Coloring Non-Uniform Hypergraphs Red and Blue

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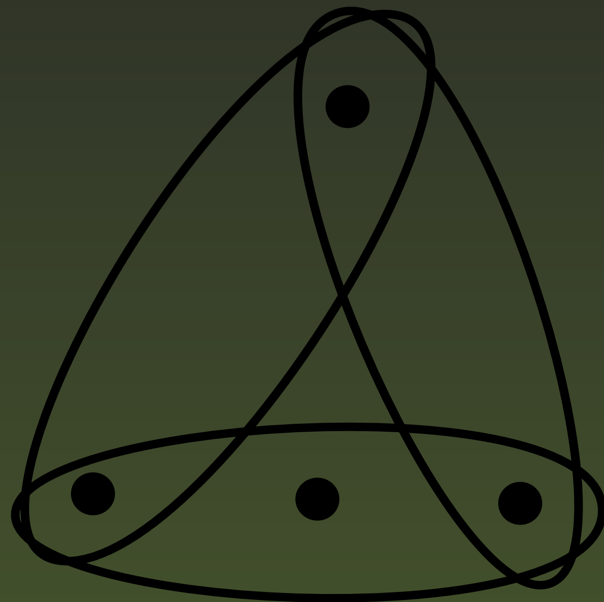
Hypergraphs

Hypergraph H:

- $V(H)$: the set of vertices.
- $E(H)$: the set of edges.
  (A edge F is a subset of $V(H)$.)



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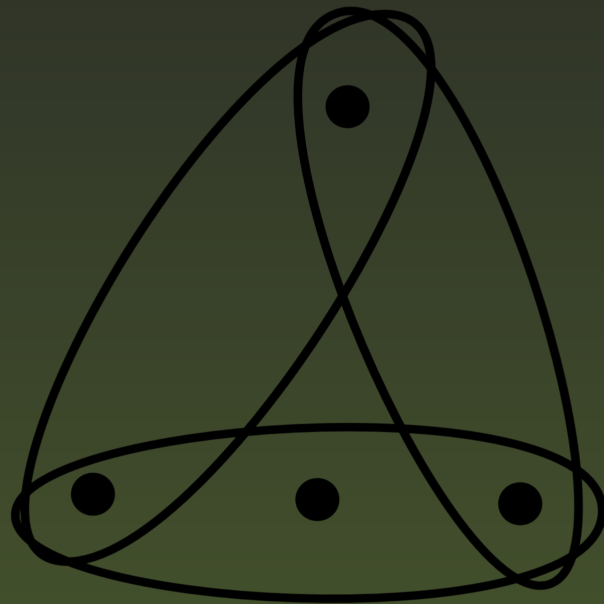

H is r-uniform if $|F| = r$ for every edge F of H.

Property B

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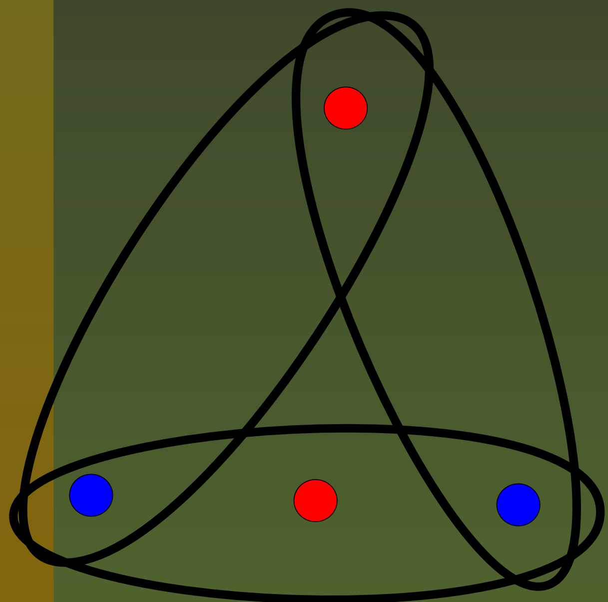

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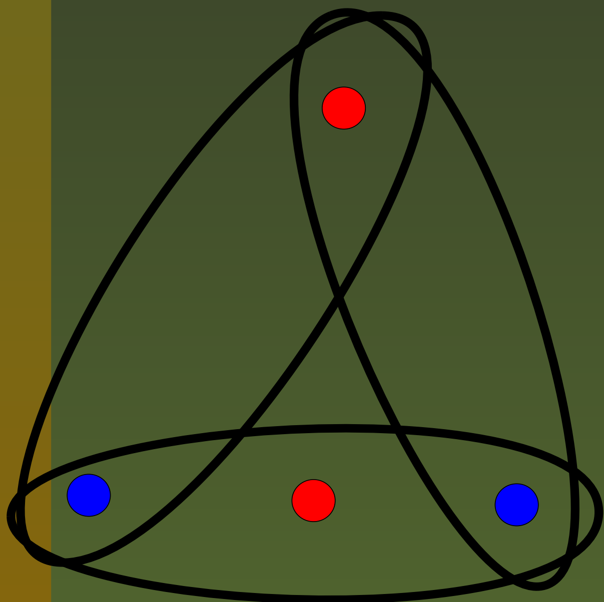

