Laplacian of Random Hypergraphs

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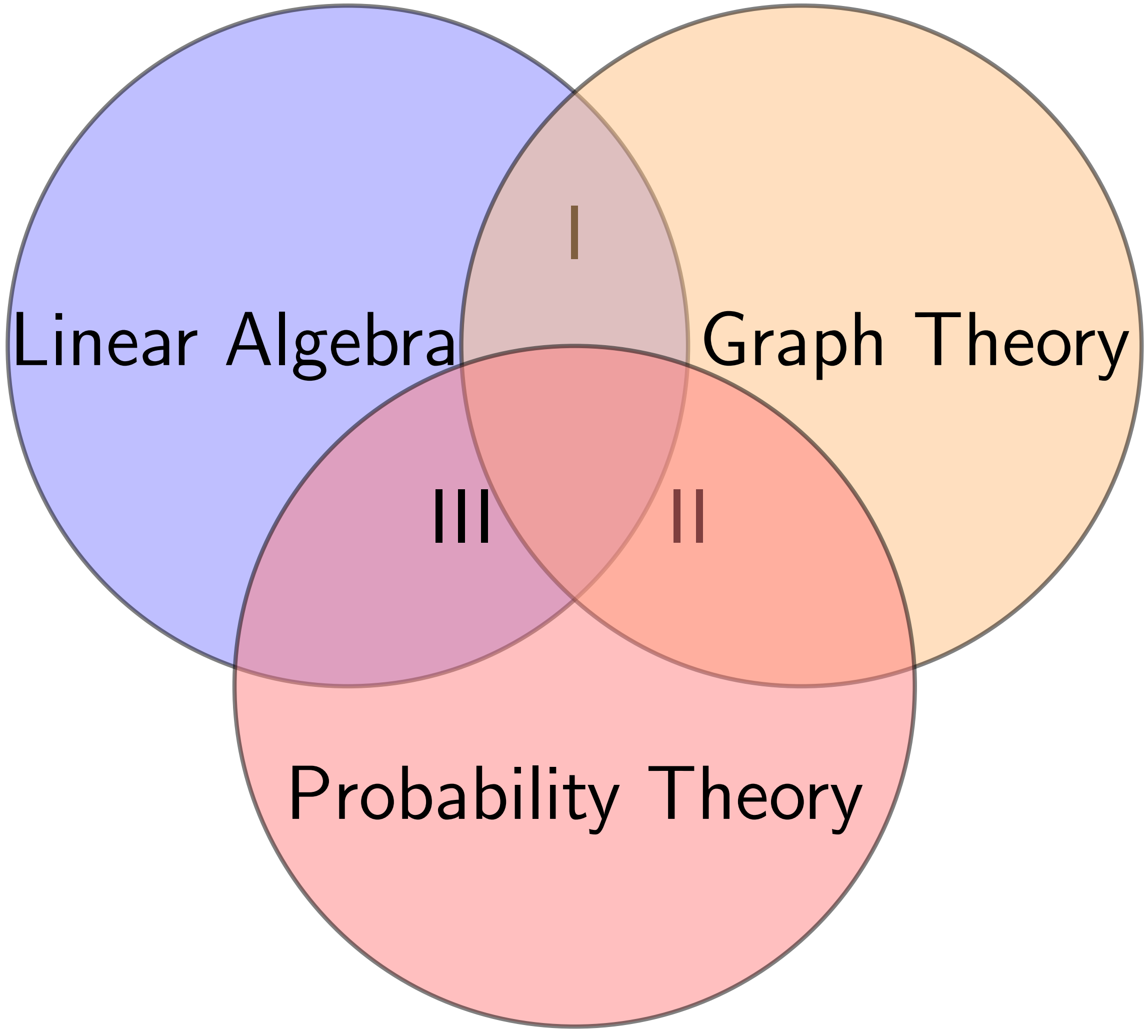
Selected Topics on Spectral Graph Theory (V)
Nankai University, Tianjin, June 12, 2014

Five talks

Selected Topics on Spectral Graph Theory

1. Graphs with Small Spectral Radius
   Time: Friday (May 16) 4pm.-5:30p.m.
2. Laplacian and Random Walks on Graphs
   Time: Thursday (May 22) 4pm.-5:30p.m.
3. Spectra of Random Graphs
   Time: Thursday (May 29) 4pm.-5:30p.m.
4. Hypergraphs with Small Spectral Radius
   Time: Friday (June 6) 4pm.-5:30p.m.
5. Laplacian of Random Hypergraphs
   Time: Thursday (June 12) 4pm.-5:30p.m.

Backgrounds



I: Spectral Graph Theory   II: Random Graph Theory
III: Random Matrix Theory

The Laplacians of graphs

- $G = (V, E)$: a weighted graph; each edge xy is associated with a positive integer weight $w(x, y)$. $(w(x, y) = 0$ if $xy \notin E(G)$.)

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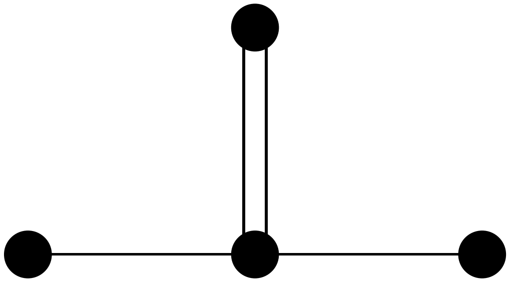
- Laplacian spectrum: $LSP(G) := \{\lambda_0, \ldots, \lambda_{n-1}\}$

$$0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1} \le 2.$$

An example



$$A = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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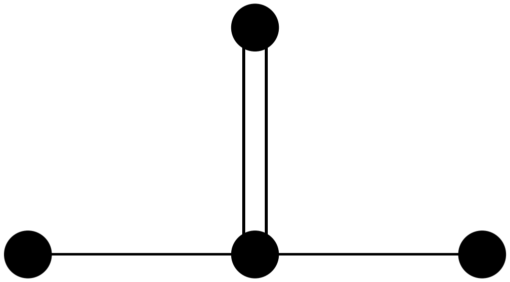

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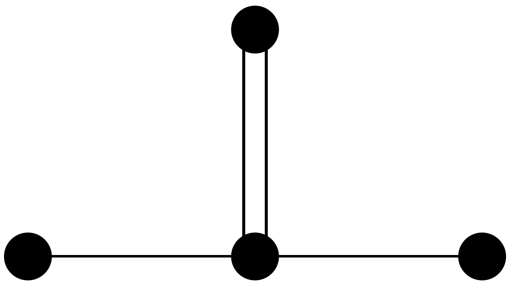

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Laplacian eigenvalues: $\lambda_0 = 0$, $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$

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- Rayleigh quotients :

$$\begin{aligned}\lambda_1 &= \inf_{f \perp T\mathbf{1}} \frac{\sum_{x \sim y} (f(x) - f(y))^2 w(x,y)}{\sum_x f(x)^2 d_x}, \\ \lambda_{n-1} &= \sup_{f \perp T\mathbf{1}} \frac{\sum_{x \sim y} (f(x) - f(y))^2 w(x,y)}{\sum_x f(x)^2 d_x}.\end{aligned}$$

An important parameter

λ_1 is related to

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- many other applications.

Random walks

A walk on a graph is a sequence of vertices together a sequence of edges:

$$v_0, v_1, v_2, v_3, \ldots, v_k, v_{k+1}, \ldots$$

$$v_0v_1, v_1v_2, v_2v_3, \ldots, v_kv_{k+1}, \ldots$$

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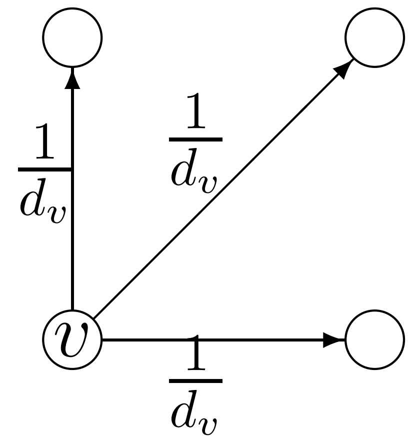
Random walks on a graph G:

$$f_{k+1} = f_k T^{-1} A.$$



$$\|(f_k - \pi)T^{-1/2}\| \leq \bar{\lambda}^k \|(f_0 - \pi)T^{-1/2}\|.$$

$$T^{-1}A \sim T^{-1/2}AT^{-1/2} = I - \mathcal{L}.$$

$\bar{\lambda}$ determines the mixing rate of random walks.

