# The Giant Component in a Random Subgraph of a Given Graph

Linyuan Lu
University of South Carolina

Coauthors: Fan Chung Graham, Paul Horn, Xing Peng

Atlanta Lecture Series in Combinatorics & Graph Theory IV
Georgia State University, November 5-6, 2011.

# Outline

- Percolation on graphs
  - Motivations
  - Previous results
  - Examples
  - Our results
  - Methods

# Outline

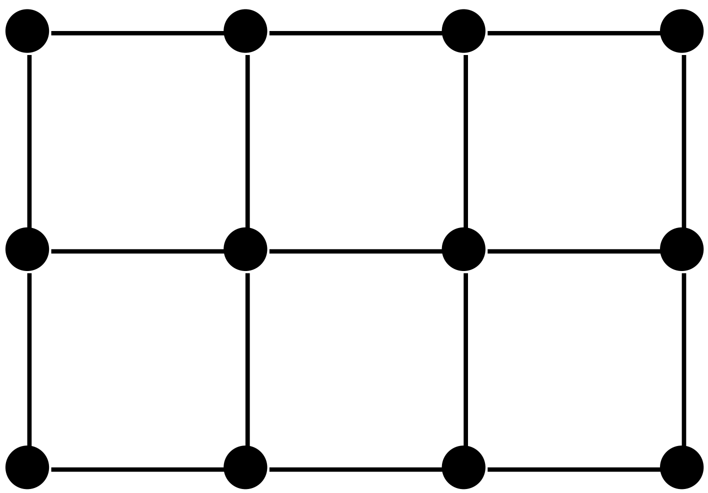
- Percolation on graphs
  - Motivations
  - Previous results
  - Examples
  - Our results
  - Methods
- Ongoing projects on hypergraphs
  - Laplacians of hypergraphs
  - Random hypergraphs

# Part I: Graph percolation

- $G$: a connected graph on $n$ vertices
- $p$: a probability $(0 \le p \le 1)$

$G_p$: a random spanning subgraph of $G$, obtained as follows: for each edge $f$ of $G$, independently,

$$\Pr(f \text{ is an edge of } G_p) = p.$$

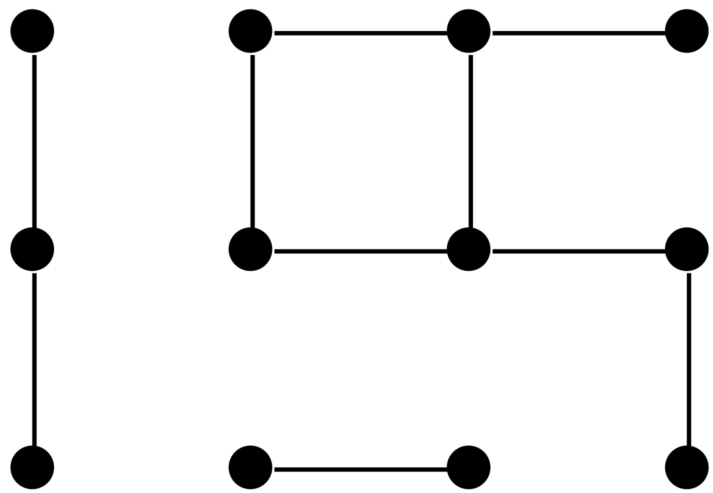


# Part I: Graph percolation

- $G$: a connected graph on $n$ vertices
- $p$: a probability $(0 \le p \le 1)$

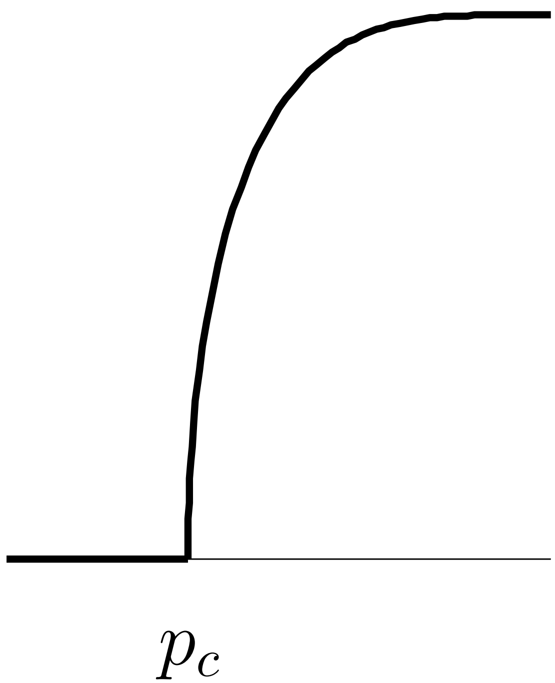
$G_p$: a random spanning subgraph of $G$, obtained as follows: for each edge $f$ of $G$, independently,

$$\Pr(f \text{ is an edge of } G_p) = p.$$



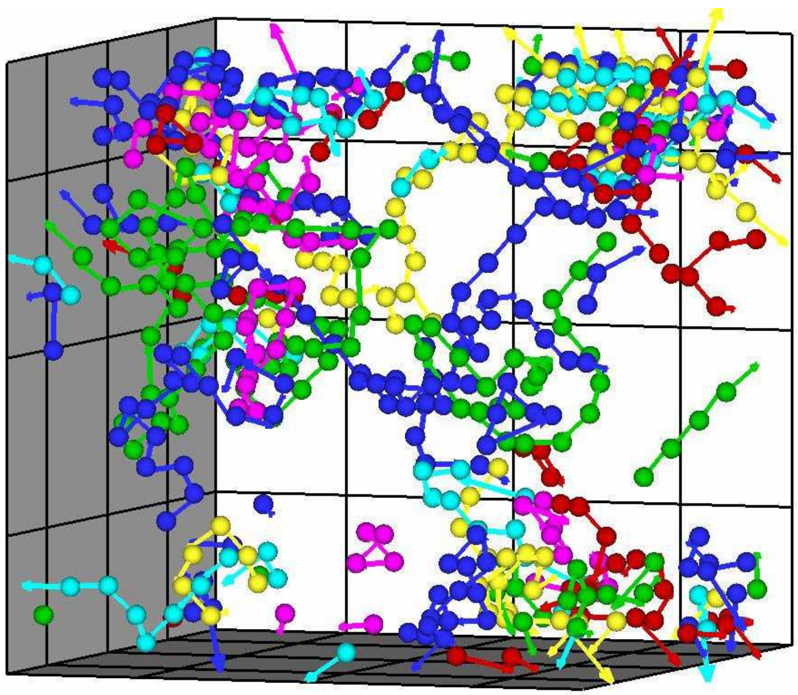
# Percolation threshold $p_c$

- For $p < p_c$, almost surely there is no giant component
- For $p > p_c$, almost surely there is a giant component.



# Motivations

- Graph theory: random graphs
- Theoretical physics: crystals melting
- Sociology: the spread of disease on contact networks



# The case $G = K_n$

For $G = K_n$, $G_p = G(n, p)$: Erdős-Rényi random graphs

# The case $G = K_n$

For $G = K_n$, $G_p = G(n, p)$: Erdős-Rényi random graphs

- n nodes

# The case $G = K_n$

For $G = K_n$, $G_p = G(n, p)$: Erdős-Rényi random graphs

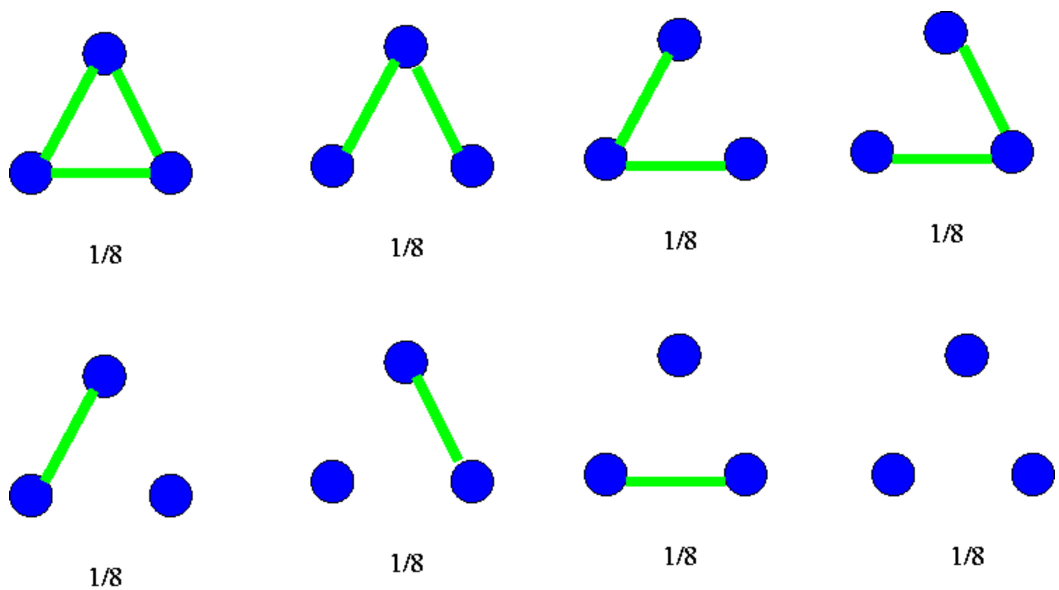
- n nodes
- For each pair of vertices, create an edge independently with probability $p$.

# The case $G = K_n$

For $G = K_n$, $G_p = G(n, p)$: Erdős-Rényi random graphs

- n nodes
- For each pair of vertices, create an edge independently with probability $p$.

An example $G(3, \frac{1}{2})$:



# The evolution of $G(n, p)$

Let $p \sim 1/n + \mu/n$.

- If $\mu < 0$, the largest component has size $(\mu - \log(1 + \mu))^{-1} \log n + O(\log \log n)$.

# The evolution of $G(n, p)$

Let $p \sim 1/n + \mu/n$.

- If $\mu < 0$, the largest component has size $(\mu - \log(1 + \mu))^{-1} \log n + O(\log \log n)$.
- If $\mu = 0$, the largest component has size of order $n^{2/3}$.

# The evolution of $G(n, p)$

Let $p \sim 1/n + \mu/n$.

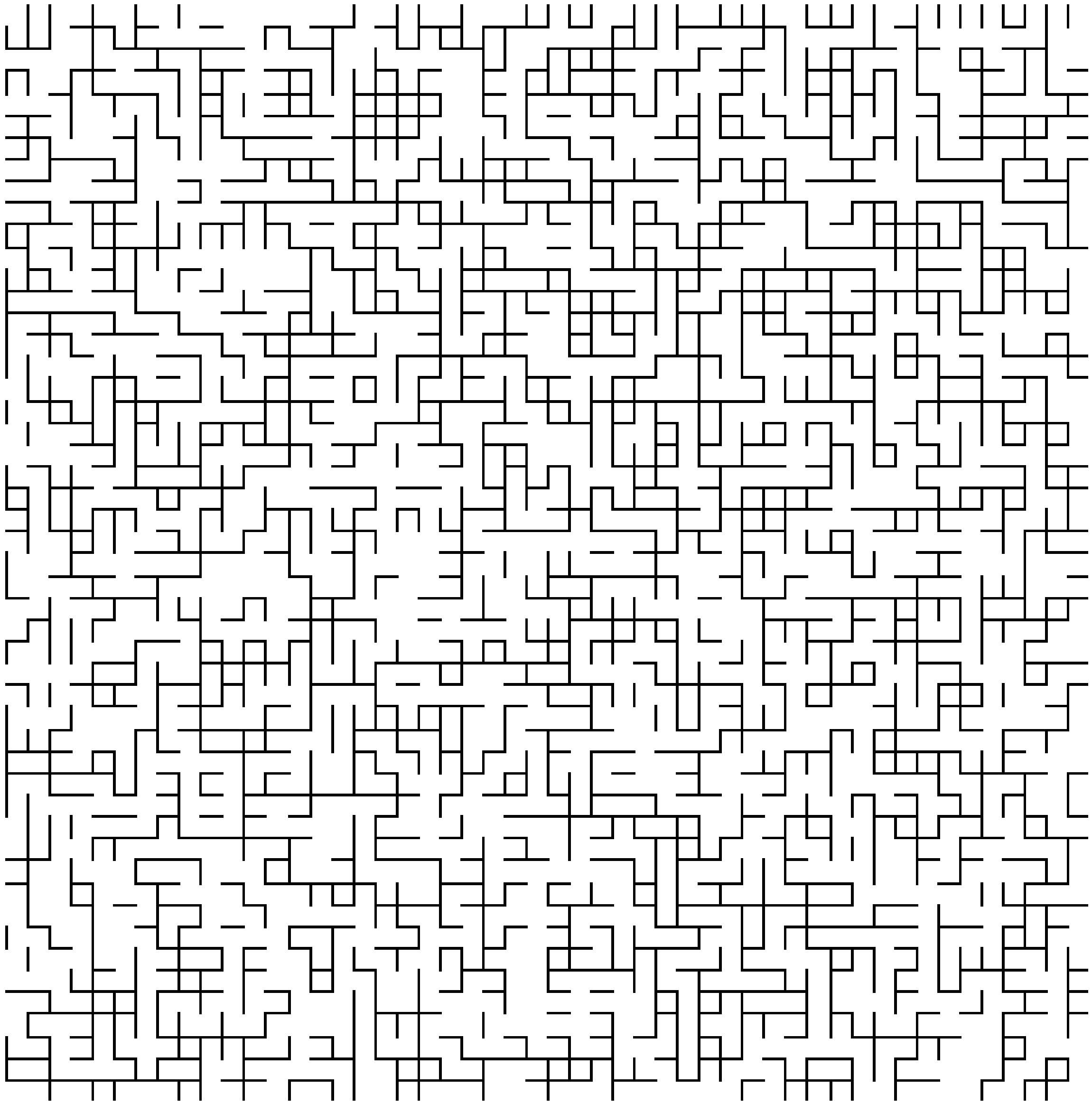
- If $\mu < 0$, the largest component has size $(\mu - \log(1+\mu))^{-1} \log n + O(\log \log n)$.
- If $\mu = 0$, the largest component has size of order $n^{2/3}$.
- If $\mu > 0$, there is a unique giant component of size $\alpha n$ where $\mu = -\alpha^{-1} \log(1-\alpha) - 1$.