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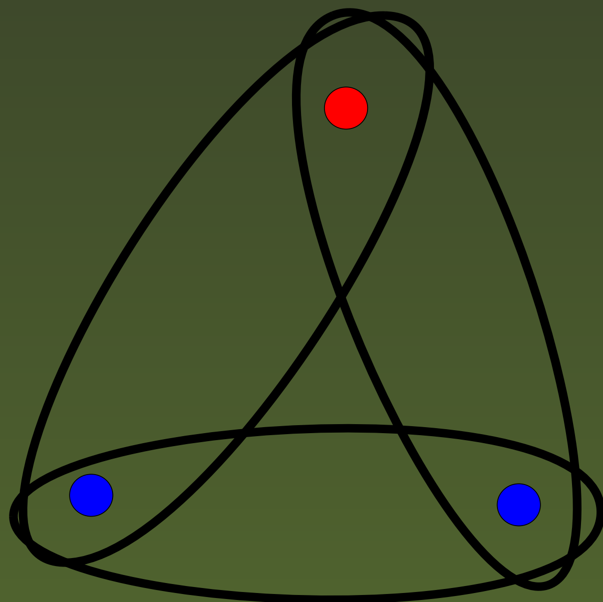

Without Property B

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Property B is first introduced by Miller in 1937.

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Erdős (1963) asked:

"What is the minimum edge number $m_2(r)$ of a r-uniform hypergraph not having property B?"

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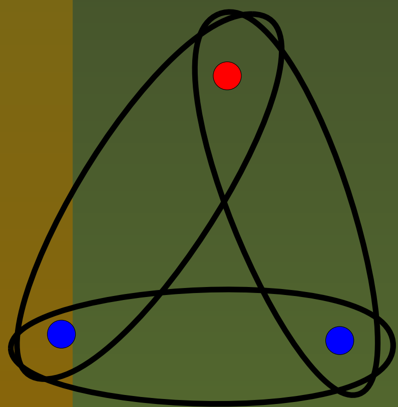

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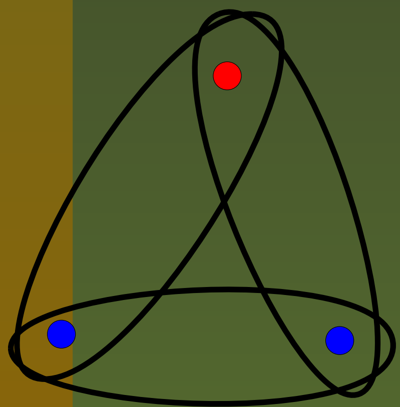
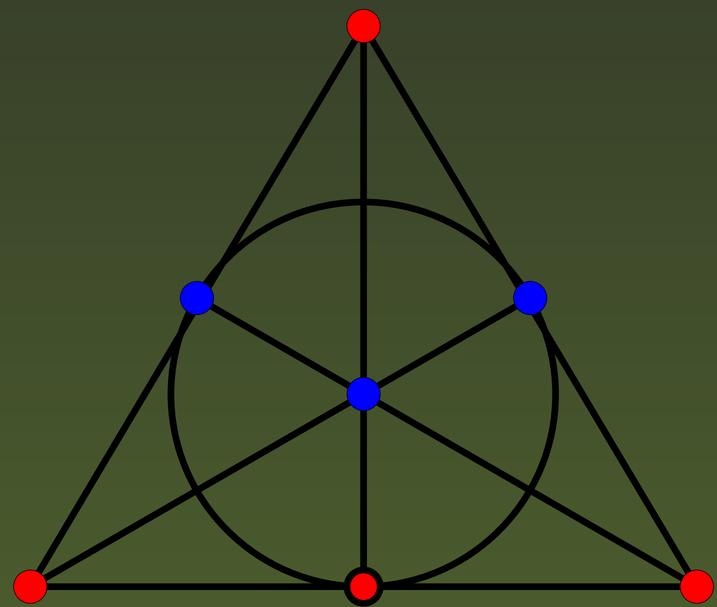


- $m_2(3) = 7$:



Fano plane

Erdős and Lovász (1975)

Perhaps $r2^r$ is the correct order of magnitude of $m_2(r)$; it seems likely that

$$\frac{m_2(r)}{2^r} \to \infty.$$

A stronger conjecture would be: Let $E_{k=1}^m$ be a 3-chromatic (not necessarily uniform) hypergraph. Let

$$f(r) = \min \sum_{k=1}^{m} \frac{1}{2^{|E_k|}},$$

where the minimum is extended over all hypergraphs with $\min |E_k| = r$. We conjecture that $f(r) \to \infty$ as $r \to \infty$.