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Proof

Theorem:

$$\|(f_k - \pi)T^{-1/2}\| \leq \bar{\lambda}_\alpha^k \|(f_0 - \pi)T^{-1/2}\|.$$

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&\le \bar{\lambda}_\alpha^k \|(f_0 - \pi)T^{-1/2}\|.
\end{aligned}$$

□

Diameter

Theorem [Chung (1989)]
If G is not a complete weighted graph, then we have

$$\mathrm{diam}(G) \leq \left\lceil \frac{\log(\mathrm{vol}(G)/\delta)}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil,$$

where δ is the minimum degree of G.

Edge discrepancy

For any two subsets X and Y, we have

$$\left| |E(X,Y)| - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leq \bar{\lambda} \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(G)}.$$

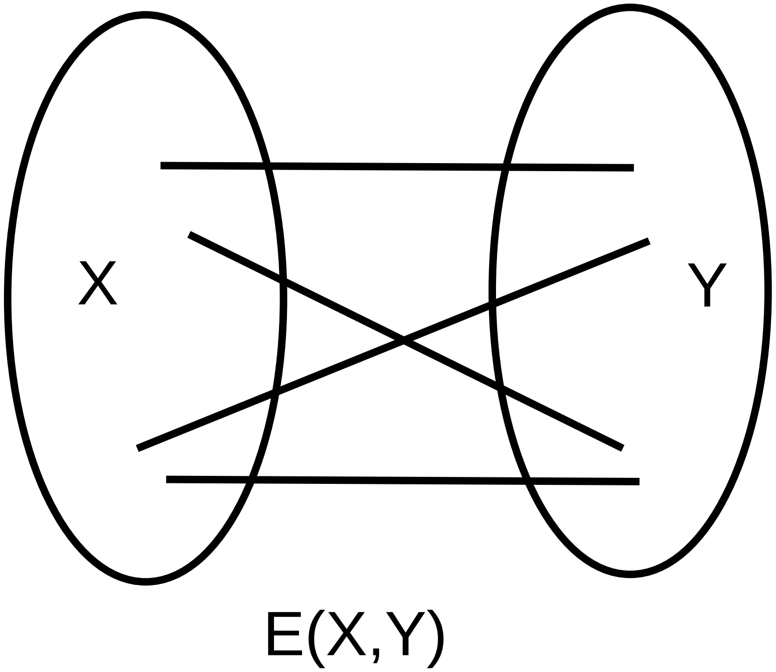
where

$\mathrm{vol}(X) = \sum_{x \in X} d_x$

$\mathrm{vol}(G) = \sum_{x \in V(G)} d_x$

$\mathrm{vol}(\bar{X}) = \mathrm{vol}(G) - \mathrm{vol}(X)$

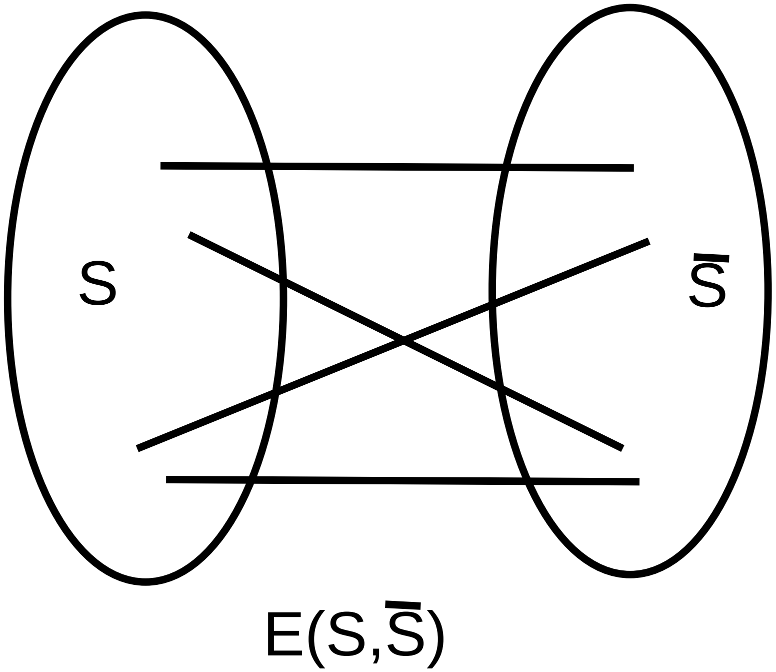
$\bar{\lambda} = \max\{|1 - \lambda_1|, |\lambda_{n-1} - 1|\}.$



Cheeger's Constant

$$h(S) := \frac{|E(S,\bar{S})|}{\min\{\mathrm{vol}(S), \mathrm{vol}(\bar{S})\}}.$$

$$h_G := \min_{S \subset V(G)} h(S).$$



Cheeger's inequality

$$2h_G \geq \lambda_1 \geq \frac{h_G^2}{2}.$$

Eulerian directed graphs

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Chung [2005] defined the Laplacian of Eulerian directed graphs.

$$\mathcal{L} = \frac{\vec{\mathcal{L}} + \vec{\mathcal{L}}'}{2}.$$

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Theorem: $\|(f_k - \pi)T^{-1/2}\| \leq \sigma_\alpha^k \|(f_0 - \pi)T^{-1/2}\|$.
Here $\sigma_\alpha := \max_{f \perp \phi_0'} \frac{\|\vec{\mathcal{L}}_\alpha f\|}{\|f\|}$ is the second largest singular value of $\vec{\mathcal{L}}_\alpha$.

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Choosing α to minimize σ_α, we get

$$\min_{0\leq\alpha<1}\{\sigma_\alpha\} \leq \begin{cases} \sigma_0 & \text{if } \lambda_1 \leq 1-\sigma_0^2; \\ \sqrt{1-\frac{\lambda_1^2}{2\lambda_1+\sigma_0^2-1}} & \text{otherwise.} \end{cases}$$

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In particular, $\min_{0\leq\alpha<1}\{\sigma_\alpha\} \leq \sqrt{1-\frac{\lambda_1}{2}}.$

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We improved it into

Theorem [Lu-Peng 2011]:

$$\mathrm{diam}(D) \leq \left\lceil \frac{\log(\mathrm{vol}(D)/\delta)}{\log \sigma_\alpha} \right\rceil,$$

for any $0 < \alpha < 1$.

Edge expansion in D

Theorem [Lu-Peng 2011]: Let D be a Eulerian directed graph. If X and Y are two subsets of $V(D)$, then we have

$$\left| \frac{|E(X,Y)| + |E(Y,X)|}{2} - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(D)} \right| \leq \bar{\lambda} \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(D)}.$$