# The evolution of $G(n, p)$

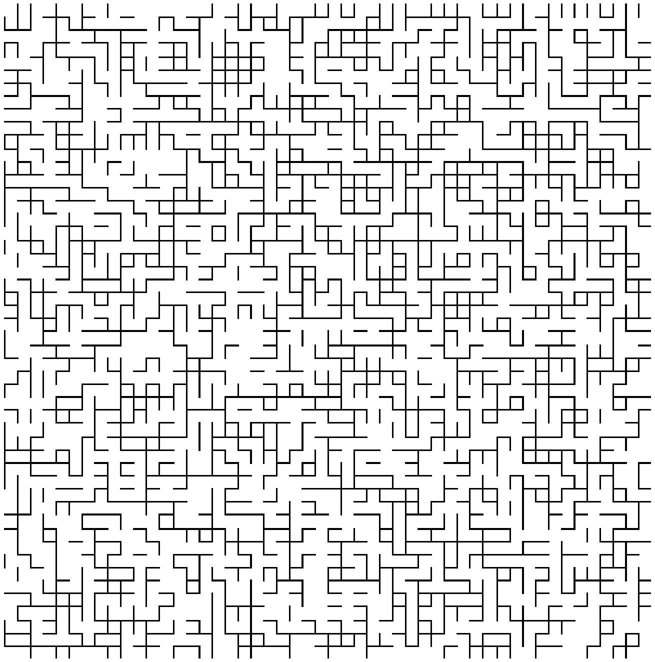
Let $p \sim 1/n + \mu/n$.

- If $\mu < 0$, the largest component has size $(\mu - \log(1+\mu))^{-1} \log n + O(\log\log n)$.
- If $\mu = 0$, the largest component has size of order $n^{2/3}$.
- If $\mu > 0$, there is a unique giant component of size $\alpha n$ where $\mu = -\alpha^{-1}\log(1-\alpha) - 1$.
- Bollobás showed that a component of size at least $n^{2/3}$ in $G_{n,p}$ is almost always unique if $p$ exceeds $1/n + 4(\log n)^{1/2} n^{-4/3}$. (Later he removed the $\log n$-factor.)

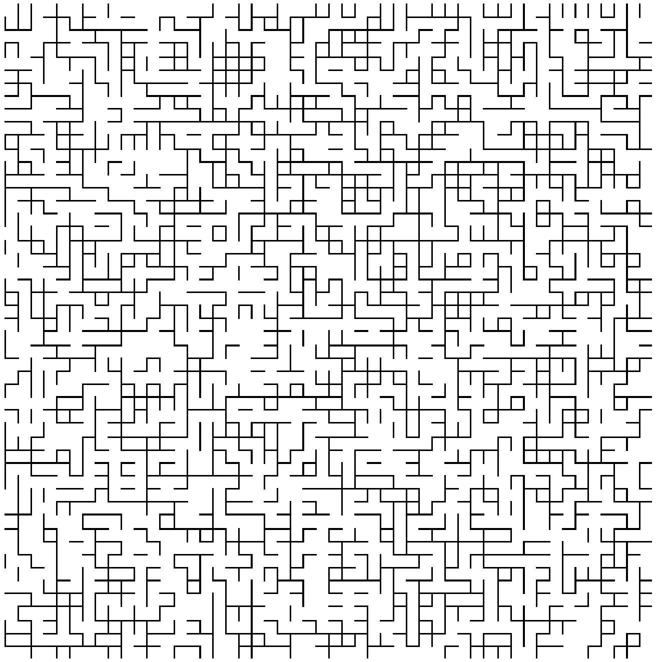
# Percolation of $Z^d$



# Percolation of $Z^d$

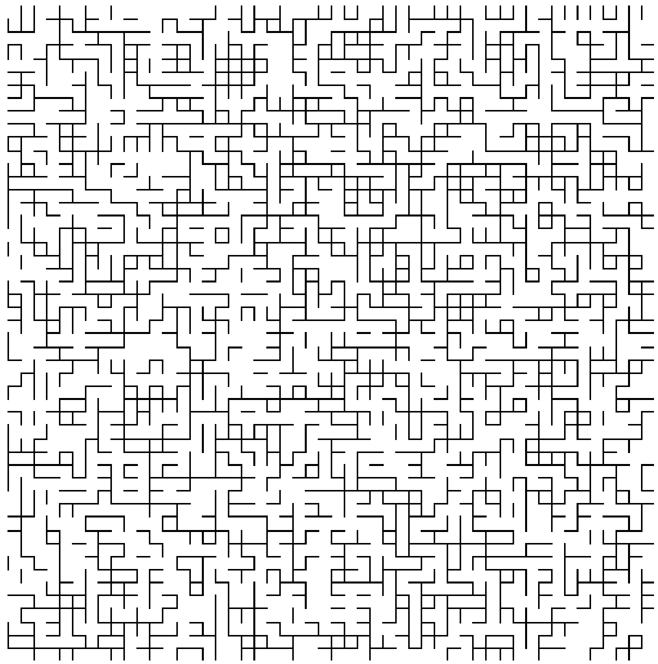


# Percolation of $Z^d$



**Kesten (1980):** $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

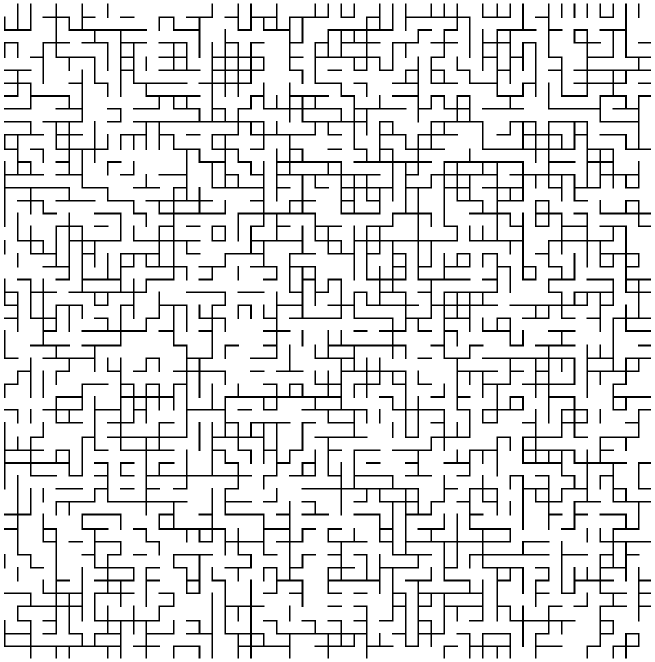
# Percolation of $Z^d$



**Kesten (1980):** $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

**Lorenz and Ziff (1997, simulation):**
$p_c(\mathbb{Z}^3) \approx 0.2488126 \pm 0.0000005$ if it exists.

# Percolation of $Z^d$



**Kesten (1980):** $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

**Lorenz and Ziff (1997, simulation):**
$p_c(\mathbb{Z}^3) \approx 0.2488126 \pm 0.0000005$ if it exists.

**Kesten (1990):** $p_c(\mathbb{Z}^d) \sim \frac{1}{2d}$ as $d \to \infty$.

# $d$-regular graphs

**Alon, Benjamini, Stacey (2004):** Suppose $d \geq 2$ and let $(G_n)$ be a sequence of $d$-regular expanders with $girth(G_n) \to \infty$, then

$$p_c = \frac{1}{d-1} + o(1).$$

# Dense graphs

**Bollobás, Borgs, Chayes, and Riordan (2008):** Suppose that $G$ is a dense graph (i.e., average degree $d = \Theta(n)$). Let $\mu$ be the largest eigenvalue of the adjacency matrix of $G$. Then

$$p_c \approx \frac{1}{\mu}.$$

# Dense graphs

**Bollobás, Borgs, Chayes, and Riordan (2008):** Suppose that $G$ is a dense graph (i.e., average degree $d = \Theta(n)$). Let $\mu$ be the largest eigenvalue of the adjacency matrix of $G$. Then

$$p_c \approx \frac{1}{\mu}.$$

**Remark:** The requirement of "dense graph" is essential. Their methods can not be extended to sparse graphs.

# Questions

Is $p_c \approx \frac{1}{\mu}$?

# Questions

Is $p_c \approx \frac{1}{\mu}$?

"Yes" for some regular graphs and for dense graphs.

# Questions

Is $p_c \approx \frac{1}{\mu}$?

"Yes" for some regular graphs and for dense graphs.

"No" for general graphs.

# Questions

Is $p_c \approx \frac{1}{\mu}$?

"Yes" for some regular graphs and for dense graphs.

"No" for general graphs.

We ask

- Is $p_c \geq \frac{1}{\mu}$?
- Under what conditions, $p_c \approx \frac{1}{\mu}$?

# Notations

- Degrees: $d_1, d_2, \ldots, d_n$.

# Notations

- Degrees: $d_1, d_2, \ldots, d_n$.
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)^*$.

# Notations

- Degrees: $d_1, d_2, \ldots, d_n$.
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)^*$.
- The volume: $\mathrm{vol}(S) = \sum_{i \in S} d_i$.

# Notations

- Degrees: $d_1, d_2, \ldots, d_n$.
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)^*$.
- The volume: $\mathrm{vol}(S) = \sum_{i \in S} d_i$.
- The $k$-th volume: $\mathrm{vol}_k(S) = \sum_{i \in S} d_i^k$.

# Notations

- Degrees: $d_1, d_2, \ldots, d_n$.
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)^*$.
- The volume: $\mathrm{vol}(S) = \sum_{i \in S} d_i$.
- The $k$-th volume: $\mathrm{vol}_k(S) = \sum_{i \in S} d_i^k$.
- Average degree: $d = \frac{\mathrm{vol}(G)}{n}$.

# Notations

- Degrees: $d_1, d_2, \ldots, d_n$.
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)^*$.
- The volume: $\mathrm{vol}(S) = \sum_{i \in S} d_i$.
- The $k$-th volume: $\mathrm{vol}_k(S) = \sum_{i \in S} d_i^k$.
- Average degree: $d = \frac{\mathrm{vol}(G)}{n}$.
- Second order average degree: $\tilde{d} = \frac{\mathrm{vol}_2(G)}{\mathrm{vol}(G)}$

# Notations

- Degrees: $d_1, d_2, \ldots, d_n$.
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)^*$.
- The volume: $\mathrm{vol}(S) = \sum_{i \in S} d_i$.
- The $k$-th volume: $\mathrm{vol}_k(S) = \sum_{i \in S} d_i^k$.
- Average degree: $d = \frac{\mathrm{vol}(G)}{n}$.
- Second order average degree: $\tilde{d} = \frac{\mathrm{vol}_2(G)}{\mathrm{vol}(G)}$
- Third order average degree: $\tilde{\tilde{d}} = \frac{\mathrm{vol}_3(G)}{\mathrm{vol}_2(G)}$.

# Notations

- Degrees: $d_1, d_2, \ldots, d_n$.
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)^*$.
- The volume: $\mathrm{vol}(S) = \sum_{i \in S} d_i$.
- The $k$-th volume: $\mathrm{vol}_k(S) = \sum_{i \in S} d_i^k$.
- Average degree: $d = \frac{\mathrm{vol}(G)}{n}$.
- Second order average degree: $\tilde{d} = \frac{\mathrm{vol}_2(G)}{\mathrm{vol}(G)}$
- Third order average degree: $\tilde{\tilde{d}} = \frac{\mathrm{vol}_3(G)}{\mathrm{vol}_2(G)}$.

A connected component is giant if its volume is $\Theta(\mathrm{vol}(G))$.