Previous results

- Erdős (1963)

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- Radhakrishnan and Srinivasan (2000)

$$m_2(r) > (\frac{\sqrt{2}}{2} - o(1))\frac{\sqrt{r}}{\sqrt{\ln r}} 2^r.$$

Non-uniform hypergraphs

Let $g_0(x) = x$, $g_k(x) = \log_2(g_{k-1}(x))$ for $k \geq 1$. For all $x > 0$, let $\log^*(x) = \min\{k \colon g_k(x) \leq 1\}$.

Beck (1978) proved

$$f(r) \geq \frac{\log^*(r) - 100}{7}.$$

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The function $\log^*(x)$ grows very slowly since it is the inverse function of the following tower function of height n

$$n \rightarrow 2^{\overbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}^{n}}.$$

Observation

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- Using probabilistic method, Spencer simplified Beck's proof for the uniform case, but not for the non-uniform case.

Main result

Theorem (Lu) For any $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$, for all $r > r_0$, we have

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An obvious upper bound:

$$f(r) \leq \frac{m_2(r)}{2^r} \leq (1+\epsilon)\frac{2 \ln 2}{4} r^2.$$

Recoloring method

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Observation: With positive probability, the recoloring process destroys all monochromatic edges and does not create any new monochromatic edge.

Type I: a red edge survives.

Let $h = |E(H)|2^{-r}$ be the expected number of red edges.

The probability of this event is

$$|E(H)|2^{-r}(1-p)^r \leq he^{-rp}.$$

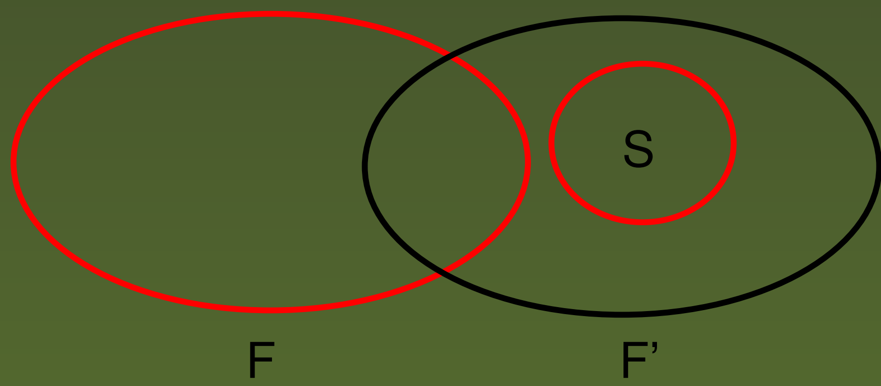
Type II: a blue edge is created.

$$\sum_{i\geq 1}\sum_{|F\cap F'|=i} 2^{-2r+i}\sum_{s\geq 0}\binom{r-i}{s}p^{i+s}$$

$$= 2^{-2r}\sum_{i\geq 1}(2p)^i \sum_{|F\cap F'|=i}(1+p)^{r-i}$$

$$\leq 2^{-2r}(1+p)^r\frac{2p}{1+p}|E(H)|^2$$

$$\leq 2ph^2e^{pr}.$$



Put together

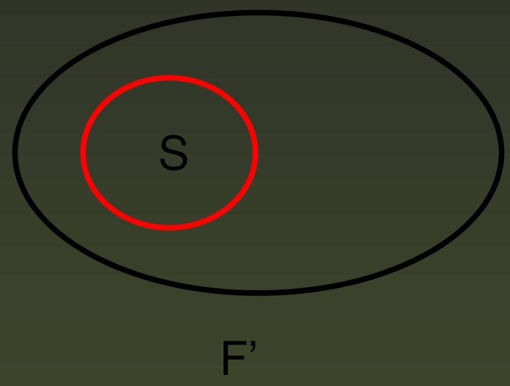
H has Property B if

$$2he^{-rp} + 4ph^2e^{pr} < 1.$$

Choose $h = r^{(1-\epsilon)/3}$ and $p = \frac{(1+\epsilon)\ln h}{r}$. Done!