Hypergraphs

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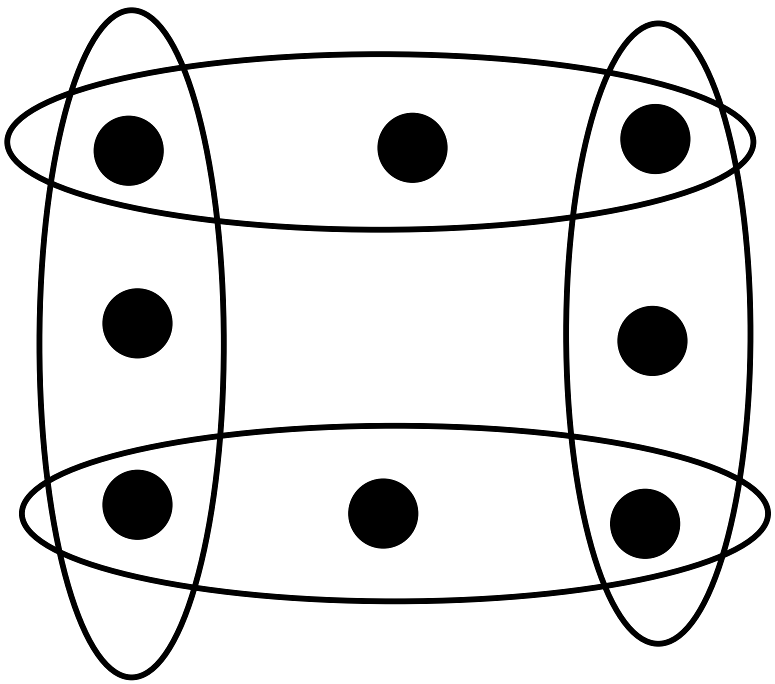
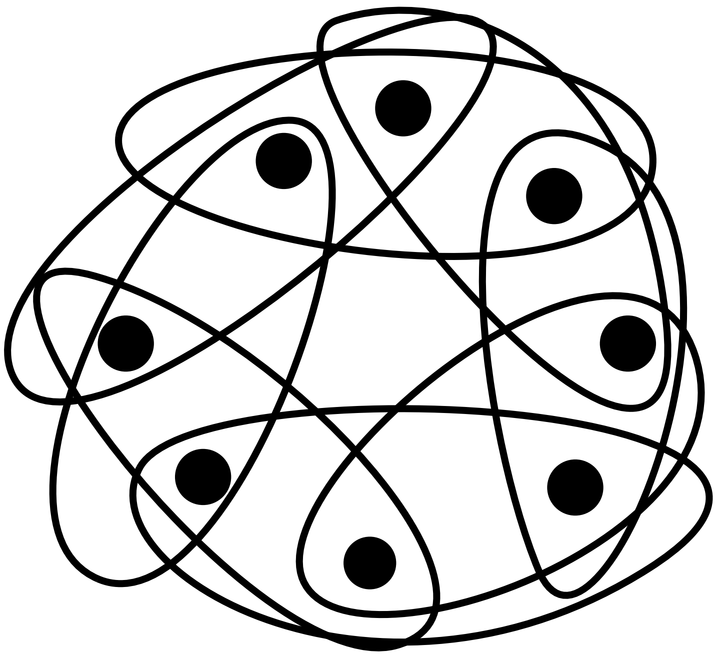


A 3-uniform loose cycle



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- S: an s-subset of V
- Degree d_S: the number of edges passing through S.

$$\sum_{S \in \binom{V}{s}} d_S = \binom{r}{s} |E(H)|.$$

s-walks on hypergraphs

For $1 \leq s \leq r-1$, an s-walk on H consists of

- a vertex sequence: $v_1, v_2, \ldots, v_{(k-1)(r-s)+r}$
- an edge sequence: $F_1, F_2, \ldots, F_k$ satisfying $F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\}$ for $1 \leq i \leq k$.

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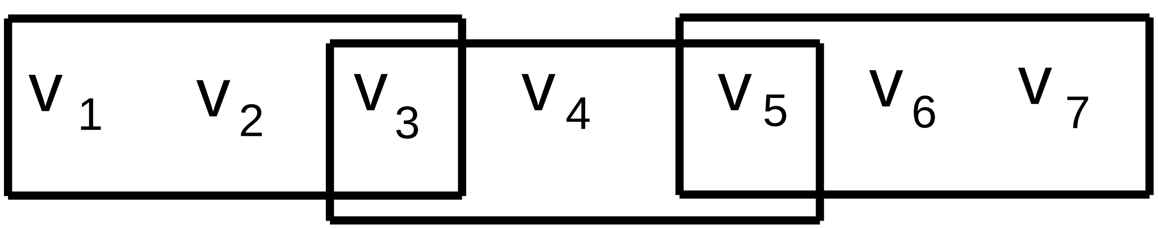
$$|F_i \cap F_{i+1}| = s$$



A 1-walk in a 3-graph

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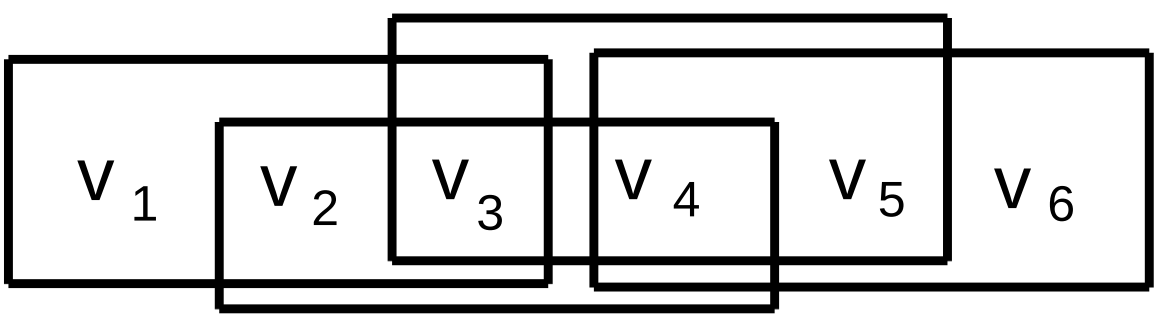

A 2-walk in a 3-graph

s-walks on hypergraphs

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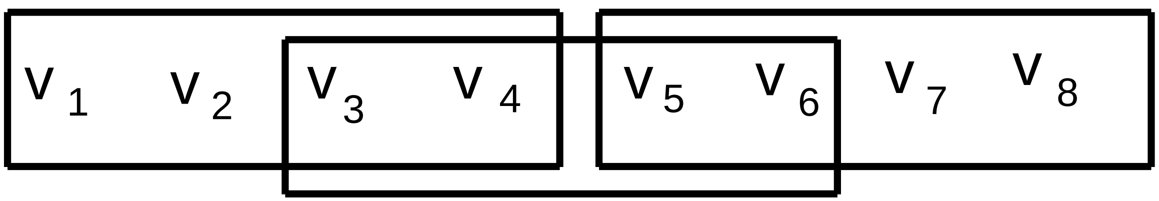
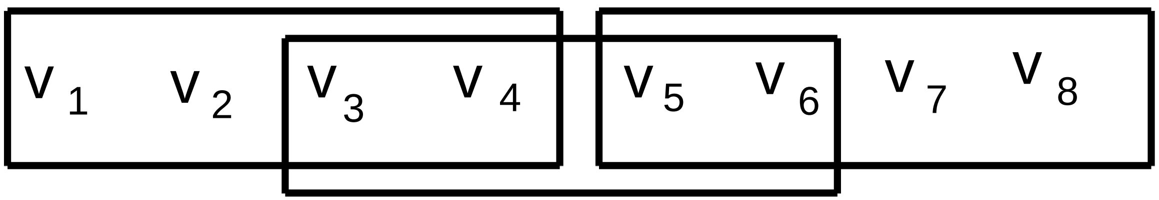


A 2-walk in a 4-graph

Loose random walks

Loose walk: $1 \leq s \leq \frac{r}{2}$.



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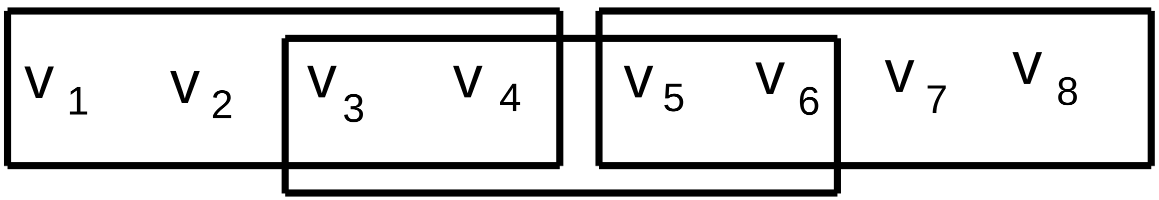

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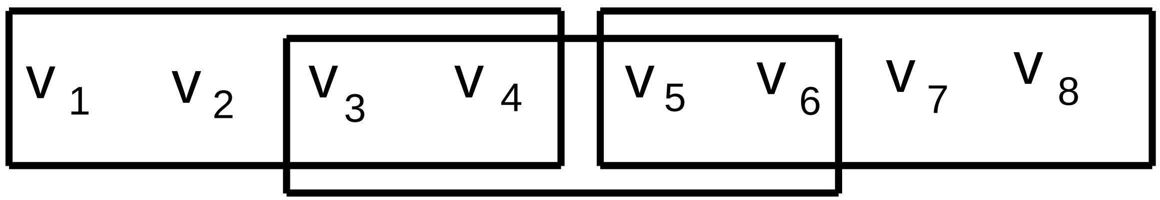

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$\mathcal{L}^{(1)}$ is the same as the Laplacian of hypergraph introduced by **Rodríguez [2009]**.