# Yes, $p_c \geq \frac{1}{\mu}$.

**Chung, Lu, Horn 2008:** *For $p < \frac{1}{\mu}$, almost surely every connected component in $G_p$ has volume at most $O(\sqrt{\mathrm{vol}_2(G)}g(n))$, where $g(n)$ is any slowly growing function as $n \to \infty$.*

# Yes, $p_c \geq \frac{1}{\mu}$.

**Chung, Lu, Horn 2008:** *For $p < \frac{1}{\mu}$, almost surely every connected component in $G_p$ has volume at most $O(\sqrt{\mathrm{vol}_2(G)}g(n))$, where $g(n)$ is any slowly growing function as $n \to \infty$.*

**Proof:** Let $A$ be the event that there exists a component $S$ in $G_p$ with $\mathrm{vol}(S) > C\sqrt{\mathrm{vol}_2(G)}$.

**Claim A**: $\Pr(A) \leq \frac{1}{C^2(1-p\mu)}$.

# Yes, $p_c \geq \frac{1}{\mu}$.

**Chung, Lu, Horn 2008:** *For $p < \frac{1}{\mu}$, almost surely every connected component in $G_p$ has volume at most $O(\sqrt{\mathrm{vol}_2(G)}g(n))$, where $g(n)$ is any slowly growing function as $n \to \infty$.*

**Proof:** Let $A$ be the event that there exists a component $S$ in $G_p$ with $\mathrm{vol}(S) > C\sqrt{\mathrm{vol}_2(G)}$.

**Claim A**: $\Pr(A) \leq \frac{1}{C^2(1-p\mu)}$.

- $u$, $v$: two random vertices selected with probability proportional to their degrees.

# Yes, $p_c \geq \frac{1}{\mu}$.

**Chung, Lu, Horn 2008:** *For $p < \frac{1}{\mu}$, almost surely every connected component in $G_p$ has volume at most $O(\sqrt{\mathrm{vol}_2(G)}g(n))$, where $g(n)$ is any slowly growing function as $n \to \infty$.*

**Proof:** Let $A$ be the event that there exists a component $S$ in $G_p$ with $\mathrm{vol}(S) > C\sqrt{\mathrm{vol}_2(G)}$.

**Claim A**: $\Pr(A) \leq \frac{1}{C^2(1-p\mu)}$.

- $u$, $v$: two random vertices selected with probability proportional to their degrees.
- $|\{k\text{-path from } u \text{ to } v \text{ in } G\}| \leq \mathbf{1}_u^* A^k \mathbf{1}_v$.

# Yes, $p_c \geq \frac{1}{\mu}$.

**Chung, Lu, Horn 2008:** *For $p < \frac{1}{\mu}$, almost surely every connected component in $G_p$ has volume at most $O(\sqrt{\mathrm{vol}_2(G)}g(n))$, where $g(n)$ is any slowly growing function as $n \to \infty$.*

**Proof:** Let $A$ be the event that there exists a component $S$ in $G_p$ with $\mathrm{vol}(S) > C\sqrt{\mathrm{vol}_2(G)}$.

**Claim A**: $\Pr(A) \leq \frac{1}{C^2(1-p\mu)}$.

- $u$, $v$: two random vertices selected with probability proportional to their degrees.
- $|\{k\text{-path from } u \text{ to } v \text{ in } G\}| \leq \mathbf{1}_u^* A^k \mathbf{1}_v$.
- The probability of a $k$-path survived in $G_p$ is $p^k$.

# Proof of Claim A:

$$\Pr(u, v \text{ are in the same component of } G_p)$$

# Proof of Claim A:

$$\Pr(u, v \text{ are in the same component of } G_p)$$
$$\leq \sum_{u,v} \frac{d_u}{\mathrm{vol}(G)} \frac{d_v}{\mathrm{vol}(G)} \sum_{k=0}^{n} p^k \mathbf{1}_u^* A^k \mathbf{1}_v$$

# Proof of Claim A:

$$\Pr(u, v \text{ are in the same component of } G_p)$$

$$\leq \sum_{u,v} \frac{d_u}{\mathrm{vol}(G)} \frac{d_v}{\mathrm{vol}(G)} \sum_{k=0}^{n} p^k \mathbf{1}_u^* A^k \mathbf{1}_v$$

$$= \sum_{k=0}^{n} \frac{1}{\mathrm{vol}(G)^2} p^k \mathbf{d}^* A^k \mathbf{d}$$

# Proof of Claim A:

$$
\begin{aligned}
&\mathrm{Pr}(u, v \text{ are in the same component of } G_p) \\
\leq\ & \sum_{u,v} \frac{d_u}{\mathrm{vol}(G)} \frac{d_v}{\mathrm{vol}(G)} \sum_{k=0}^{n} p^k \mathbf{1}_u^* A^k \mathbf{1}_v \\
=\ & \sum_{k=0}^{n} \frac{1}{\mathrm{vol}(G)^2} p^k \mathbf{d}^* A^k \mathbf{d} \\
\leq\ & \sum_{k=0}^{\infty} \frac{p^k \mu^k \mathrm{vol}_2(G)}{\mathrm{vol}(G)^2}
\end{aligned}
$$

# Proof of Claim A:

$$
\begin{aligned}
&\Pr(u, v \text{ are in the same component of } G_p) \\
\leq\ & \sum_{u,v} \frac{d_u}{\mathrm{vol}(G)} \frac{d_v}{\mathrm{vol}(G)} \sum_{k=0}^{n} p^k \mathbf{1}_u^* A^k \mathbf{1}_v \\
=\ & \sum_{k=0}^{n} \frac{1}{\mathrm{vol}(G)^2} p^k \mathbf{d}^* A^k \mathbf{d} \\
\leq\ & \sum_{k=0}^{\infty} \frac{p^k \mu^k \mathrm{vol}_2(G)}{\mathrm{vol}(G)^2} \\
\leq\ & \frac{\tilde{d}}{(1 - p\mu)\mathrm{vol}(G)}.
\end{aligned}
$$

# Continue

$\Pr(u, v$ are in the same component of $G_p)$

# Continue

$$\Pr(u, v \text{ are in the same component of } G_p) \\ \geq \Pr(A \text{ and } u, v \in S)$$

# Continue

$$
\begin{aligned}
&\Pr(u, v \text{ are in the same component of } G_p) \\
&\geq \Pr(A \text{ and } u, v \in S) \\
&> \Pr(A) \frac{C\sqrt{\mathrm{vol}_2(G)}}{\mathrm{vol}(G)} \frac{C\sqrt{\mathrm{vol}_2(G)}}{\mathrm{vol}(G)}
\end{aligned}
$$

# Continue

$$
\begin{aligned}
&\Pr(u, v \text{ are in the same component of } G_p) \\
&\geq \Pr(A \text{ and } u, v \in S) \\
&> \Pr(A)\frac{C\sqrt{\mathrm{vol}_2(G)}}{\mathrm{vol}(G)}\frac{C\sqrt{\mathrm{vol}_2(G)}}{\mathrm{vol}(G)} \\
&= \Pr(A)\frac{C^2\tilde{d}}{\mathrm{vol}(G)}.
\end{aligned}
$$

# Continue

$$
\begin{aligned}
&\Pr(u, v \text{ are in the same component of } G_p) \\
&\geq \Pr(A \text{ and } u, v \in S) \\
&> \Pr(A) \frac{C\sqrt{\mathrm{vol}_2(G)}}{\mathrm{vol}(G)} \frac{C\sqrt{\mathrm{vol}_2(G)}}{\mathrm{vol}(G)} \\
&= \Pr(A) \frac{C^2 \tilde{d}}{\mathrm{vol}(G)}.
\end{aligned}
$$

Thus, $$\frac{\tilde{d}}{(1 - p\mu)\mathrm{vol}(G)} > \Pr(A) \frac{C^2 \tilde{d}}{\mathrm{vol}(G)}$$

# Continue

$$
\begin{aligned}
&\Pr(u, v \text{ are in the same component of } G_p) \\
&\geq \Pr(A \text{ and } u, v \in S) \\
&> \Pr(A) \frac{C\sqrt{\mathrm{vol}_2(G)}}{\mathrm{vol}(G)} \frac{C\sqrt{\mathrm{vol}_2(G)}}{\mathrm{vol}(G)} \\
&= \Pr(A) \frac{C^2 \tilde{d}}{\mathrm{vol}(G)}.
\end{aligned}
$$

Thus, $$\frac{\tilde{d}}{(1 - p\mu)\mathrm{vol}(G)} > \Pr(A) \frac{C^2 \tilde{d}}{\mathrm{vol}(G)}$$

$$\Pr(A) < \frac{1}{C^2(1 - p\mu)}. \qquad \square$$

# The target graphs

- Not dense graphs.
- Unevenly distributed degree sequence, like power law graphs.
- Some bounds on spectra, like expanders.

# Model $G(w_1, w_2, \ldots, w_n)$

Random graph model with given expected degree sequence

- $n$ nodes with weights $w_1, w_2, \ldots, w_n$.

# Model $G(w_1, w_2, \ldots, w_n)$

Random graph model with given expected degree sequence

- $n$ nodes with weights $w_1, w_2, \ldots, w_n$.
- For each pair $(i, j)$, create an edge independently with probability $p_{ij} = w_i w_j \rho$, where $\rho = \frac{1}{\sum_{i=1}^n w_i}$.

# **Model** $G(w_1, w_2, \ldots, w_n)$

Random graph model with given expected degree sequence

- $n$ nodes with weights $w_1, w_2, \ldots, w_n$.
- For each pair $(i, j)$, create an edge independently with probability $p_{ij} = w_i w_j \rho$, where $\rho = \frac{1}{\sum_{i=1}^n w_i}$.
- The graph $H$ has probability

$$\prod_{ij \in E(H)} p_{ij} \prod_{ij \notin E(H)} (1 - p_{ij}).$$

# Model $G(w_1, w_2, \ldots, w_n)$

Random graph model with given expected degree sequence

- $n$ nodes with weights $w_1, w_2, \ldots, w_n$.
- For each pair $(i, j)$, create an edge independently with probability $p_{ij} = w_i w_j \rho$, where $\rho = \frac{1}{\sum_{i=1}^n w_i}$.
- The graph $H$ has probability

$$\prod_{ij \in E(H)} p_{ij} \prod_{ij \notin E(H)} (1 - p_{ij}).$$

- The expected degree of vertex $i$ is $w_i$.

# Model $G(w_1, w_2, \ldots, w_n)$

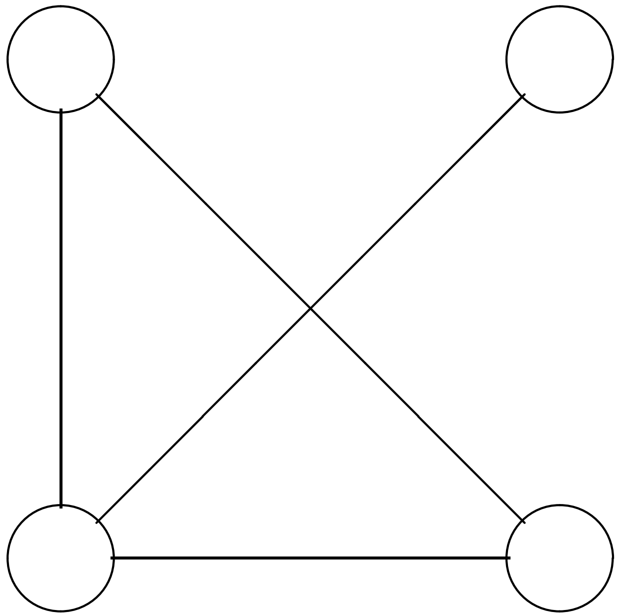
Random graph model with given expected degree sequence

- $n$ nodes with weights $w_1, w_2, \ldots, w_n$.
- For each pair $(i, j)$, create an edge independently with probability $p_{ij} = w_i w_j \rho$, where $\rho = \frac{1}{\sum_{i=1}^n w_i}$.
- The graph $H$ has probability

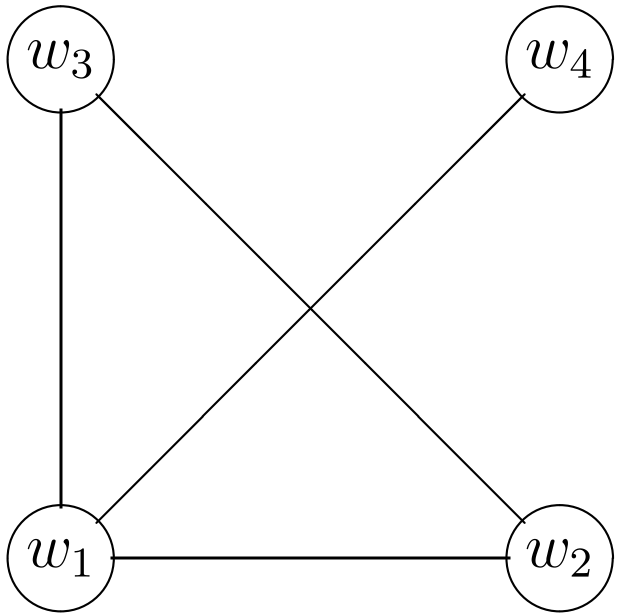
$$\prod_{ij \in E(H)} p_{ij} \prod_{ij \notin E(H)} (1 - p_{ij}).$$

- The expected degree of vertex $i$ is $w_i$.
- Erdős-Rényi model $G(n, p) = G(np, \ldots, np)$.