The difficulty

A critical case:



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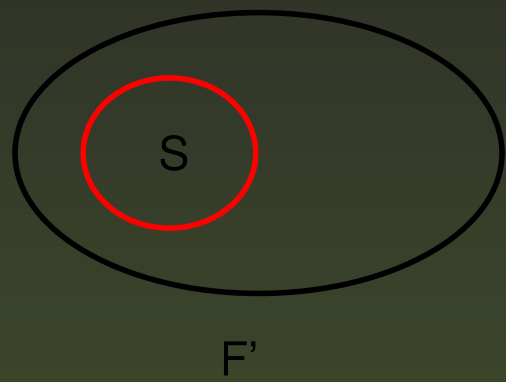

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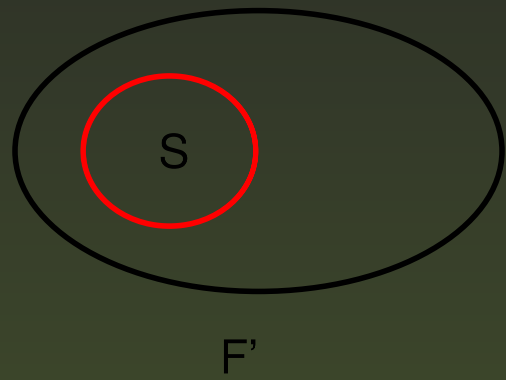

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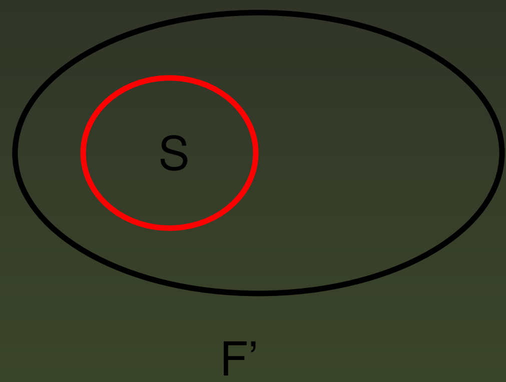

- S is red while $F' \setminus S$ is blue.
- For any $v \in S$, there exists an red edge F containing v.
- The size of F' is unbounded.
- There are too many choices of S.

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- Carefully separate independence relations between random variables.

Twin-hypergraphs

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The twin-hypergraph (H_1, H_2) is said *to have Property B* if there exists a red-blue vertex-coloring satisfying

- H_1 has no red edge.
- H_2 has no blue edge.

Residue twin-hypergraphs

Let C be a coloring of $H = (H_1, H_2)$.
The red residue $R_C(H)$ is a twin-hypergraph (H_1', H_2') satisfying

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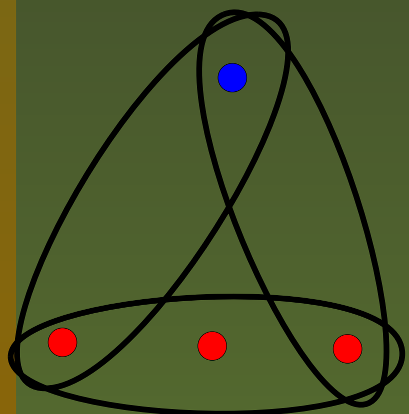

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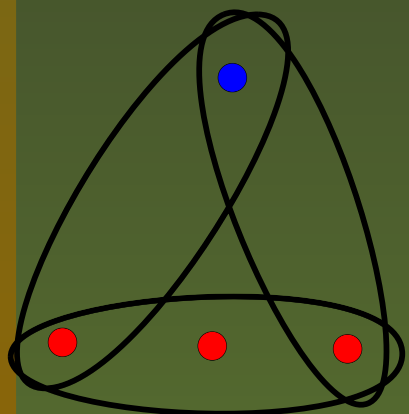

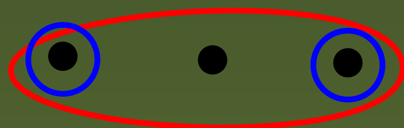

$R_C(H)$

Recoloring Lemma

Blue residue $B_C(H)$ is defined similarly.

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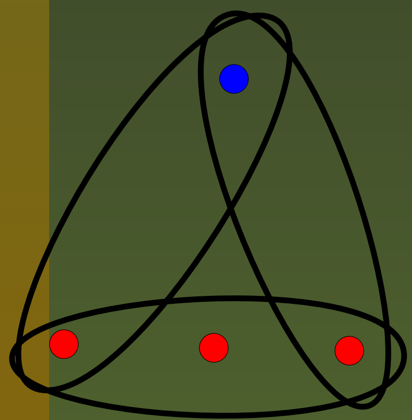

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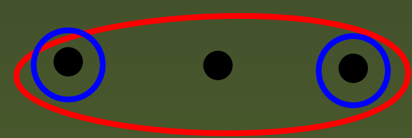