Tight random walks

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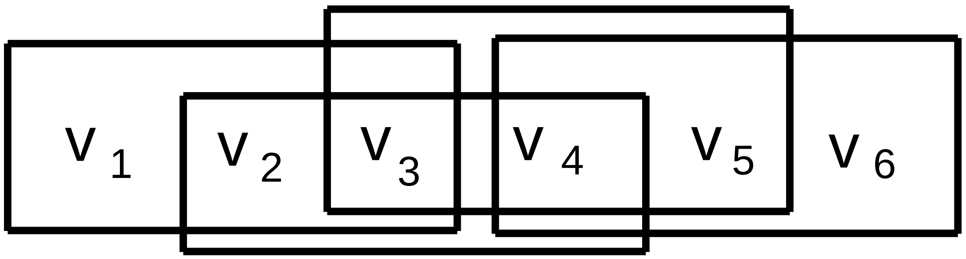

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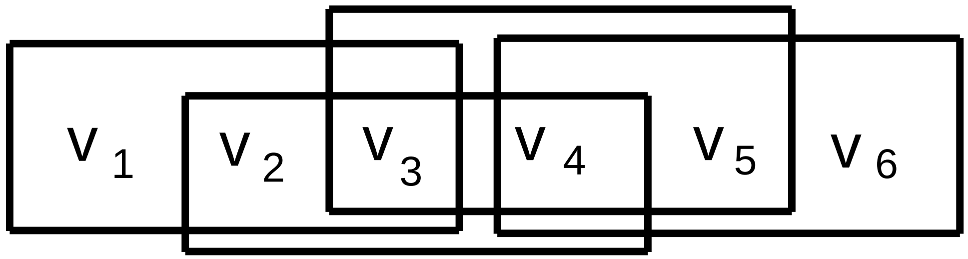

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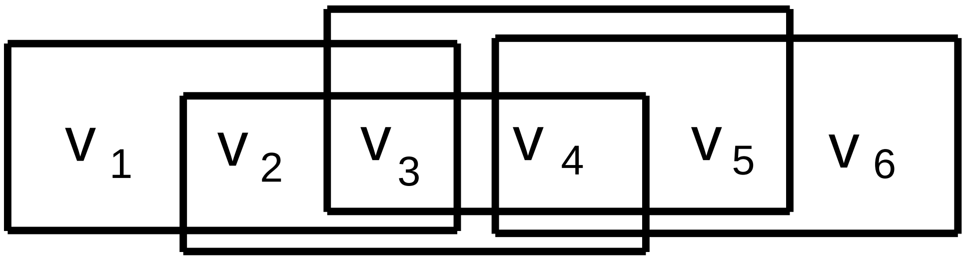

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- Vertex set $V(G^{(s)}) = \mathrm{V}^{\underline{s}}$
- For $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$, xy is a directed edge if
  - $x_{r-s+j} = y_j$ for $1 \le j \le 2s - r$.
  - $\{x_1, \ldots, x_s, y_{2s-r+1}, y_s\}$ is an edge of H.

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$\lambda_1^{(s)}$, $\lambda_{max}^{(s)}$, and $\bar{\lambda}^{(s)}$ are defined in the same way.

Examples

Some eigenvalues of Laplacians of complete hypergraph K_n^r:

H	$\lambda_1^{(4)}$	$\lambda_1^{(3)}$	$\lambda_1^{(2)}$	$\lambda_1^{(1)}$	$\lambda_{\max}^{(1)}$	$\lambda_{\max}^{(2)}$	$\lambda_{\max}^{(3)}$	$\lambda_{\max}^{(4)}$
K_6^3			$3/4$	$6/5$	$6/5$	$3/2$		
K_7^3			$7/10$	$7/6$	$7/6$	$3/2$		
K_6^4		$1/3$	$5/6$	$6/5$	$6/5$	$3/2$	1.76759	
K_7^4		$3/8$	$9/10$	$7/6$	$7/6$	$7/5$	$7/4$	
K_6^5	0.1464	$1/2$	$5/6$	$6/5$	$6/5$	$3/2$	$3/2$	1.809
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$\lambda_1^{(s)}$ (and/or) $\bar{\lambda}_{max}^{(s)}$ is related to

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Each application is divided into the loose case and the tight case.

Random walks (I)

Theorem [Lu-Peng 2011]: For $1 \leq s \leq r/2$, suppose that H is an s-connected r uniform hypergraph. For $0 \leq \alpha < 1$, the joint distribution f_k at the k-th stop of the α-lazy random walk at time k converges to the stationary distribution π in probability. In particular, we have

$$\|(f_k - \pi)T^{-1/2}\| \leq (\bar{\lambda}_\alpha^{(s)})^k \|(f_0 - \pi)T^{-1/2}\|,$$

where $\bar{\lambda}_\alpha^{(s)} = \max\{|1 - (1-\alpha)\lambda_1^{(s)}|, |(1-\alpha)\lambda_{\max}^{(s)} - 1|$, and f_0 is the probability distribution at the initial stop.

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$$\|(f_k - \pi)T^{-1/2}\| \leq (\sigma_\alpha^{(s)})^k \|(f_0 - \pi)T^{-1/2}\|,$$

where $\sigma_\alpha^{(s)} \leq \sqrt{1 - 2\alpha(1-\alpha)\lambda_1^{(s)}}$, and f_0 is the probability distribution at the initial stop.

s-Diameter (I)

Theorem [Lu-Peng 2011]: Let H be a r-uniform hypergraph. For $1 \leq s \leq \frac{r}{2}$, if $\lambda^{(s)}_{\max} > \lambda^{(s)}_1 > 0$, then the s-diameter of an r-uniform hypergraph H satisfies

$$\text{diam}^{(s)}(H) \leq \left\lceil \frac{\log \frac{\text{vol}(V^{\underline{s}})}{\delta^{(s)}}}{\log \frac{\lambda^{(s)}_{\max}+\lambda^{(s)}_1}{\lambda^{(s)}_{\max}-\lambda^{(s)}_1}} \right\rceil .$$

Here $\text{vol}(V^{\underline{s}}) = \sum_{x \in V^{\underline{s}}} d_x = |E(H)| \frac{r!}{(r-2s)!}$ and $\delta^{(s)}$ is the minimum degree in $G^{(s)}$.

s-Diameter (II)

Theorem [Lu-Peng 2011]: Let H be a r-uniform hypergraph. For $r/2 < s \leq r-1$, if $\lambda_1^{(s)} > 0$, then the s-diameter of H satisfies

$$\text{diam}^{(s)}(H) \leq \left\lceil \frac{2\log \frac{\text{vol}(V^{\underline{s}})}{\delta^{(s)}}}{\log \frac{2}{2-\lambda_1^{(s)}}} \right\rceil .$$

Here $\text{vol}(V^{\underline{s}}) = \sum_{x \in V^{\underline{s}}} d_x = |E(H)| r!$ and $\delta^{(s)}$ is the minimum degree in $D^{(s)}$.