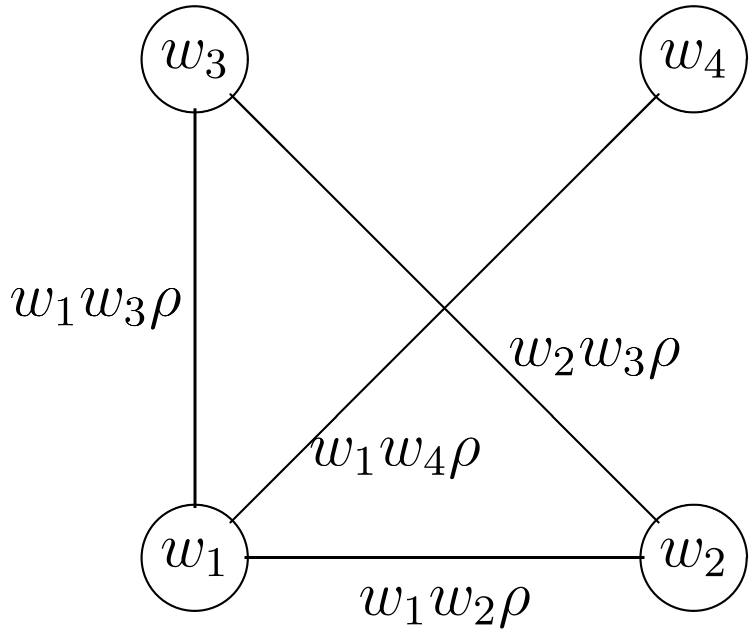
# An example: $G(w_1, w_2, w_3, w_4)$



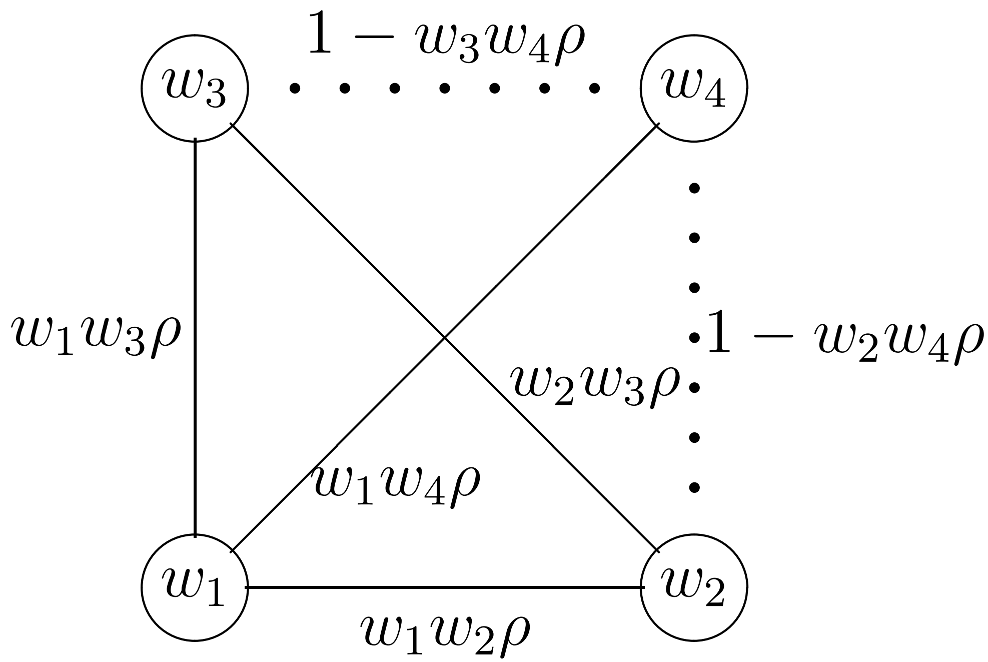
# An example: $G(w_1, w_2, w_3, w_4)$



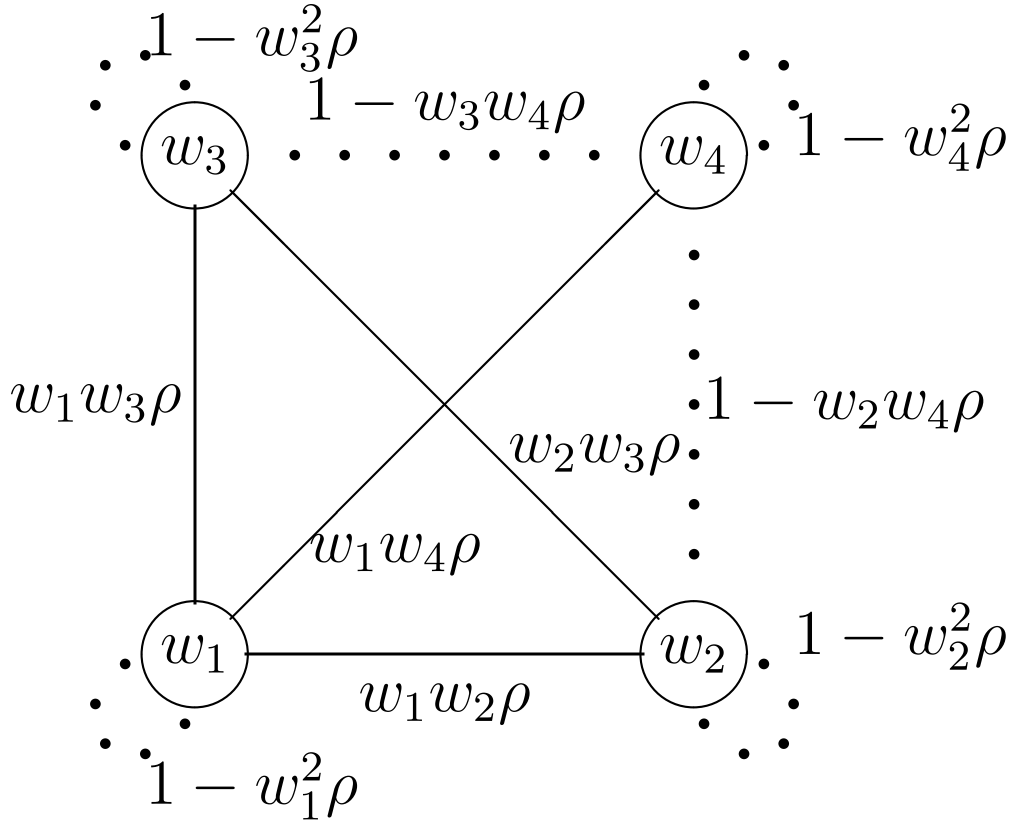
# An example: $G(w_1, w_2, w_3, w_4)$



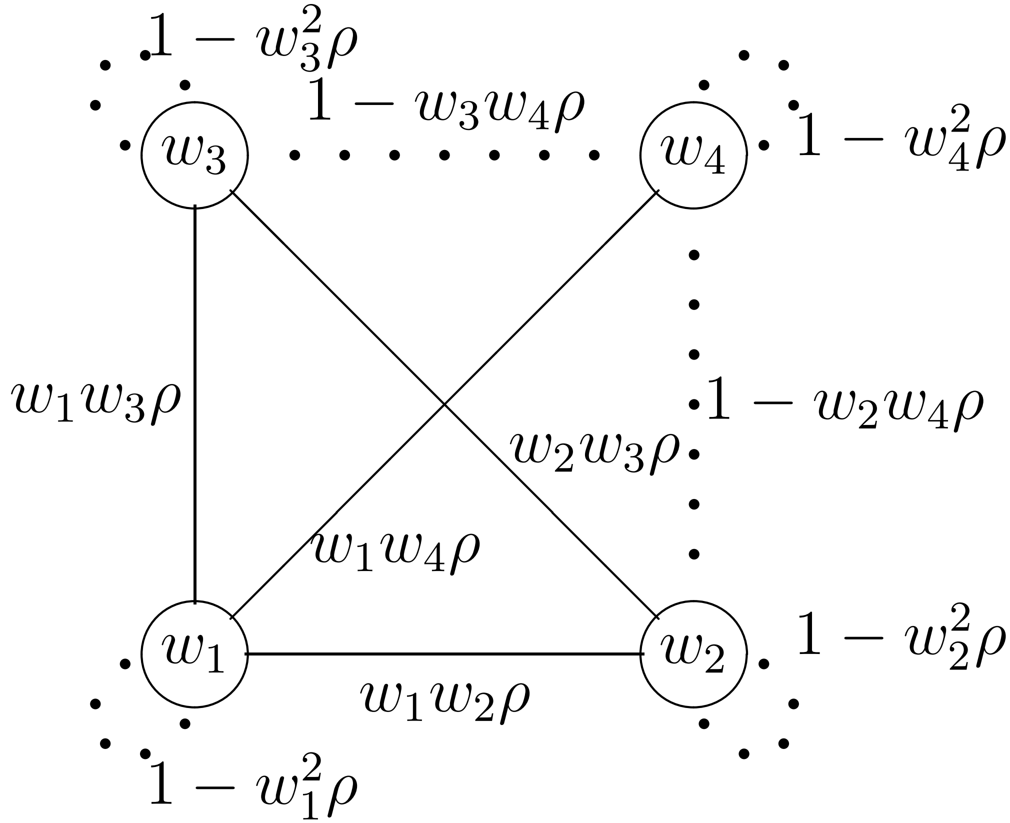
# An example: $G(w_1, w_2, w_3, w_4)$



# An example: $G(w_1, w_2, w_3, w_4)$



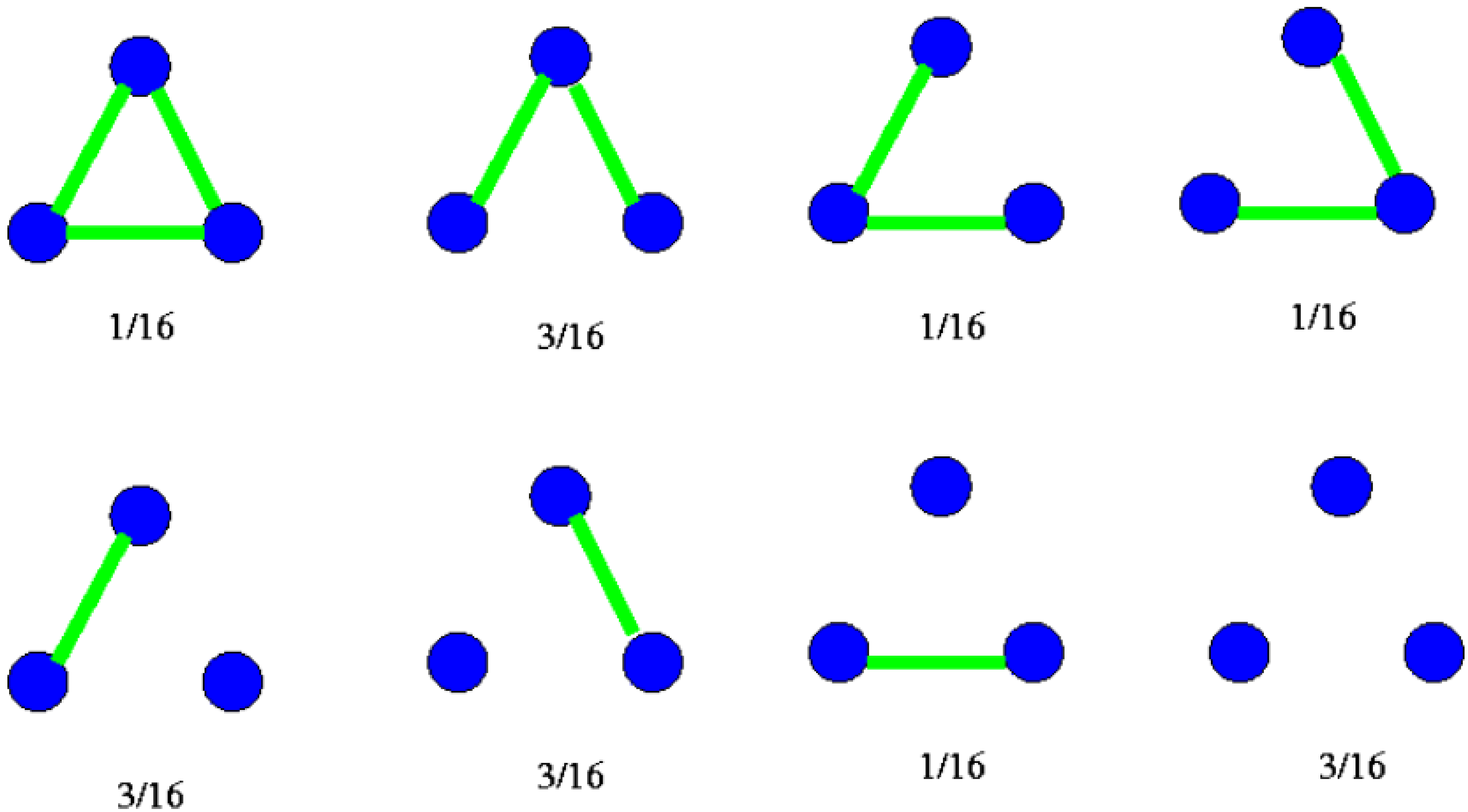
# An example: $G(w_1, w_2, w_3, w_4)$




The probability of the graph is

$$w_1^3 w_2^2 w_3^2 w_4 \rho^4 (1 - w_2 w_4 \rho) \times (1 - w_3 w_4 \rho) \prod_{i=1}^{4} (1 - w_i^2 \rho).$$

# A example: $G(1,2,1)$




Loops are omitted here.