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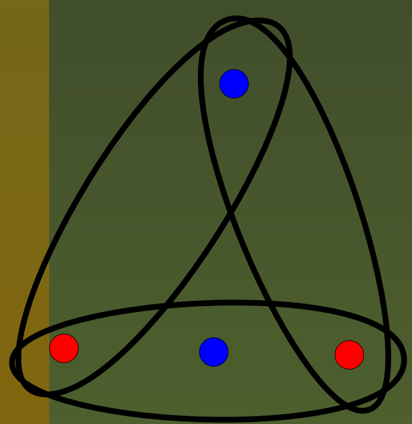

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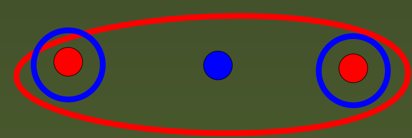

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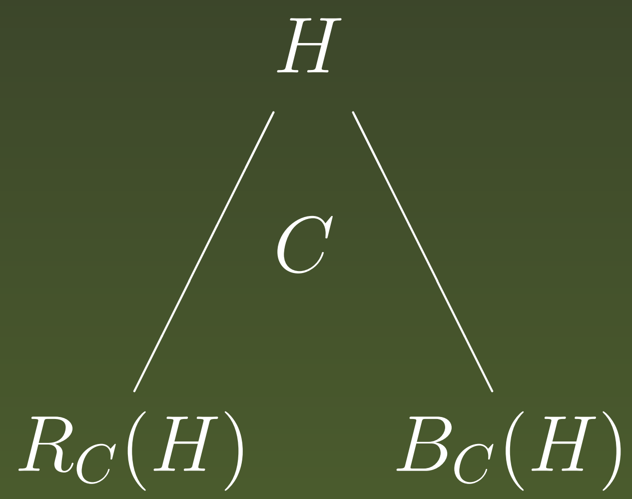
Can not apply it recursively unless one of the residues is empty.



Irreducibility

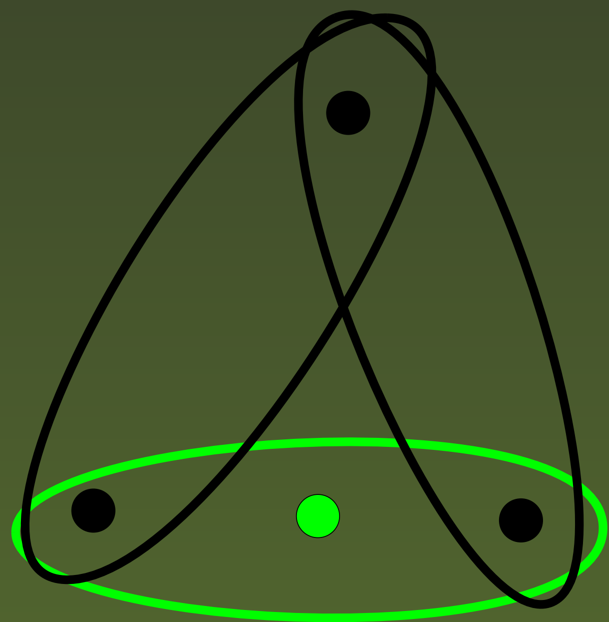
A twin-hypergraph $H = (H_1, H_2)$ is called *irreducible* if

1. $\forall F_1 \in E(H_1)$ and $v \in F_1$, $\exists F_2 \in E(H_2)$ such that $F_1 \cap F_2 = \{v\}$.
2. $\forall F_2 \in E(H_2)$ and $v \in F_2$, $\exists F_1 \in E(H_1)$ such that $F_1 \cap F_2 = \{v\}$.

Reducibility

A twin-hypergraph $H = (H_1, H_2)$ is called *reducible* if there is an evidence (F, v) satisfying

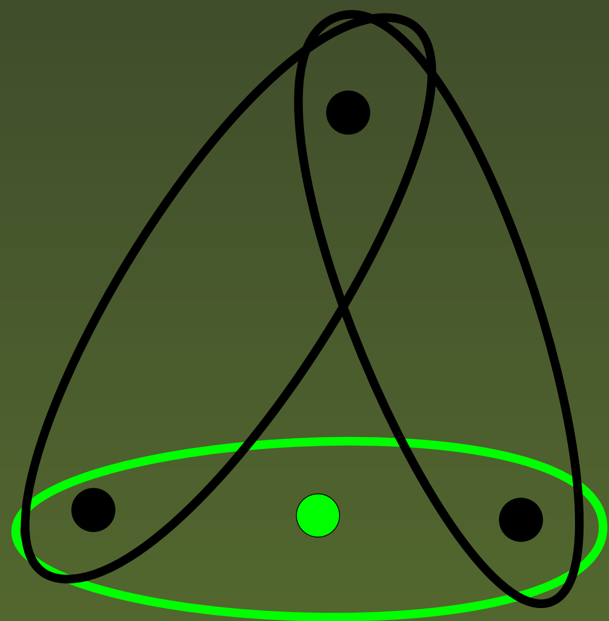
1. $v \in F$, and $F \in E(H_i)$ for $i = 1$ or 2.
2. $\forall F' \in E(H_{3-i})$, if $v \in F'$ then $|F \cap F'| \geq 2$.