Edge expansion (I)

Theorem [Lu-Peng 2011]: For $1 \leq t \leq s \leq r-t$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let

$$E(S,T) = \{F \in E(H) : \exists x \in S, \exists y \in T, x \cap y = \emptyset, \text{ and } x \cup y \subseteq$$

Let $e(S) = \frac{\mathrm{vol}(S)}{\mathrm{vol}(\binom{V}{s})}$ and $e(S,T) = \frac{|E(S,T)|}{|E(\binom{V}{s},\binom{V}{t})|}$.

$$|e(S,T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$

Edge expansion (II)

Theorem [Lu-Peng 2011]: For $1 \leq t < \frac{r}{2} < s < s+t \leq r$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let $e(S,T) = \frac{|E(S,T)|}{|E(\binom{V}{s},\binom{V}{t})|}$. If $|x \cap y| \neq \min\{t, 2s-r\}$ for any $x \in S$ and $y \in T$, then we have

$$|\frac{1}{2}e(S,T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$

Edge expansion (III)

Theorem [Lu-Peng 2011]: Suppose $\frac{r}{2} < s \leq r-1$. For $S, T \subseteq \binom{V}{s}$, let

$$E'(S,T) = \{F \in E(H) \mid \exists x \in S, \exists y \in T, F = x \cup y\}$$

and $e'(S,T) = \frac{|E'(S,T)|}{|E'(\binom{V}{s},\binom{V}{s})|}$. We have

$$|e'(S,T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$

Connections of different $\mathcal{L}^{(s)}$

Theorem [Lu, Peng 2011] We have the following inequalities for the "loose" Laplacian eigenvalues.

$$\lambda_1^{(1)} \geq \lambda_1^{(2)} \geq \ldots \geq \lambda_1^{(\lfloor r/2 \rfloor)};$$
$$\lambda_{\max}^{(1)} \leq \lambda_{\max}^{(2)} \leq \ldots \leq \lambda_{\max}^{(\lfloor r/2 \rfloor)}.$$

Reduced Laplacian (I)

For $1 \leq s \leq r/2$, let $G^{(s)'}$ be the weighted graph defined as

- Vertex set $V(G^{(s)'}) = \binom{V}{s}$
- Weight function $w\colon \binom{V}{s} \times \binom{V}{s} \to \mathbb{Z}$:

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Since $G^{(s)}$ is a blow-up of $G^{(s)'}$, we have
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Complete hypergraph K_n^r

Theorem: For $1 \leq s \leq r/2$, the reduced s-th Laplacian eigenvalues of K_n^r is the eigenvalues of s-th reduced Lapacian of K_n^r are given by

$$1 - \frac{(-1)^i \binom{n-s-i}{s-i}}{\binom{n-s}{s}} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1}$$

for $0 \leq i \leq s$.

Keneser graph $K(n, s)$

The Kneser graph $K(n, s)$ is a graph over the vertex set $\binom{[n]}{s}$; two s-sets S and T form an edge of $K(n, s)$ if and only if $S \cap T = 0$.

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Note $K(n, s)$ is a regular graph; so the Laplacian eigenvalues can be determined from the eigenvalues of its adjacency matrix.

Proof

We observe that $G^{(s')}(K_n^r)$ is essentially the Kneser graph $K(n,s)$ with each edge associated with a weight $\binom{n-2s}{r-2s}$. Note the multiplicative factor $\binom{n-2s}{r-2s}$ is canceled after normalization. The $\mathcal{L}^{(s)}$ (for K_n^r) is exactly the Laplacian of Kneser graph. Thus, the eigenvalues of s-th Lapacian of K_n^r are given by

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Here we present a "new" proof using the s-th Laplacian eigenvalues of K_n^r.(Actually it is due to **Calderbank-Frankl [1992]**.)

Calderbank-Frankl's proof

It suffices to any intersecting family U has size at most $\binom{n-1}{s-1}$.

Note U is an independent set in $G^{(s)'}(K_n^r)$. Let $\mathcal{L}$ be the Laplacian of $G^{(s)'}(K_n^r)$. We have $\mathcal{L}|_U = I$. By Cauchy's interlace theorem, we have

$$\lambda_k^{(s)} \leq 1 \leq \lambda^{(s)}_{\binom{n}{s}-|U|+k}$$

for $0 \leq k \leq |U| - 1$.

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Let N^+ (or N^-) be the number of eigenvalues of $\mathcal{L}^{(s)}$ which is ≥ 1 (or ≤ 1) respectively. We have $|U| \leq N^+$ and $|U| \leq N^-$.

continue...

The eigenvalues of $\mathcal{L}$ are

$$1 - \frac{(-1)^i \binom{n-s-i}{s-i}}{\binom{n-s}{s}} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1}$$

for $0 \le i \le s$.

$$N^+ = \sum_{i=0}^{\lfloor (s-1)/2 \rfloor} \left(\binom{n}{2i+1} - \binom{n}{2i} \right)$$

$$N^- = \sum_{i=0}^{\lfloor s/2 \rfloor} \left(\binom{n}{2i} - \binom{n}{2i-1} \right).$$

continue...

We have

$$
\begin{aligned}
|U| &\leq \min\{N^+, N^-\} \\
&= \sum_{i=0}^{s-1} (-1)^{s-1-i} \binom{n}{i} \\
&= \binom{n}{s-1} - \binom{n}{s-2} + \binom{n}{s-3} - \binom{n}{s-4} + \cdots \\
&= \binom{n-1}{s-1}.
\end{aligned}
$$

□

Random graphs

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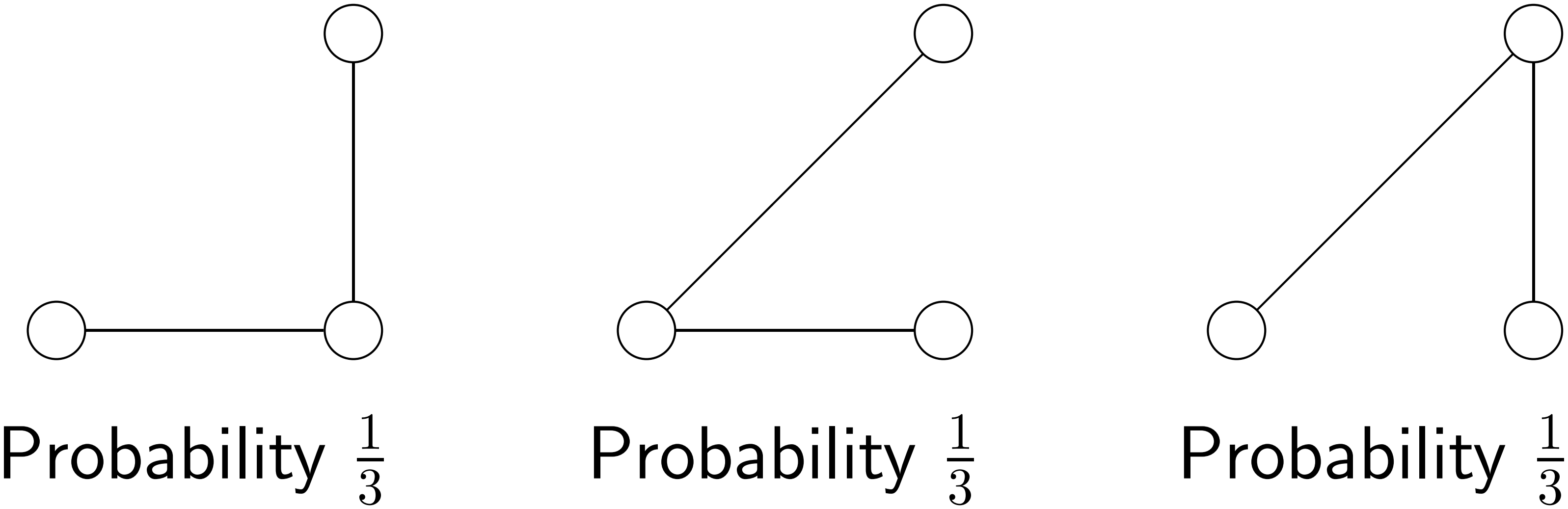
Example: A random graph on 3 vertices and 2 edges with the uniform distribution on it.