# Notations

For $G = G(w_1, \ldots, w_n)$, let

- $d = \frac{1}{n}\sum_{i=1}^{n} w_i$
- $\tilde{d} = \frac{\sum_{i=1}^{n} w_i^2}{\sum_{i=1}^{n} w_i}$.
- The volume of $S$: $\mathrm{vol}(S) = \sum_{i\in S} w_i$.

# Notations

For $G = G(w_1, \ldots, w_n)$, let

- $d = \frac{1}{n}\sum_{i=1}^n w_i$
- $\tilde{d} = \frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n w_i}$.
- The volume of $S$: $\mathrm{vol}(S) = \sum_{i \in S} w_i$.

We have

$$\tilde{d} \geq d$$

"$=$" holds if and only if $w_1 = \cdots = w_n$.

# Notations

For $G = G(w_1, \ldots, w_n)$, let

- $d = \frac{1}{n}\sum_{i=1}^n w_i$
- $\tilde{d} = \frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n w_i}$.
- The volume of $S$: $\mathrm{vol}(S) = \sum_{i \in S} w_i$.

We have

$$\tilde{d} \geq d$$

"$=$" holds if and only if $w_1 = \cdots = w_n$.

A connected component $S$ is called a giant component if

$$\mathrm{vol}(S) = \Theta(\mathrm{vol}(G)).$$

# Connected components

**Chung and Lu (2001)** For $G = G(w_1, \ldots, w_n)$,

- If $\tilde{d} < 1 - \epsilon$, then almost surely, all components have volume at most $O(\sqrt{n}\log n)$.

# Connected components

**Chung and Lu (2001)** For $G = G(w_1, \ldots, w_n)$,

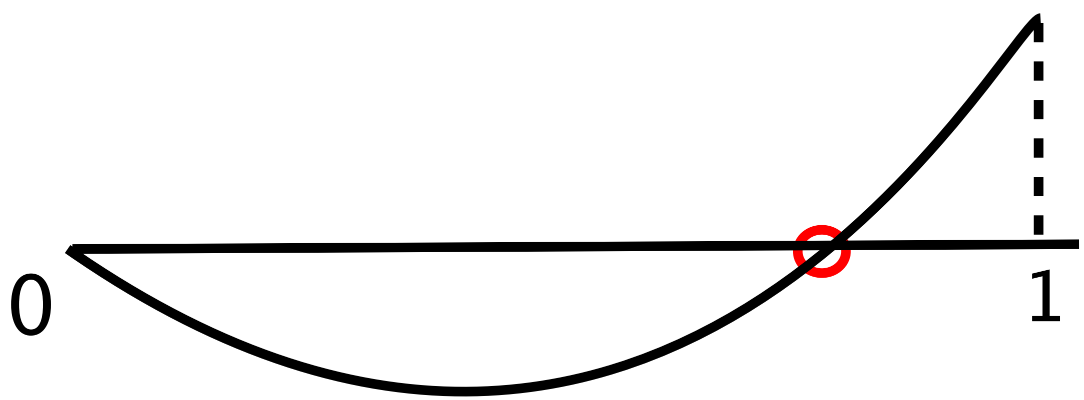
- If $\tilde{d} < 1 - \epsilon$, then almost surely, all components have volume at most $O(\sqrt{n}\log n)$.
- If $d > 1 + \epsilon$, then almost surely there is a unique giant component of volume $\Theta(\mathrm{vol}(G))$. All other components have size at most

$$\begin{cases} \frac{\log n}{d-1-\log d-\epsilon d} & \text{if } \frac{1}{1-\epsilon} < d < \frac{2}{1-\epsilon} \\ \frac{\log n}{1+\log d-\log 4+2\log(1-\epsilon)} & \text{if } d > \frac{4}{e(1-\epsilon)^2}. \end{cases}$$

# Volume of Giant Component

**Chung and Lu (2004)**
If the average degree is strictly greater than $1$, then almost surely the giant component in a graph $G$ in $G(\mathbf{w})$ has volume $(\lambda_0 + O(\sqrt{n}\frac{\log^{3.5} n}{\mathrm{vol}(G)}))\mathrm{vol}(G)$, where $\lambda_0$ is the unique positive root of the following equation:

$$\sum_{i=1}^{n} w_i e^{-w_i\lambda} = (1-\lambda)\sum_{i=1}^{n} w_i.$$



# Percolation on $G(w_1, w_2, \ldots, w_n)$

$$G_p = G(w_1 p, \ldots, w_n p).$$

# Percolation on $G(w_1, w_2, \ldots, w_n)$

$$G_p = G(w_1 p, \ldots, w_n p).$$

We have

- If $p < \frac{1}{\tilde{d}}$, there is no giant component.

# Percolation on $G(w_1, w_2, \ldots, w_n)$

$$G_p = G(w_1 p, \ldots, w_n p).$$

We have

- If $p < \frac{1}{\tilde{d}}$, there is no giant component.
- If $p > \frac{1}{\tilde{d}}$, there is a giant component.

# Percolation on $G(w_1, w_2, \ldots, w_n)$

$$G_p = G(w_1 p, \ldots, w_n p).$$

We have

- If $p < \frac{1}{\tilde{d}}$, there is no giant component.
- If $p > \frac{1}{d}$, there is a giant component.
- For $\frac{1}{\tilde{d}} < p < \frac{1}{d}$, no conclusion.

# Sub-dense graphs

For any big constant $C$ and any $\epsilon_n \to 0$, there exists a graph $G = G_n$ satisfying

- $d = \Omega(\epsilon_n n)$

# Sub-dense graphs

For any big constant $C$ and any $\epsilon_n \to 0$, there exists a graph $G = G_n$ satisfying

- $d = \Omega(\epsilon_n n)$
- $\mu \approx \Theta(\sqrt{\epsilon_n} n)$

# Sub-dense graphs

For any big constant $C$ and any $\epsilon_n \to 0$, there exists a graph $G = G_n$ satisfying

- $d = \Omega(\epsilon_n n)$
- $\mu \approx \Theta(\sqrt{\epsilon_n} n)$
- For $p = \frac{C}{\mu}$, the volume of all components is at most $O(\epsilon_n n)$.

**Conclusion:** In general, $\frac{1}{\mu}$ is not the threshold function for the giant component appearing.

# $\tilde{d}$ and $\mu$

- $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$: normalized Laplacian

# $\tilde{d}$ and $\mu$

- $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$: normalized Laplacian
- $0 = \lambda_0 \leq \lambda_1 \leq \ldots, \leq \lambda_{n-1}$: Laplacian spectra

# $\tilde{d}$ and $\mu$

- $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$: normalized Laplacian
- $0 = \lambda_0 \leq \lambda_1 \leq \ldots, \leq \lambda_{n-1}$: Laplacian spectra
- $\sigma = \max_{i=1}^{n-1} |1 - \lambda_i|$: spectral bound

# $\tilde{d}$ and $\mu$

- $\mathcal{L} = I - D^{-1/2} A D^{-1/2}$: normalized Laplacian
- $0 = \lambda_0 \leq \lambda_1 \leq \ldots, \leq \lambda_{n-1}$: Laplacian spectra
- $\sigma = \max_{i=1}^{n-1} |1 - \lambda_i|$: spectral bound
- $\Delta$: maximum degree

# $\tilde{d}$ and $\mu$

- $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$: normalized Laplacian
- $0 = \lambda_0 \leq \lambda_1 \leq \ldots, \leq \lambda_{n-1}$: Laplacian spectra
- $\sigma = \max_{i=1}^{n-1} |1 - \lambda_i|$: spectral bound
- $\Delta$: maximum degree

**Lemma:**

$$|\mu - \tilde{d}| \leq \sigma\Delta.$$

# Admissible conditions

A set $U$ is a $(\epsilon, M)$-admissible set if

**(i)** $\mathrm{vol}_2(U) \geq (1-\epsilon)\mathrm{vol}_2(G)$.

**(ii)** $\mathrm{vol}_3(U) \leq Md\mathrm{vol}_2(G)$

# Admissible conditions

A set $U$ is a $(\epsilon, M)$-admissible set if

**(i)** $\mathrm{vol}_2(U) \geq (1-\epsilon)\mathrm{vol}_2(G)$.
**(ii)** $\mathrm{vol}_3(U) \leq Md\mathrm{vol}_2(G)$

We say $G$ is $(\epsilon, M)$-admissible.

# Admissible conditions

A set $U$ is a $(\epsilon, M)$-admissible set if

**(i)** $\mathrm{vol}_2(U) \geq (1-\epsilon)\mathrm{vol}_2(G)$.
**(ii)** $\mathrm{vol}_3(U) \leq Md\,\mathrm{vol}_2(G)$

We say $G$ is $(\epsilon, M)$-admissible.

"$\Delta \leq Md$" implies "$\tilde{\tilde{d}} \leq Md$".

# Admissible conditions

A set $U$ is a $(\epsilon, M)$-admissible set if

**(i)** $\mathrm{vol}_2(U) \geq (1-\epsilon)\mathrm{vol}_2(G)$.
**(ii)** $\mathrm{vol}_3(U) \leq Md\,\mathrm{vol}_2(G)$

We say $G$ is $(\epsilon, M)$-admissible.

"$\Delta \leq Md$" implies "$\tilde{\tilde{d}} \leq Md$".

"$\tilde{\tilde{d}} \leq Md$" implies "$G$ is $(\epsilon, M)$-admissible".

# Our result (II)

**Chung, Lu, Horn (2008):**
*Suppose* $p \geq \frac{1+c}{\tilde{d}}$ *for some* $c \leq \frac{1}{20}$*. Suppose* $G$ *satisfies* $\Delta = o(\frac{\tilde{d}}{\sigma})$, $\Delta = o(\frac{d\sqrt{n}}{\log n})$ *and* $\sigma = o(n^{-\kappa})$ *for some* $\kappa > 0$*, and* $G$ *is* $(\frac{c\kappa}{10}, M)$*-admissible. Then almost surely there is a unique giant connected component in* $G_p$ *with volume* $\Theta(\mathrm{vol}(G))$*, and no other component has volume more than* $\max(2d\log n, \omega(\sigma\sqrt{\mathrm{vol}(G)}))$*.*

# Our result (II)

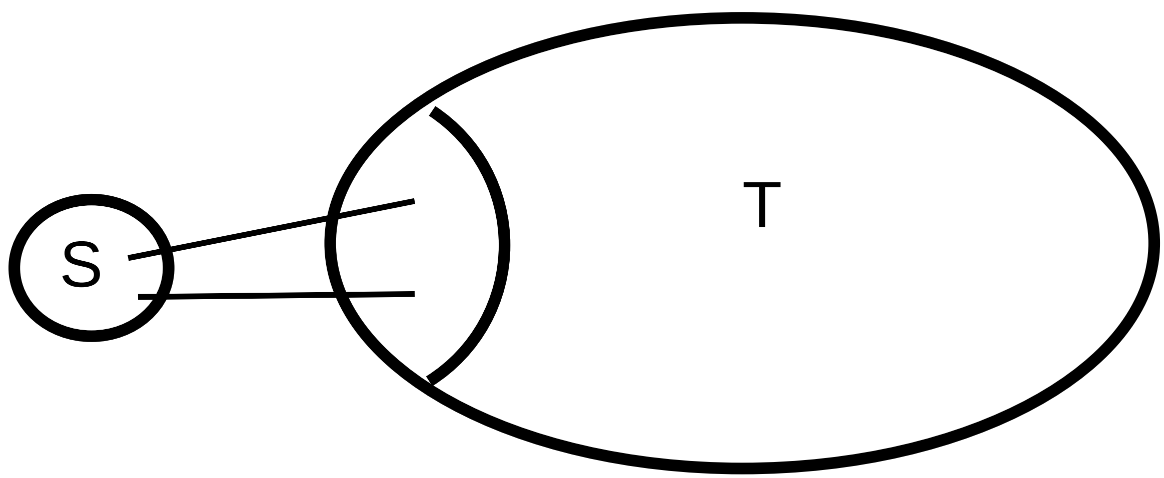
**Chung, Lu, Horn (2008):**
*Suppose* $p \geq \frac{1+c}{\tilde{d}}$ *for some* $c \leq \frac{1}{20}$. *Suppose* $G$ *satisfies* $\Delta = o(\frac{\tilde{d}}{\sigma})$, $\Delta = o(\frac{d\sqrt{n}}{\log n})$ *and* $\sigma = o(n^{-\kappa})$ *for some* $\kappa > 0$, *and* $G$ *is* $(\frac{c\kappa}{10}, M)$*-admissible. Then almost surely there is a unique giant connected component in* $G_p$ *with volume* $\Theta(\mathrm{vol}(G))$, *and no other component has volume more than* $\max(2d\log n, \omega(\sigma\sqrt{\mathrm{vol}(G)}))$.