Reducing twin-hypergraphs

If H is reducible, there is an evidence (F, v). Removing F from H we get a twin-hypergraph with one edge less. Repeat this process until an irreducible twin-hypergraph is reached.

$$H = H^{(0)} \supset H^{(1)} \supset \cdots \supset H^{(s)}.$$



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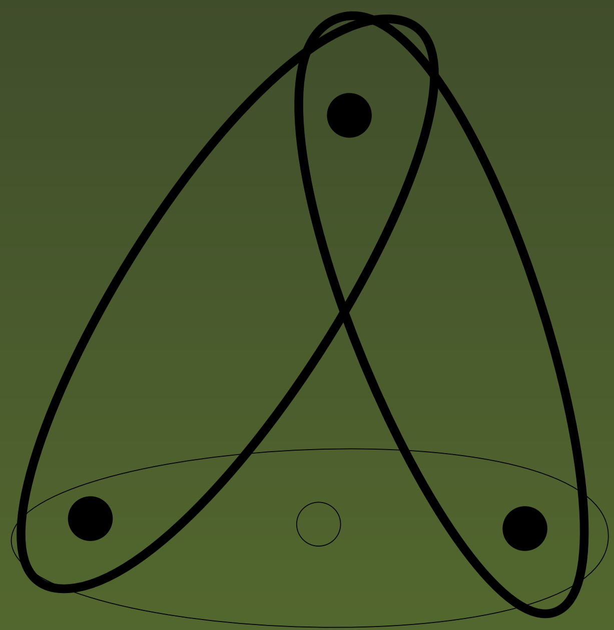

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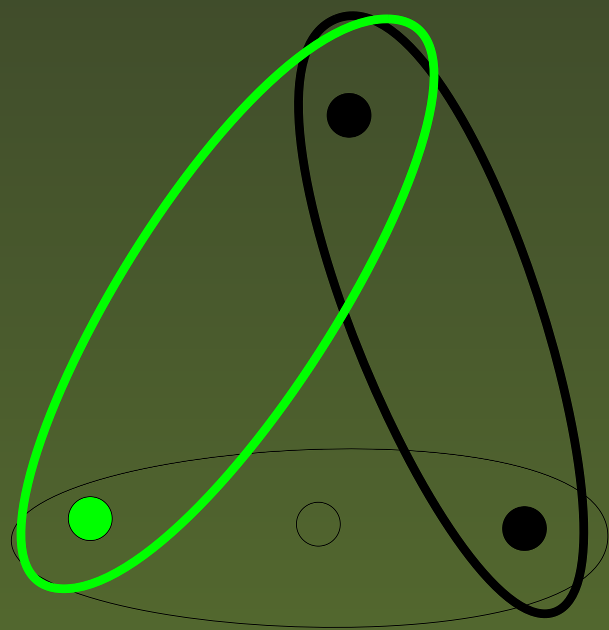

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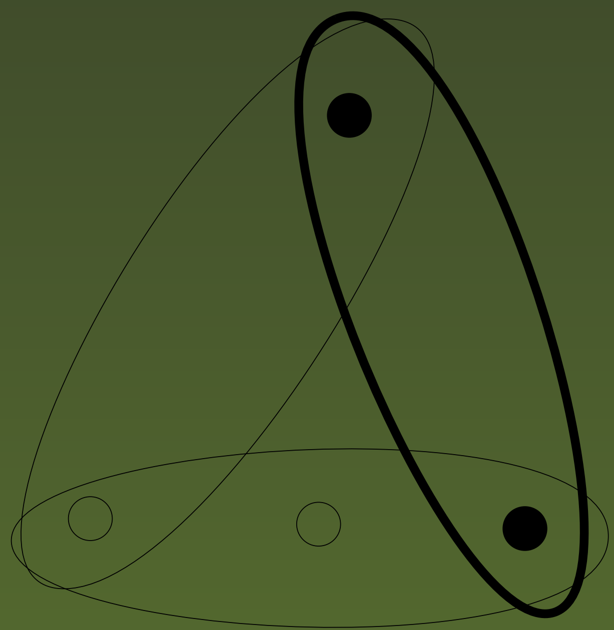