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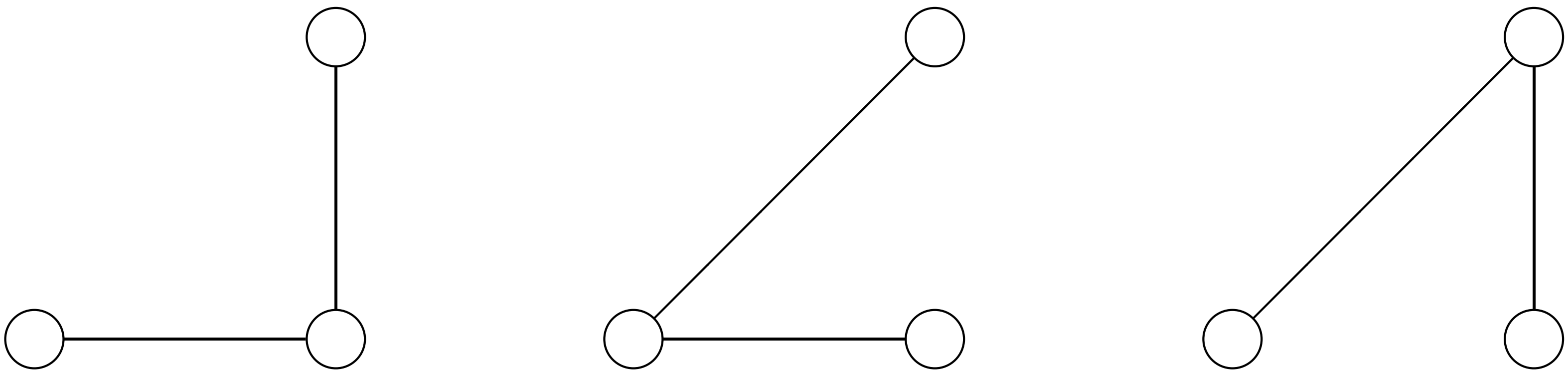

Probability $\frac{1}{3}$ Probability $\frac{1}{3}$ Probability $\frac{1}{3}$

A random graph G *almost surely* satisfies a property P, if

$$Pr(G \text{ satisfies } P) \to 1 \text{ as } |V(G)| \to \infty.$$

Erdős-Rényi model $G(n, p)$

For each pair of vertices in the vertex set of size n, an edge is created independently with probability p.

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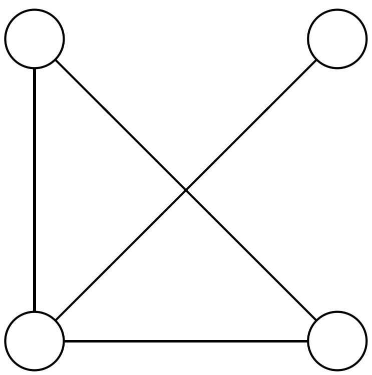

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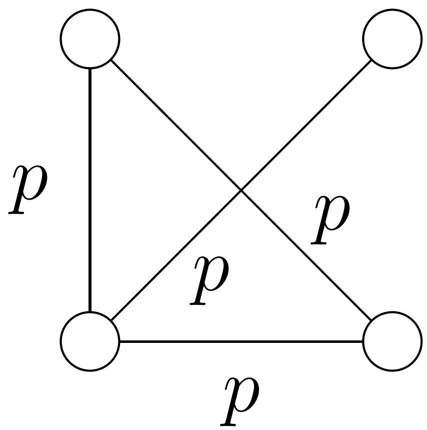

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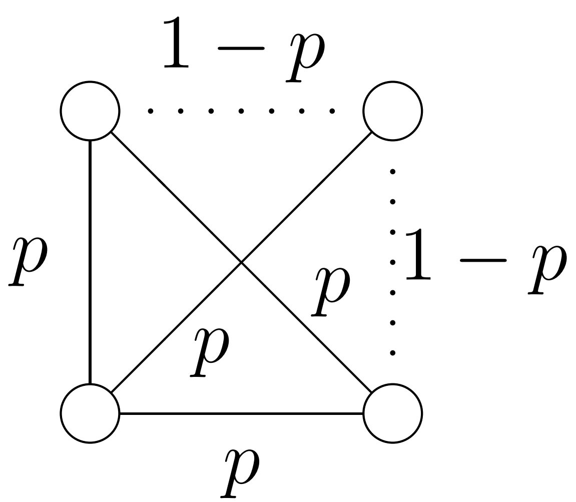

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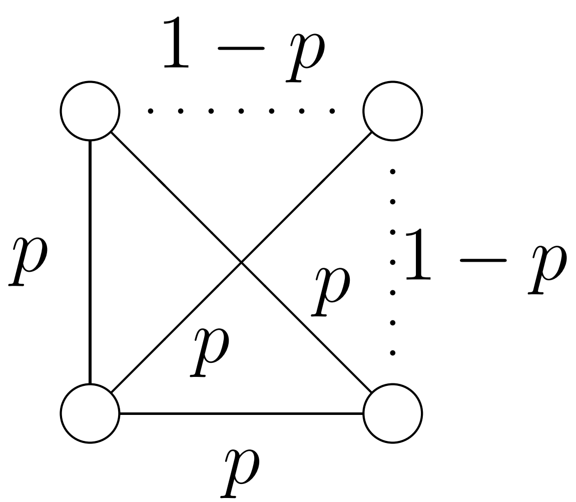

The probability of this graph is

$$p^4(1-p)^2.$$

The birth of random graph theory

ON THE EVOLUTION OF RANDOM GRAPHS

by

P. ERDÖS and A. RÉNYI
Institute of Mathematics
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1. *Definition of a random graph*

Let $E_{n,N}$ denote the set of all graphs having n given labelled vertices $V_1, V_2, \cdots, V_n$ and N edges. The graphs considered are supposed to be not oriented, without parallel edges and without slings (such graphs are sometimes called linear graphs). Thus a graph belonging to the set $E_{n,N}$ is obtained by choosing N out of the possible $\binom{n}{2}$ edges between the points $V_1, V_2, \cdots, V_n$, and therefore the number of elements of $E_{n,N}$ is equal to $\binom{\binom{n}{2}}{N}$. A random graph $\Gamma_{n,N}$ can be defined as an element of $E_{n,N}$ chosen at random, so that each of the elements of $E_{n,N}$ have the same probability to be chosen, namely $1/\binom{\binom{n}{2}}{N}$. There is however an other slightly

Evolution of $G(n, p)$

p

p	
0	the empty graph.
	disjoint union of trees.
$\frac{c}{n}$	cycles of any size.
$\frac{1}{n}$	the double jumps.
$\frac{c'}{n}$	one giant component, others are trees.
$\frac{\log n}{n}$	$G(n, p)$ is connected.
$\Omega(\frac{\log n}{n})$	connected and almost regular.
$\Omega(n^{\epsilon-1})$	finite diameter.
$\Theta(1)$	dense graphs, diameter is 2.
1	the complete graph.

Eigenvalues of $G(n,p)$

Füredi and Komlós (1981): If $np(1-p) \gg \log^6 n$, then almost surely

$$\begin{aligned} \mu_n(G(n,p)) &= (1+o(1))np \\ \max_{1\le i\le n-1}\{|\mu_i(G(n,p))|\} &\le (2+o(1))\sqrt{np(1-p)}. \end{aligned}$$

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What about the Laplacian eigenvalues of $G(n,p)$?

Model $G(w_1, w_2, \ldots, w_n)$

Random graph with expected degree sequence:

- Each vertex i is associated with a weight w_i.
- The probability that ij is an edge is $w_i w_j \frac{1}{\sum_{k=1}^n w_k}$.
- The expected degree of i is w_i.

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- The expected degree of i is w_i.

Chung-Lu-Van (2003):

$$\bar{\lambda} \leq (1 + o(1)) \frac{4}{\sqrt{\bar{w}}} + \frac{g(n) \log^2 n}{w_{min}},$$

where w_{min} is the minimum weight and $\bar{w}$ is the average weight.

Laplacian of $G(n, p)$

$G(n, p)$ is a special case of $G(w_1, w_2, \ldots, w_n)$ with $w_1 = w_2 = \cdots = w_n = np$.