If $\Delta = O(d)$ and $\sigma = o(\frac{1}{\log n})$, then $p_c = (1 + o(1))\frac{1}{\mu}$.

# Neighborhood expansion in $G_p$

If $\mathrm{vol}(S) \geq \Theta(\sigma^2 \mathrm{vol}(G))$, and $\mathrm{vol}_2(T) > (1-\delta)\mathrm{vol}_2(G)$, then

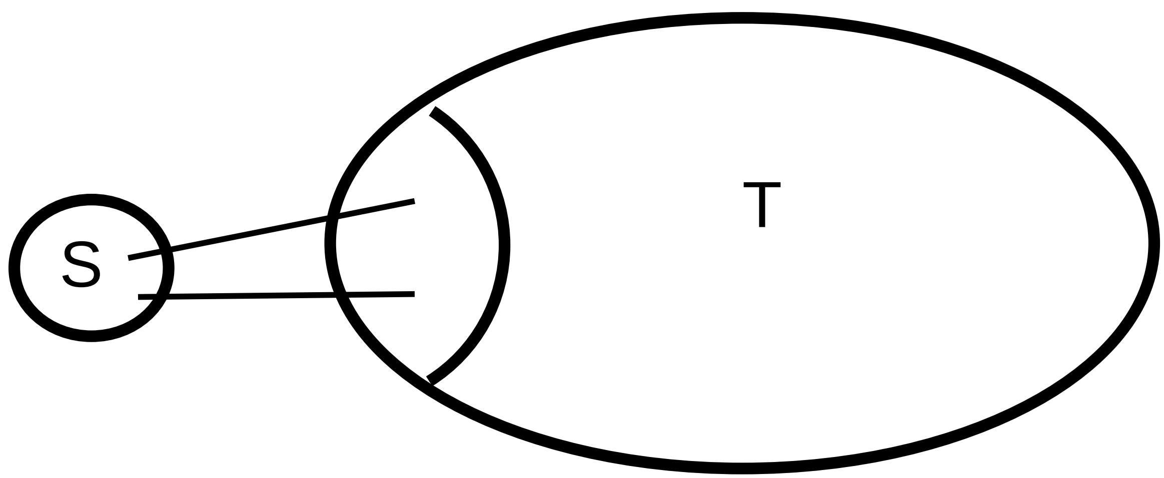
$$\mathrm{vol}(\Gamma(S) \cap T) \geq (1-\delta)p\tilde{d}\mathrm{vol}(S).$$



# Neighborhood expansion in $G_p$

If $\mathrm{vol}(S) \geq \Theta(\sigma^2 \mathrm{vol}(G))$, and $\mathrm{vol}_2(T) > (1-\delta)\mathrm{vol}_2(G)$, then

$$\mathrm{vol}(\Gamma(S) \cap T) \geq (1-\delta)p\tilde{d}\mathrm{vol}(S).$$




Since $p > \frac{1+c}{\tilde{d}}$, $(1-\delta)p\tilde{d} > 1$ for small $\delta$.

# The difficulty

The spectral bound gives a good control of neighborhood expansion of $S$ if $\mathrm{vol}(S) = \Omega(\sigma^2 \mathrm{vol}(G))$.

# The difficulty

The spectral bound gives a good control of neighborhood expansion of $S$ if $\mathrm{vol}(S) = \Omega(\sigma^2 \mathrm{vol}(G))$.

**Question:** How to find a large connect set $S$ with $\mathrm{vol}(S) = \Omega(\sigma^2 \mathrm{vol}(G))$?

# Main ideas

- Choose $S_0$ with $\mathrm{vol}(S_0) = \Theta(\sigma^2 \mathrm{vol}(G))$.

# Main ideas

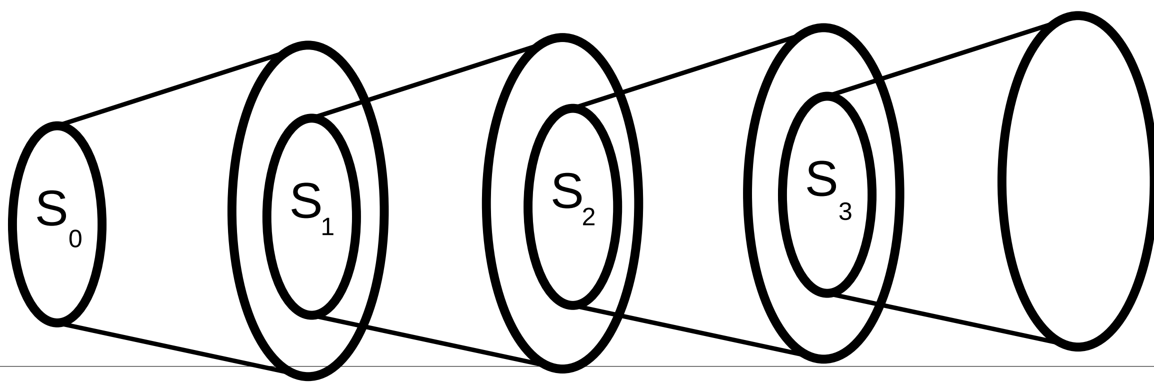
- Choose $S_0$ with $\mathrm{vol}(S_0) = \Theta(\sigma^2 \mathrm{vol}(G))$.
- Grow the neighborhood of $S_0$ so that the second volume of exposed vertices is about $x\sigma \mathrm{vol}_2(G)$. By the admissible condition, its volume reaches $\frac{1}{C}x^2\sigma^2\mathrm{vol}(S)$.

# Main ideas

- Choose $S_0$ with $\mathrm{vol}(S_0) = \Theta(\sigma^2 \mathrm{vol}(G))$.
- Grow the neighborhood of $S_0$ so that the second volume of exposed vertices is about $x\sigma\mathrm{vol}_2(G)$. By the admissible condition, its volume reaches $\frac{1}{C}x^2\sigma^2\mathrm{vol}(S)$.
- Pick up vertices in big components with volume about $\Theta(\sigma^2\mathrm{vol}(G))$ and call this set $S_1$.

# Main ideas

- Choose $S_0$ with $\mathrm{vol}(S_0) = \Theta(\sigma^2 \mathrm{vol}(G))$.
- Grow the neighborhood of $S_0$ so that the second volume of exposed vertices is about $x\sigma \mathrm{vol}_2(G)$. By the admissible condition, its volume reaches $\frac{1}{C} x^2 \sigma^2 \mathrm{vol}(S)$.
- Pick up vertices in big components with volume about $\Theta(\sigma^2 \mathrm{vol}(G))$ and call this set $S_1$.
- Repeat this process until we we find the giant connected component.



# Analysis

Let $f(S)$ be the number of connected pieces in $S$. We have

$$\begin{aligned} f(S_0) &\leq \frac{\sigma^2}{\sqrt{\epsilon}} n \\ f(S_1) &\leq f(S_0) C x^{-2} \\ f(S_2) &\leq f(S_1) C x^{-2} \\ &\vdots \end{aligned}$$

After at most $t = \frac{\log n}{\log(x^2/C)}$ step, $S_t$ is connected

$$\mathrm{vol}(S_t) = \Theta(\sigma^2 \mathrm{vol}(G)).$$

# Continue

The unexposed vertices have second volume at least

$$(1 - tx\sigma)\mathrm{vol}_2(G) > (1 - \delta)\mathrm{vol}_2(G).$$

Here we assume $\sigma = o(\frac{1}{\log n})$.

Apply the neighborhood expansion lemma once again. We find the giant connected component. $\square$

# Summary

- If $p < \frac{1}{\mu}$, almost surely $G_p$ has no giant component.
- Suppose $p \geq \frac{1+c}{\tilde{d}}$ for some $c \leq \frac{1}{20}$. Suppose $G$ satisfies $\Delta = o(\frac{\tilde{d}}{\sigma})$, $\Delta = o(\frac{d\sqrt{n}}{\log n})$ and $\sigma = o(n^{-\kappa})$ for some $\kappa > 0$, and $G$ is $(\frac{c\kappa}{10}, M)$-admissible. Then almost surely $G_p$ has giant components.
- If $\Delta = O(d)$ and $\sigma = o(\frac{1}{\log n})$, then $p_c \approx \frac{1}{\mu}$.

# Summary

- If $p < \frac{1}{\mu}$, almost surely $G_p$ has no giant component.
- Suppose $p \geq \frac{1+c}{\tilde{d}}$ for some $c \leq \frac{1}{20}$. Suppose $G$ satisfies $\Delta = o(\frac{\tilde{d}}{\sigma})$, $\Delta = o(\frac{d\sqrt{n}}{\log n})$ and $\sigma = o(n^{-\kappa})$ for some $\kappa > 0$, and $G$ is $(\frac{c\kappa}{10}, M)$-admissible. Then almost surely $G_p$ has giant components.
- If $\Delta = O(d)$ and $\sigma = o(\frac{1}{\log n})$, then $p_c \approx \frac{1}{\mu}$.

**Question:** For any dense graph $G$, is it true that there exists a dense subgraph $H$ with $|V(H)| > \epsilon |V(G)|$ and $\sigma_H < \epsilon$?

# Part II: Hypergraphs

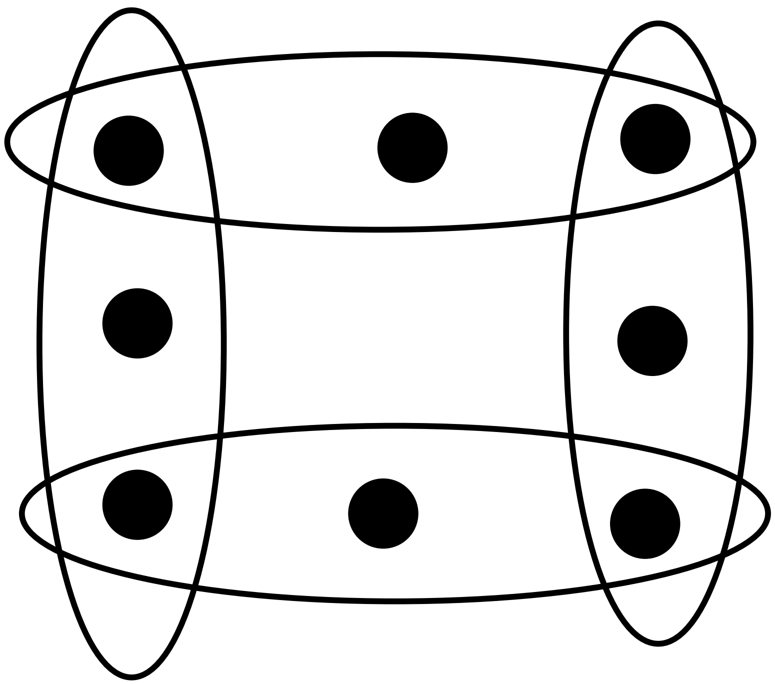
$H = (V, E)$ is an $r$-uniform hypergraph ($r$-graph, for short).

- $V$: the set of vertices
- $E$: the set of edges, each edge has cardinality $r$.

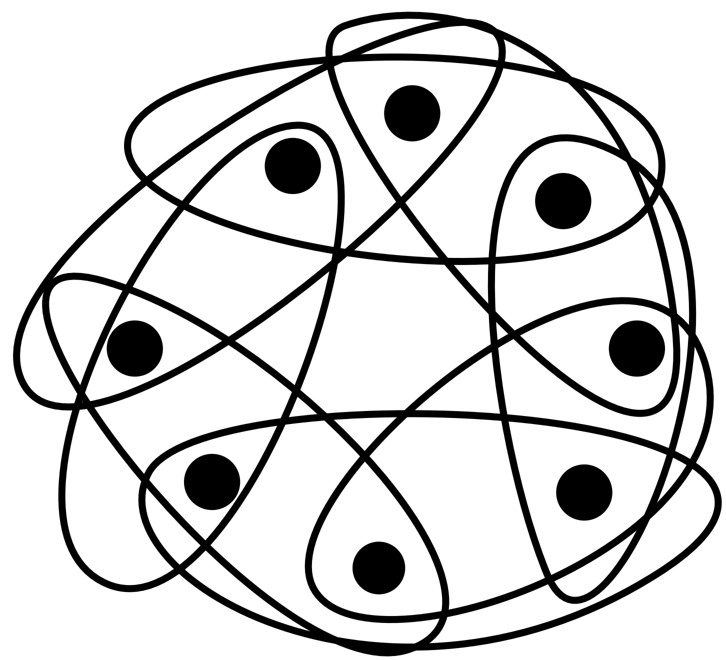
# Part II: Hypergraphs

$H = (V, E)$ is an $r$-uniform hypergraph ($r$-graph, for short).

- $V$: the set of vertices
- $E$: the set of edges, each edge has cardinality $r$.