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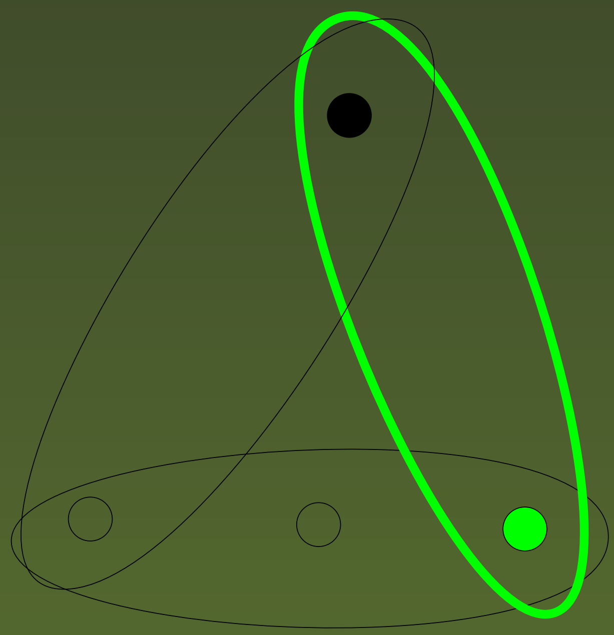

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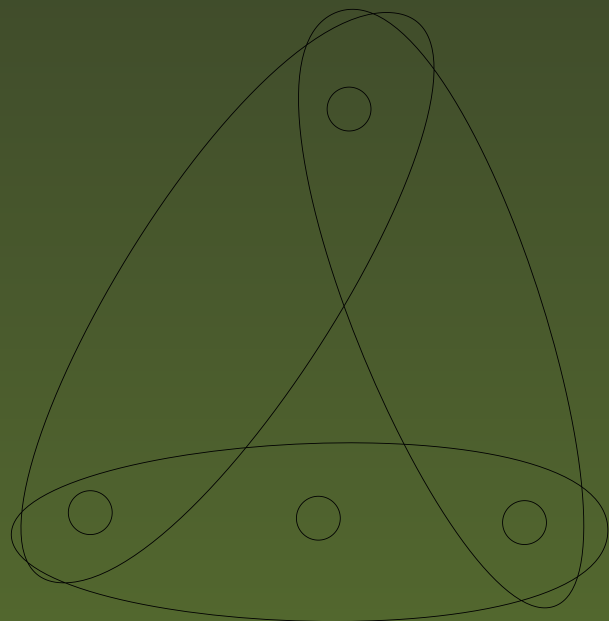

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Such a unique H^s is called the irreducible core of H.

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Proof: It suffices to add a removed edge F back.

- If F is not monochromatic, do nothing.
- Otherwise, flip the color of v. For any F' containing v, F' contains another vertex of F. Thus, F' is not monochromatic.

Randomized testing algorithm

- Let $C\colon V(H) \to red, blue$: (independently)

$$\Pr(C(v) = red) = \frac{1}{2}, \quad \Pr(C(v) = blue) = \frac{1}{2}$$

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Procedure to test $R_C(H)$

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- Suppose a twin-hypergraph $H = (H_1, H_2)$ with minimum edge-cardinality r satisfies

$$h_i \stackrel{def}{=} \sum_{F \in E(H_i)} \frac{1}{2^{|F|}} \leq (\frac{1}{16} - o(1))\frac{\ln r}{\ln \ln r}$$

for $i = 1, 2$. Then the program succeeds with positive probability.

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- $X = \sum_{i \geq r} \frac{X^{(i)}}{i}.$

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Lemma 4 With probability at least $1 - \frac{1}{M}$, we have

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- Markov's inequality.

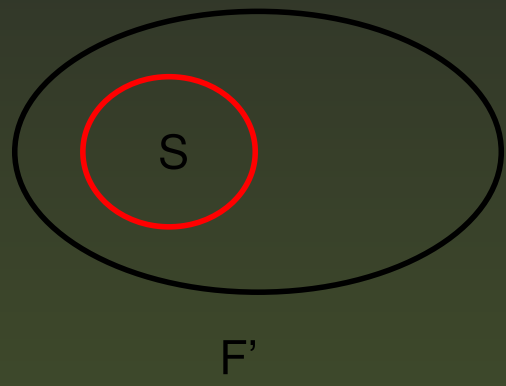
Type I: a red edge survives.

The failure probability of type I event is at most

$$
\begin{aligned}
& \sum_{F\in E(H_1)} \frac{1}{2^{|F|}}\left(1-\frac{q}{r2^{\lfloor \log_2 \frac{|F|}{r}\rfloor}}\right)^{|F|} \\
\leq & \sum_{F\in E(H_1)} \frac{1}{2^{|F|}}\left(1-\frac{q}{|F|}\right)^{|F|} \\
\leq & \sum_{F\in E(H_1)} \frac{1}{2^{|F|}} e^{-q} \\
= & \ h_1 e^{-q}.
\end{aligned}
$$

Type II: a blue edge is created.

- S is red while $F' \setminus S$ is blue in C.



