \mid \text{there is a } u, v\text{-path of even length}\}.$$

Since G is connected, we have $X \cup Y = V(G)$. We will show $X \cap Y = \emptyset$. Otherwise, let $u \in X \cap Y$. There are two u, v -paths. One path has even length while the other one has odd length. Put them together. We have an odd closed walk. By lemma, G has an odd cycle. Contradiction. Therefore, $V(G) = X \cup Y$ is a partition.

Next, we will show there are no edges with both ends in X or Y . Otherwise, suppose that vw is such an edge. The u, v -path, the edge vw , and the w, u -path together form an odd closed walk. By previous lemma, G contains an odd cycle. Contradiction.

Hence, G is a bipartite graph. \square