A 3-uniform loose cycle



A 3-uniform tight cycle

# Ongoing projects

- How to define “connected components” for hypergraphs?

# Ongoing projects

- How to define “connected components” for hypergraphs?
- How to define “Laplacians” for hypergraphs?

# Ongoing projects

- How to define "connected components" for hypergraphs?
- How to define "Laplacians" for hypergraphs?
- Phase transition for random hypergraphs?

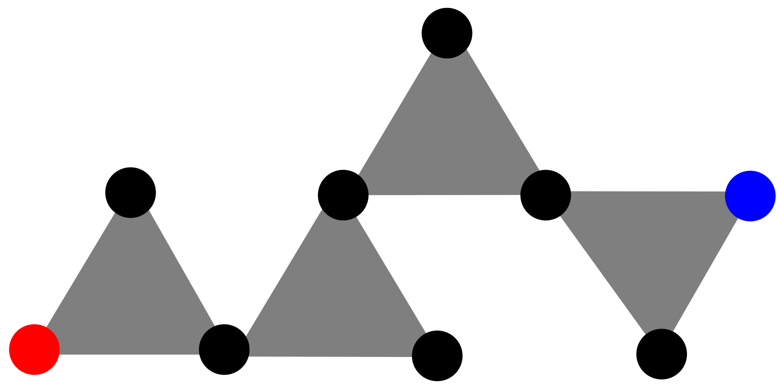
# Ongoing projects

- How to define “connected components” for hypergraphs?
- How to define “Laplacians” for hypergraphs?
- Phase transition for random hypergraphs?
- Percolation on hypergraphs?

The remaining talk will focus on Question 1 and 2.

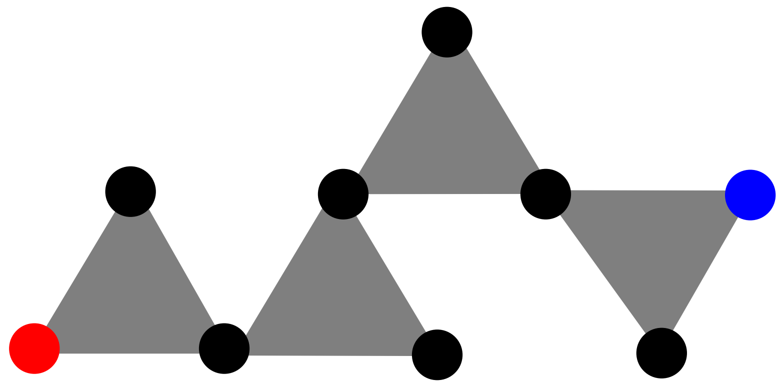
# Connectivities in $3$-graphs

- Vertex to Vertex

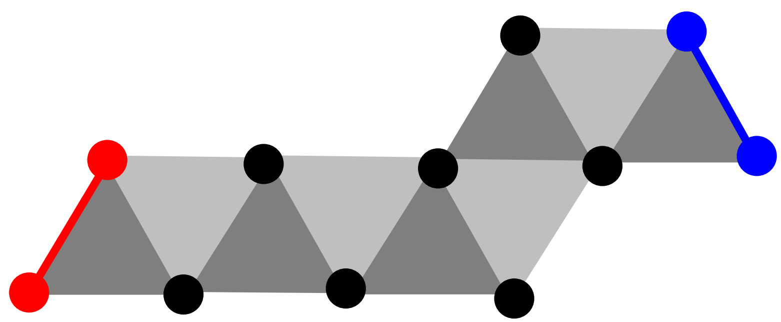


# Connectivities in $3$-graphs

- Vertex to Vertex



- Pair to Pair



# $s$-walks on hypergraphs

For $1 \le s \le r-1$, an $s$-walk on $H$ consists of

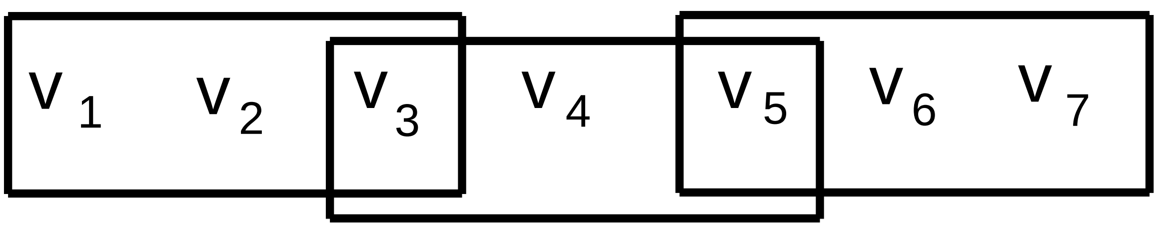
- a vertex sequence: $v_1, v_2, \ldots, v_{(k-1)(r-s)+r}$
- an edge sequence: $F_1, F_2, \ldots, F_k$ satisfying $F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\}$ for $1 \le i \le k$.

# $s$-walks on hypergraphs

For $1 \le s \le r-1$, an $s$-walk on $H$ consists of

- a vertex sequence: $v_1, v_2, \ldots, v_{(k-1)(r-s)+r}$
- an edge sequence: $F_1, F_2, \ldots, F_k$ satisfying $F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\}$ for $1 \le i \le k$.

$$|F_i \cap F_{i+1}| = s$$



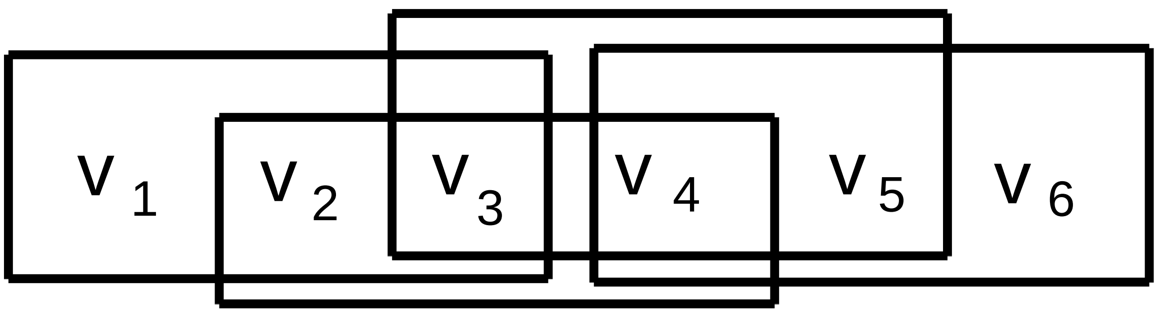

A 1-walk in a 3-graph

# $s$-walks on hypergraphs

For $1 \leq s \leq r-1$, an $s$-walk on $H$ consists of

- a vertex sequence: $v_1, v_2, \ldots, v_{(k-1)(r-s)+r}$
- an edge sequence: $F_1, F_2, \ldots, F_k$ satisfying $F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\}$ for $1 \leq i \leq k$.

$$|F_i \cap F_{i+1}| = s$$



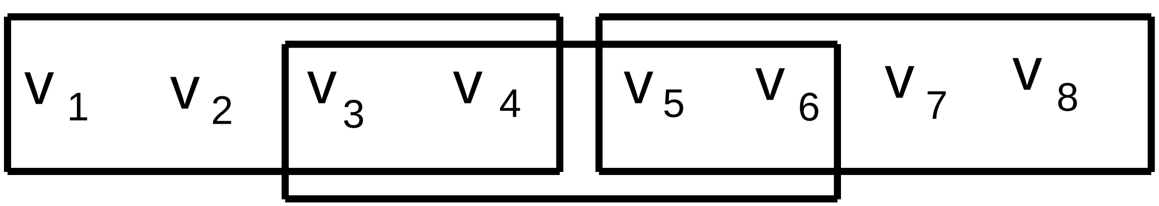

A 2-walk in a $3$-graph

# $s$-walks on hypergraphs

For $1 \leq s \leq r-1$, an $s$-walk on $H$ consists of

- a vertex sequence: $v_1, v_2, \ldots, v_{(k-1)(r-s)+r}$
- an edge sequence: $F_1, F_2, \ldots, F_k$ satisfying $F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\}$ for $1 \leq i \leq k$.

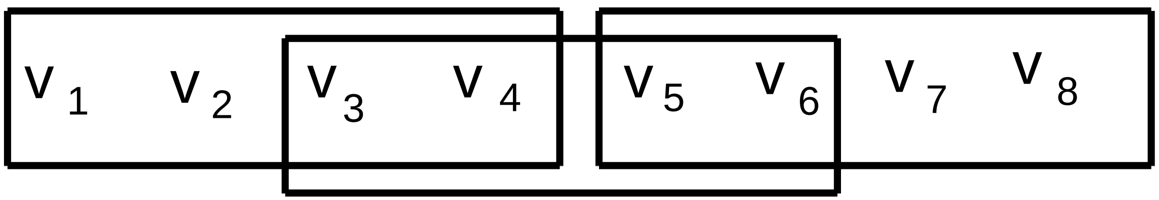
$$|F_i \cap F_{i+1}| = s$$




A 2-walk in a $4$-graph

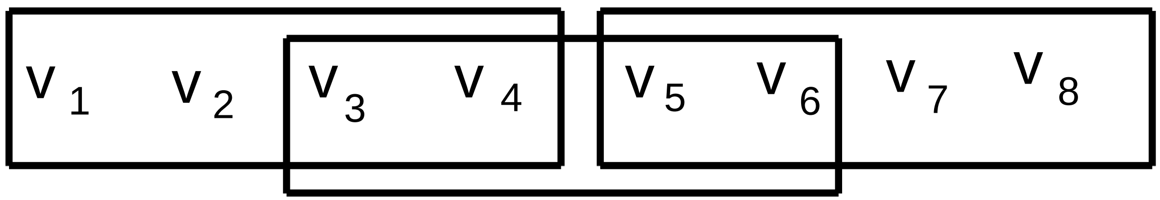
# Loose random walks

**Loose walk:** $1 \leq s \leq \frac{r}{2}$.



# Loose random walks

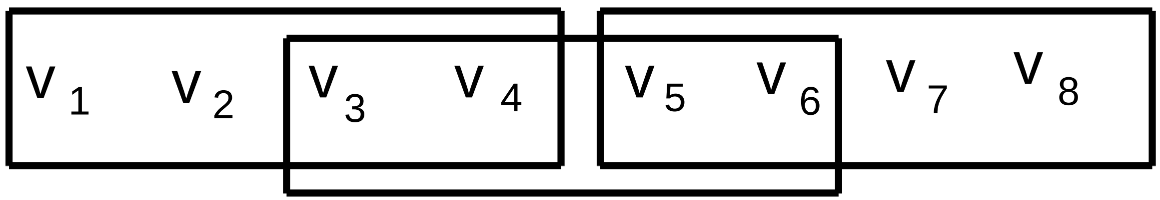
**Loose walk:** $1 \le s \le \frac{r}{2}$.




Observation: an $s$-th random walk on $H$ is essentially a random walk on an auxiliary weighted graph $G^{(s)}$.

# Loose random walks

**Loose walk:** $1 \leq s \leq \frac{r}{2}$.




Observation: an $s$-th random walk on $H$ is essentially a random walk on an auxiliary weighted graph $G^{(s)}$.

- Vertex set $V(G^s) = \binom{V}{s}$
- Weight function $w$: $\binom{V}{s} \times \binom{V}{s} \to \mathbb{Z}$:

$$w(S,T) = \begin{cases} 0 & \text{if } S \cap T \neq \emptyset \\ d_{[S]\cup[T]} & \text{if } S \cap T = \emptyset. \end{cases}$$

# Laplacians of hypergraph (I)

For $1 \leq s \leq r/2$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $G^{(s)}$.

# Laplacians of hypergraph (I)

For $1 \le s \le r/2$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $G^{(s)}$.

- $\lambda_1^{(s)}$: the smallest non-trivial eigenvalue of $\mathcal{L}^{(s)}$.

# Laplacians of hypergraph (I)

For $1 \leq s \leq r/2$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $G^{(s)}$.

- $\lambda_1^{(s)}$: the smallest non-trivial eigenvalue of $\mathcal{L}^{(s)}$.
- $\lambda_{max}^{(s)}$: the largest eigenvalue of $\mathcal{L}^{(s)}$.

# Laplacians of hypergraph (I)

For $1 \leq s \leq r/2$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $G^{(s)}$.

- $\lambda_1^{(s)}$: the smallest non-trivial eigenvalue of $\mathcal{L}^{(s)}$.
- $\lambda_{max}^{(s)}$: the largest eigenvalue of $\mathcal{L}^{(s)}$.
- $\bar{\lambda}^{(s)}$: the spectral bound $\max\{|1-\lambda_1^{(s)}|, |\lambda_{max}^{(s)}-1|\}$.