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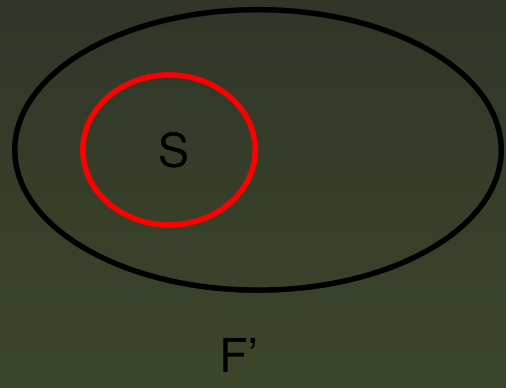
- S is red while $F' \setminus S$ is blue in C.
- For any $v \in S$, $\exists F_v$ such that $F_v \cap F = \{v\}$. Moreover, F_v survives until v is recolored into blue.



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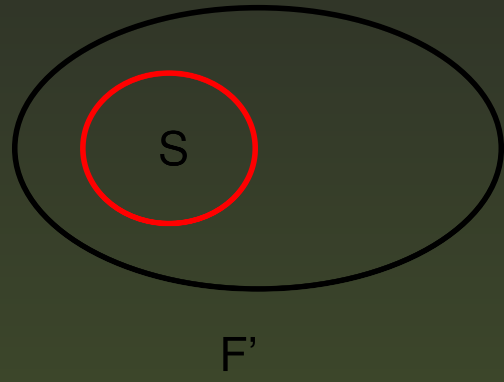

- All vertices in S are changed into blue eventually. Let x be the rank of F_v. For any $v \in S$,

$$\begin{aligned}\Pr(v \text{ is recolored into blue}) &< \sum_{s=x}^{\infty} \frac{q}{r2^{s-1}} \\ &= \frac{4q}{r2^x} < \frac{4q}{|F_v|}.\end{aligned}$$

Random variable Z

Let $\mathcal{F}_v = \{F \mid F \cap F' = \{v\}, F \setminus \{v\} \text{is red}\}$.

$$
\begin{aligned}
Z &\stackrel{def}{=} \sum_{\substack{S \subset F' \\ |S| \geq 1}} \prod_{v \in S} \sum_{F \in \mathcal{F}_v} \frac{4q}{|F|} \\
&= \prod_{v \in F'} (1 + \sum_{F \in \mathcal{F}_v} \frac{4q}{|F|}) - 1 \\
&\leq e^{\sum_{v \in F'} \sum_{F \in \mathcal{F}_v} \frac{4q}{|F|}} - 1 \\
&= e^{4q \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i}} - 1.
\end{aligned}
$$

Upper-bound Z over $A_{F'} = (\sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} \leq 2Mh_1)$

$$
\begin{aligned}
\mathbf{1}_{A_{F'}} Z &= \mathbf{1}_{A_{F'}} \frac{e^{4q \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i}} - 1}{\sum_{i \geq r} \frac{X_{F'}^{(i)}}{i}} \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} \\
&\leq \frac{e^{8Mh_1 q} - 1}{2Mh_1} \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} \\
&\leq \frac{e^{8Mh_1 q}}{2Mh_1} \frac{1}{r} \sum_{i \geq r} X_{F'}^{(i)} \\
&= \frac{e^{8Mh_1 q}}{2Mh_1 r} X_{F'}.
\end{aligned}
$$

Probability of type II event

$$
\begin{aligned}
\sum_{F' \in E(H_2)} \mathrm{E}(\mathbf{1}_A Z)\frac{1}{2^{|F'|}} &\leq \sum_{F' \in E(H_2)} \frac{1}{2^{|F'|}}\mathrm{E}\left(\frac{e^{8Mh_1q}}{2Mh_1r}X_{F'}\right) \\
&= \sum_{F' \in E(H_2)} \frac{1}{2^{|F'|}}\frac{e^{8Mh_1q}}{2Mh_1r}\mathrm{E}(X_{F'}) \\
&\leq h_2\frac{e^{8Mh_1q}}{2Mh_1r}2h_1 \\
&= \frac{h_2e^{8Mh_1q}}{Mr}.
\end{aligned}
$$

Put together

The probability of success is at least

$$1 - \frac{2}{M} - 2he^{-q} - \frac{2he^{8Mhq}}{Mr}.$$

Choose $M = 2(1+\epsilon)$, $q = \ln\ln r$, and $h = \frac{1-\epsilon}{16}\frac{\ln r}{\ln\ln r}$. The above probability is

$$\frac{\epsilon}{1+\epsilon} - \frac{2h}{\ln r} - \frac{2h}{Mr^{\epsilon^2}} > 0$$

for sufficiently large r.

Therefore, H has Property B.

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- Prove of disprove Erdős-Lovász's stronger conjecture $m_2(r) = \Theta(r2^r)$.

The End