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Applying Chung-Lu-Van's result to $G(n,p)$, we have

Chung-Lu-Van (2003): For $1 - \epsilon > p \gg \frac{\log^6 n}{n}$,

$$\bar{\lambda}(G(n,p)) \leq (4 + o(1))\frac{1}{\sqrt{np}}.$$

Random d-regular graphs

Random d-regular graphs $G_{n,d}$

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Friedman (2002) For random d-regular graph with $d \ge 4$, almost surely

$$\max_{1 \le i \le n-1} \{|\mu_i(G_{n,d})|\} = (2 + o(1))\sqrt{d-1}.$$

Random hypergraphs

Random r-uniform hypergraph $H^r(n,p)$:

- n: the number of vertices
- p: probability, $0 < p < 1$.
  For any $F \in \binom{[n]}{r}$, F is an edge with probability p independently.

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- n: the number of vertices
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Question: What is Laplacian eigenvalues of $H^r(n,p)$?

Our result (I)

Theorem [Lu, Peng 2011] Let $H^r(n,p)$ be a random r-uniform hypergraph. For $1 \leq s \leq r/2$, if $p(1-p) \gg \frac{\log^4 n}{n^{r-s}}$ and $1-p \gg \frac{\log n}{n^2}$, then almost surely

$$\bar{\lambda}^{(s)}(H^r(n,p)) \leq \frac{s}{n-s} + \left(\frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1) \right) \sqrt{\frac{1-p}{\binom{n-s}{r-s} p}}.$$

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Moreover, for $1 \le k \le \binom{n}{s} - 1$ almost surely we have

$$|\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^r)| \le \left(\frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1)\right)\sqrt{\frac{1-p}{\binom{n-s}{r-s}p}}.$$

Applied to $G(n,p)$

Our result: If $p(1-p) \gg \frac{\log^4 n}{n}$, then

$$\bar{\lambda}(G(n,p)) \leq (3+o(1))\frac{1}{\sqrt{np}}.$$

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Chung-Lu-Van (2003): If $1-\epsilon > p \gg \frac{\log^6 n}{n}$, then

$$\bar{\lambda}(G(n,p)) \leq (4+o(1))\frac{1}{\sqrt{np}}.$$

Lemma 1

Lemma 1: Given any two $(N \times N)$-Hermitian matrices A and B, for $1 \leq k \leq N$, let $\mu_k(A)$ (or $\mu_k(B)$) be the k-th eigenvalues of A (or B) in the increasing order. We have

$$|\mu_k(A) - \mu_k(B)| \leq \|A - B\|.$$

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Proof: By the Min-Max Theorem,

$$\begin{aligned}
\mu_k(A) &= \min_{S_k} \max_{x \in S_k, \|x\|=1} x'Ax \\
&= \min_{S_k} \max_{x \in S_k, \|x\|=1} (x'Bx + x'(A-B)x) \\
&\leq \min_{S_k} \max_{x \in S_k, \|x\|=1} (x'Bx + \|A-B\|) \\
&= \mu_k(B) + \|A-B\|.
\end{aligned}$$

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Similarly we have

$$\mu_k(A) \geq \mu_k(B) - \|A - B\|. \qquad \square$$

Sketch proof

Write $\mathcal{L}^{(s)}(K_n^r) - \mathcal{L}^{(s)}(H^r(n,p)) = M_1 + M_2 + M_3 + M_4$, where

$$
\begin{aligned}
M_1 &= \frac{1}{\binom{r-s}{s}} \left(D^{-1/2}(W - \mathrm{E}(W))D^{-1/2} - d^{-1}(W - \mathrm{E}(W)) \right), \\
M_2 &= \frac{1}{\binom{r-s}{s} d}(W - \mathrm{E}(W)), \\
M_3 &= \frac{1}{\binom{r-s}{s}} D^{-1/2}\mathrm{E}(W)D^{-1/2} - \frac{d}{\binom{n}{s}} D^{-1/2} J D^{-1/2} \\
&\quad - \frac{1}{\binom{n-s}{s}} K + \frac{1}{\binom{n}{s}} J, \\
M_4 &= \frac{1}{\binom{n}{s}} (dD^{-1/2} J D^{-1/2} - J).
\end{aligned}
$$

Continue

$$\begin{aligned}
\|M_1\| &= O\left(\frac{\sqrt{(1-p)\log N}}{d}\right), && \text{easy!} \\
\|M_2\| &\leq \frac{(2+o(1))\sqrt{1-p}}{\sqrt{\binom{r-s}{s}d}}, && \text{hard!} \\
\|M_3\| &= O\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right), && \text{easy!} \\
\|M_4\| &\leq (1+o(1))\sqrt{\frac{1-p}{d}}. && \text{tricky!}
\end{aligned}$$

Putting together, $\|M\| \leq \left(\frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1)\right)\sqrt{\frac{1-p}{\binom{n-s}{r-s}p}}$. $\square$

Tools

Chenoff's inequality: Let $X_1, \ldots, X_n$ be independent 0-1 random variables with We consider the sum $X = \sum_{i=1}^{n} X_i$. Then we have

$$\begin{aligned} \text{(Lower tail)} \qquad \Pr(X \le \mathrm{E}(X) - \lambda) &\le e^{-\lambda^2/2\mathrm{E}(X)}, \\ \text{(Upper tail)} \qquad \Pr(X \ge \mathrm{E}(X) + \lambda) &\le e^{-\frac{\lambda^2}{2(\mathrm{E}(X)+\lambda/3)}}. \end{aligned}$$

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Lemma: If $d := \binom{n}{s} p \geq \log N$, then with probability at least $1 - \frac{1}{N^3}$, for any $S \in \binom{V}{s}$, we have

$$d_S \in (d - 3\sqrt{d \log N}, d + 3\sqrt{d \log N}).$$

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Let $C = W - \mathrm{E}(W)$. One of the major task is to estimate $\|C\|$. We have the following Lemma.

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Recall $M_2 = \frac{1}{\binom{r-s}{s} d} C$. We get

$$\|M_2\| \leq \frac{(2+o(1))\sqrt{1-p}}{\sqrt{\binom{r-s}{s} d}}$$

Main Lemma on $\text{Trace}(C^t)$

Lemma 3: For any $k \ll (n^{r-s}p(1-p))^{1/4}$, we have

$$\begin{aligned} \mathrm{E}\left(\text{Trace}(C^{2k})\right) &= (1+o(1))\frac{n^{s+k(r-s)}\binom{2k}{k}p^k(1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k}, \\ \mathrm{E}\left(\text{Trace}(C^{2k+1})\right) &= O\left(\frac{k(2k+1)n^{s+k(r-s)}\binom{2k}{k}p^k(1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k}\right). \end{aligned}$$

Proof of Lemma 2

Let $U := \frac{n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k$. By Markov's inequality,

$$\begin{aligned}
\Pr\left(\|C\| \geq (1+\epsilon)\sqrt[2k]{U}\right) &= \Pr\left(\|C\|^{2k} \geq (1+\epsilon)^{2k}U\right) \\
&\leq \frac{\mathrm{E}(\|C\|^{2k})}{(1+\epsilon)^{2k}U} \\
&\leq \frac{\mathrm{E}(\mathrm{Trace}(C^{2k}))}{(1+\epsilon)^{2k}U} \\
&= \frac{1+o(1)}{(1+\epsilon)^{2k}}.
\end{aligned}$$

Continue

Choose $k = sg(n)\log n$ and $\epsilon = 1/g(n)$.

$$\begin{aligned}
\|\|C\| &\leq (1+o(1))\sqrt[2k]{U} \\
&= (1+o(1))\left(\frac{n^{s+k(r-s)}\binom{2k}{k}p^k(1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k}\right)^{\frac{1}{2k}} \\
&< n^{\frac{s}{2k}}2\sqrt{\frac{n^{r-s}p(1-p)}{s!(r-2s)!}} \\
&= (2+o(1))\sqrt{\binom{r-s}{s}d(1-p)}.
\end{aligned}$$

□

Wigner's semicircle law

Wigner (1958)

- A is a real symmetric $N \times N$ matrix.
- Entries a_{ij} are independent random variables.
- $E(a_{ij}^{2k+1}) = 0.$
- $E(a_{ij}^2) = m^2.$
- $E(a_{ij}^{2k}) < M.$

The distribution of eigenvalues of A converges into a semicircle distribution of radius $2m\sqrt{N}$.