# Laplacians of hypergraph (I)

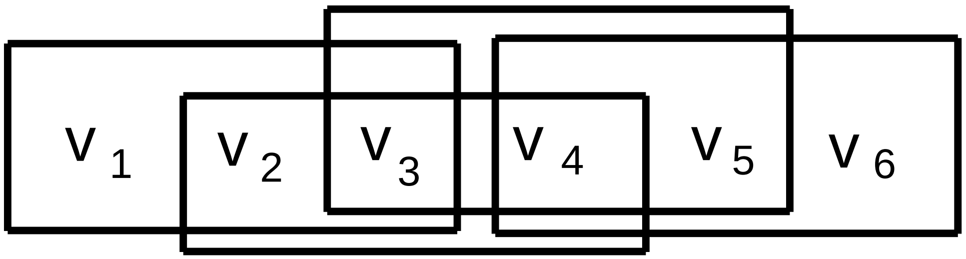
For $1 \leq s \leq r/2$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $G^{(s)}$.

- $\lambda_1^{(s)}$: the smallest non-trivial eigenvalue of $\mathcal{L}^{(s)}$.
- $\lambda_{max}^{(s)}$: the largest eigenvalue of $\mathcal{L}^{(s)}$.
- $\bar{\lambda}^{(s)}$: the spectral bound $\max\{|1-\lambda_1^{(s)}|, |\lambda_{max}^{(s)} - 1|\}$.

$\mathcal{L}^{(1)}$ is the same as the Laplacian of hypergraph introduced by **Rodríguez [2009]**.

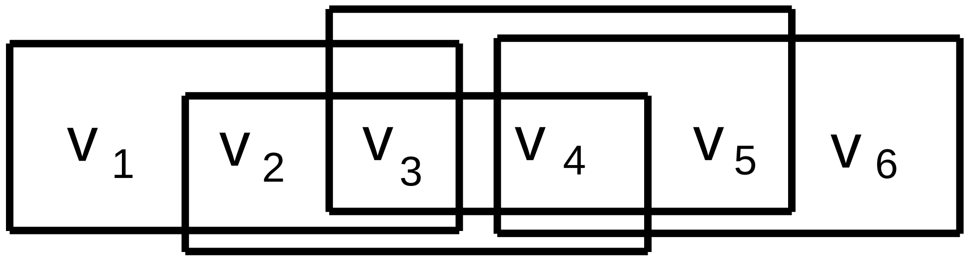
# Tight random walks

**Tight walk:** $\frac{r}{2} < s \leq r - 1.$



# Tight random walks

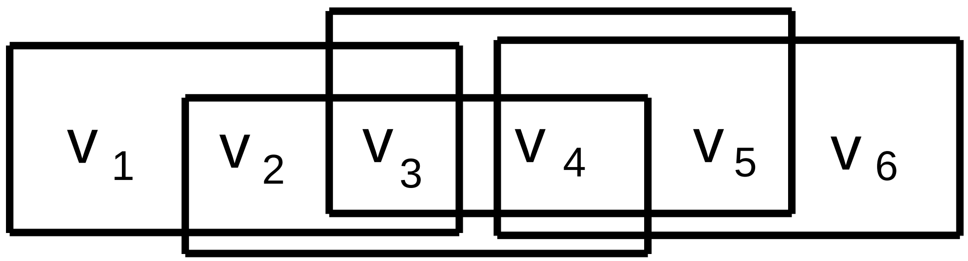
**Tight walk:** $\frac{r}{2} < s \leq r - 1.$




Observation: an $s$-th random walk on $H$ is "essentially" a random walk on an auxiliary directed graph $D^{(s)}$.

# Tight random walks

**Tight walk:** $\frac{r}{2} < s \leq r - 1$.




Observation: an $s$-th random walk on $H$ is "essentially" a random walk on an auxiliary directed graph $D^{(s)}$.

- Vertex set $V(G^{(s)}) = \mathrm{V}^{\underline{s}}$
- For $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$, $xy$ is a directed edge if
  - $x_{r-s+j} = y_j$ for $1 \leq j \leq 2s - r$.
  - $\{x_1, \ldots, x_s, y_{2s-r+1}, y_s\}$ is an edge of $H$.

# Laplacians of hypergraph (II)

For $r/2 < s \leq r-1$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $D^{(s)}$.

# Laplacians of hypergraph (II)

For $r/2 < s \leq r-1$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $D^{(s)}$.

- $D^{(s)}$ is Eulerian, i.e., indegree=outdegree at any vertex.

# Laplacians of hypergraph (II)

For $r/2 < s \leq r-1$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $D^{(s)}$.

- $D^{(s)}$ is Eulerian, i.e., indegree=outdegree at any vertex.
- **Chung [2005]** defined the Laplacian of directed graphs. Let $\vec{\mathcal{L}} = T^{-1/2}AT^{-1/2}$. Define the Laplacian

$$\mathcal{L}^{(s)} = \frac{\mathcal{L} + \mathcal{L}'}{2}.$$

# Laplacians of hypergraph (II)

For $r/2 < s \le r-1$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $D^{(s)}$.

- $D^{(s)}$ is Eulerian, i.e., indegree=outdegree at any vertex.
- **Chung [2005]** defined the Laplacian of directed graphs. Let $\vec{\mathcal{L}} = T^{-1/2}AT^{-1/2}$. Define the Laplacian

$$\mathcal{L}^{(s)} = \frac{\mathcal{L} + \mathcal{L}'}{2}.$$

- $\mathcal{L}^{(r-1)}$ is close related to the Laplacian of a regular hypergraph introduced by **Chung [1993]**.

# Laplacians of hypergraph (II)

For $r/2 < s \leq r-1$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $D^{(s)}$.

- $D^{(s)}$ is Eulerian, i.e., indegree=outdegree at any vertex.
- **Chung [2005]** defined the Laplacian of directed graphs. Let $\vec{\mathcal{L}} = T^{-1/2} A T^{-1/2}$. Define the Laplacian

$$\mathcal{L}^{(s)} = \frac{\mathcal{L} + \mathcal{L}'}{2}.$$

- $\mathcal{L}^{(r-1)}$ is close related to the Laplacian of a regular hypergraph introduced by **Chung [1993]**.

$\lambda_1^{(s)}$, $\lambda_{max}^{(s)}$, and $\bar{\lambda}^{(s)}$ are defined in the same way.

# Application: Edge expansion

- $S \subseteq \binom{V}{s}$, $T \subseteq \binom{V}{t}$.
- $E(S,T)$: the set of edges containing $x \cup y$ for some $x \in S$, $y \in T$ with $x \cap y = \emptyset$.
- $e(S) := \frac{\mathrm{vol}(S)}{\mathrm{vol}(\binom{V}{s})}$, $e(T) := \frac{\mathrm{vol}(T)}{\mathrm{vol}(\binom{V}{t})}$.
- $e(S,T) := \frac{|E(S,T)|}{|E(\binom{V}{s},\binom{V}{t})|}$.

# Application: Edge expansion

- $S \subseteq \binom{V}{s}$, $T \subseteq \binom{V}{t}$.
- $E(S,T)$: the set of edges containing $x \cup y$ for some $x \in S$, $y \in T$ with $x \cap y = \emptyset$.
- $e(S) := \frac{\mathrm{vol}(S)}{\mathrm{vol}(\binom{V}{s})}$, $e(T) := \frac{\mathrm{vol}(T)}{\mathrm{vol}(\binom{V}{t})}$.
- $e(S,T) := \frac{|E(S,T)|}{|E(\binom{V}{s},\binom{V}{t})|}$.

**Theorem [Lu-Peng 2011]:** If $1 \le t \le s \le r/2$, then

$$|e(S,T) - e(S)e(T)| \le \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$

# Connections of different $\mathcal{L}^{(s)}$

**Theorem [Lu, Peng 2011]** We have the following inequalities for the “loose” Laplacian eigenvalues.

$$\lambda_1^{(1)} \geq \lambda_1^{(2)} \geq \ldots \geq \lambda_1^{(\lfloor r/2 \rfloor)};$$
$$\lambda_{\max}^{(1)} \leq \lambda_{\max}^{(2)} \leq \ldots \leq \lambda_{\max}^{(\lfloor r/2 \rfloor)}.$$

# Complete hypergraph $K_n^r$

**Theorem:** For $1 \leq s \leq r/2$, the $s$-th Laplacian eigenvalues of $K_n^r$ is the eigenvalues of $s$-th Laplacian of $K_n^r$ are given by

$$1 - \frac{(-1)^i \binom{n-s-i}{s-i}}{\binom{n-s}{s}} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1}$$

for $0 \leq i \leq s$.

# Complete hypergraph $K_n^r$

**Theorem:** For $1 \le s \le r/2$, the $s$-th Laplacian eigenvalues of $K_n^r$ is the eigenvalues of $s$-th Laplacian of $K_n^r$ are given by

$$1 - \frac{(-1)^i \binom{n-s-i}{s-i}}{\binom{n-s}{s}} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1}$$

for $0 \le i \le s$.

**Observation:** $G^{(s)'}$ is essentially the Kneser graph $K(n,s)$. The vertices are $s$-sets in $[n]$. A pair of $s$-sets $S$ and $T$ forms an edge of $S \cap T = \emptyset$.

# Laplacians of $H^r(n,p)$

Random $r$-uniform random hypergraph $H^r(n,p)$:

- $n$: the number of vertices.
- $p$: the probability; each $r$-set is an edge with probability $p$ independently.

$\{\lambda_k^{(s)}(H)\}$: the $s$-the Laplacian spectrum of $H$.

# Laplacians of $H^r(n,p)$

Random $r$-uniform random hypergraph $H^r(n,p)$:

- $n$: the number of vertices.
- $p$: the probability; each $r$-set is an edge with probability $p$ independently.

$\{\lambda_k^{(s)}(H)\}$: the $s$-the Laplacian spectrum of $H$.

**Theorem [Lu and Peng 2011+]**: For $1 \leq s \leq r/2$, if $p(1-p) \gg \frac{\log^4 n}{\binom{n}{s}}$ and $1-p \gg \frac{\log^2 n}{n^2}$, the almost surely for $0 \leq k \leq \binom{n}{s} - 1$,
$|\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^r)| \leq (3+o(1))\sqrt{\frac{1-p}{\binom{n-s}{r-s}p}}$.

# Semicircle Law

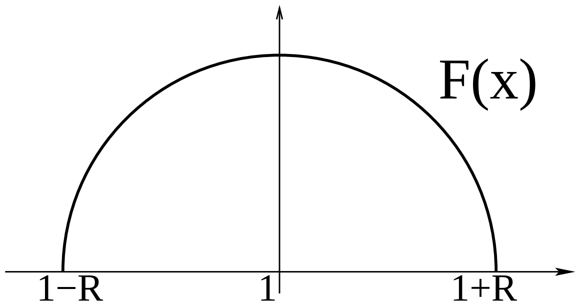
- $F_n(x)$: $\frac{1}{\binom{n}{s}}$ of the number of $s$-th Laplacian of $H^r(n,p)$ less than $x$.
- $R := (2 + o(1))\sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s}p}}$.

# Semicircle Law

- $F_n(x)$: $\frac{1}{\binom{n}{s}}$ of the number of $s$-th Laplacian of $H^r(n,p)$ less than $x$.
- $R := (2+o(1))\sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s}p}}$.

**Theorem [Lu and Peng 2011+]**:
For $1 \leq s \leq r/2$, if $p(1-p) \gg \frac{\log n}{n^{r-s}}$, then almost surely

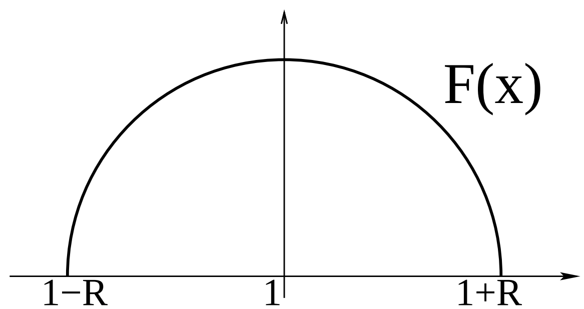
$$F_n(x) \xrightarrow{p} F(x).$$



# Semicircle Law

- $F_n(x)$: $\frac{1}{\binom{n}{s}}$ of the number of $s$-th Laplacian of $H^r(n,p)$ less than $x$.
- $R := (2+o(1))\sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s}p}}$.

**Theorem [Lu and Peng 2011+]**:
For $1 \leq s \leq r/2$, if $p(1-p) \gg \frac{\log n}{n^{r-s}}$, then almost surely




$$F_n(x) \xrightarrow{p} F(x).$$

Previously known similar results:

- on $G(n,p)$ **[Füredi-Komlós 1981]**.
- on $G(w_1,\ldots,w_n)$ **Chung-Lu-Vu 2002**.

# Phase transition of $H^r(n,p)$

- Vertex to vertex :
  **Karoński-Luczak (2002)** determines the threshold

$$p_c \approx \frac{1}{(r-1)\binom{n-1}{r-1}}.$$

# Phase transition of $H^r(n,p)$

- Vertex to vertex :
  **Karoński-Luczak (2002)** determines the threshold

$$p_c \approx \frac{1}{(r-1)\binom{n-1}{r-1}}.$$

- Pair to pair? In general, $s$-tuple to $s$-tuple? (Ongoing project with Peng.)

Thank you.