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Füredi and Komlós (1981): The eigenvalues of $G(n, p)$ follows Wigner's semicircle law.

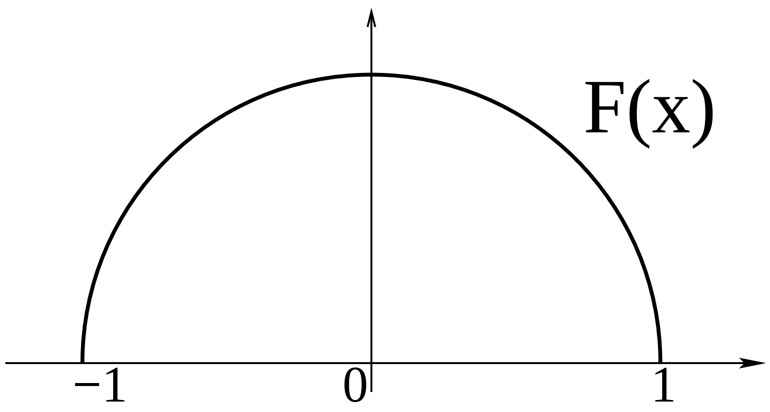
Definition

Let A be a Hermitian matrix of dimension $N \times N$. The *empirical distribution* of the eigenvalues of A is

$$F(A, x) := \frac{1}{N}|\{ \text{ eigenvalues of } A \text{ less than } x\}|.$$

We say, the empirical distribution of the eigenvalues of A follows the Semicircle Law centered at c with radius R if

$$F(\frac{1}{R}(A - cI), x) \xrightarrow{p} F(x).$$



Our result (II)

Theorem [Lu, Peng 2011] For $1 \leq s \leq r/2$, if $p(1-p)n^{r-s} \gg \log n$, then almost surely the empirical distribution of eigenvalues of the s-th Laplacian of $H^r(n,p)$ follows the Semicircle Law centered at 1 and with radius $(2+o(1))\sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s}p}}$.

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Corollary: If $p(1-p)n^{r-s} \gg \log n$, then

$$\max_{1 \leq i \leq \binom{n}{s}-1} |\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^r)| \geq \left(\frac{2}{\sqrt{\binom{r-s}{s}}} + o(1)\right)\sqrt{\frac{1-p}{\binom{n-s}{r-s}p}}.$$

Proof of Semicircle Law

Theorem: If $n^{r-s}p(1-p) \to \infty$, then the empirical distribution of the eigenvalues of C follows the semicircle law centered at 0 with radius $R := 2\sqrt{\binom{r-s}{s}\binom{n-s}{r-s}p(1-p)}$.

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Proof: Let $C_{nor} := \frac{1}{R}C$. For any k, we have

$$\begin{aligned} \mathrm{E}(\mathrm{Trace}(C_{nor}^{2k})) &= (1+o(1))\frac{(2k)!}{2^{2k}k!(k+1)!} \\ \mathrm{E}(\mathrm{Trace}(C_{nor}^{2k+1})) &= o(1). \end{aligned}$$

It converges to the $2k$-th (and $2k+1$-th) moment of the Semicircle distribution. $\square$

Another Lemma

Lemma 4: If

- A: an $(N \times N)$-Hermitian matrices satisfying the Semicircle Law centered at c with radius R,
- B: an $(N \times N)$-Hermitian matrices either $\|B\| = o(R)$ or $\mathrm{rank}(B) = o(N)$,

then $A + B$ satisfies the Semicircle Law centered at c with radius R.

Case $\|B\| = o(R)$

$$\left|\mu_k\left(\frac{1}{R}(A+B-cI)\right)-\mu_k\left(\frac{1}{R}(A-cI)\right)\right| \leq \frac{\|B\|}{R} = o(1).$$

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Thus, we have

$F\left(\frac{1}{R}(A-cI), x - \frac{\|B\|}{R}\right) \leq F\left(\frac{1}{R}(A+B-cI), x\right) \leq F\left(\frac{1}{R}(A-cI), x + \frac{\|B\|}{R}\right).$

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$F\left(\frac{1}{R}(A-cI), x+\frac{\|B\|}{R}\right)$.
Since $F\left(\frac{1}{R}(A-cI), x-\frac{\|B\|}{R}\right) \xrightarrow{p} F(x)$ and
$F\left(\frac{1}{R}(A-cI), x+\frac{\|B\|}{R}\right) \xrightarrow{p} F(x)$. By the Squeeze theorem, we have

$$F(\frac{1}{R}(A+B-cI), x) \xrightarrow{p} F(x).$$

Case $\mathrm{rank}(B) = o(N)$

Let U be the kernel of B (i.e. $B|_U = 0$) and $Z := \frac{1}{R}(A - cI)|_U = \frac{1}{R}(A + B - cI)|_U$.

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Let U be the kernel of B (i.e. $B|_U = 0$) and $Z := \frac{1}{R}(A - cI)|_U = \frac{1}{R}(A + B - cI)|_U$. By Cauchy's interlace theorem, for $1 \leq j \leq N - \mathrm{rank}(B)$, we have

$$\mu_j\left(\frac{1}{R}(A - cI)\right) \leq \mu_j(Z) \leq \mu_{j+\mathrm{rank}(B)}\left(\frac{1}{R}(A - cI)\right),$$

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Thus, for $\mathrm{rank}(B) + 1 \leq j \leq N - \mathrm{rank}(B)$, we have $\mu_{j-\mathrm{rank}(B)}\left(\frac{1}{R}(A - cI)\right) \leq \mu_j\left(\frac{1}{R}(A + B - cI)\right) \leq \mu_{j+\mathrm{rank}(B)}\left(\frac{1}{R}(A - cI)\right)$.

continue

It implies

$$F\left(\frac{1}{R}(A+B-cI),x\right) \;\geq\; F\left(\frac{1}{R}(A-cI),x\right)-\frac{\mathrm{rank}(B)}{N},$$
$$F\left(\frac{1}{R}(A+B-cI),x\right) \;\leq\; F\left(\frac{1}{R}(A-cI),x\right)+\frac{\mathrm{rank}(B)}{N}.$$

Since $\mathrm{rank}(B)=o(N)$, we have $F\left(\frac{1}{R}(A-cI),x\right) \pm \frac{\mathrm{rank}(B)}{N} \xrightarrow{p} F(x)$. By the Squeeze theorem, we have

$$F(\frac{1}{R}(A+B-cI),x) \xrightarrow{p} F(x). \qquad \square$$

From C to $\mathcal{L}^{(s)}(H^r(n,p))$

Recall $\mathcal{L}^{(s)}(K_n^r) - \mathcal{L}^{(s)}(H^r(n,p)) = M_1 + M_2 + M_2 + M_4$.
Let $c := 1 - \frac{(-1)^s}{\binom{n}{s}}$ and $R := (2 + o(1))\sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s}p}}$.

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Hence $\mathcal{L}^{(s)}(K_n^r)$ satisfies the Semicircle Law centered at 1 with radius R. □

Remaining task

It remains to prove the following Lemma.
Lemma 3: For any $k \ll (n^{r-s}p(1-p))^{1/4}$, we have

$$\mathrm{E}\left(\mathrm{Trace}(C^{2k})\right) = (1+o(1))\frac{n^{s+k(r-s)}\binom{2k}{k}p^k(1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k},$$

$$\mathrm{E}\left(\mathrm{Trace}(C^{2k+1})\right) = O\left(\frac{k(2k+1)n^{s+k(r-s)}\binom{2k}{k}p^k(1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k}\right).$$

Estimating the Trace

$$\mathrm{E}(Trace(C^t)) = \sum_{\text{closed } s\text{-walks}} \mathrm{E}(c^{F_1}_{S_1 S_2} c^{F_2}_{S_2 S_3} \ldots c^{F_t}_{S_t S_1}),$$

The sum is over all closed s-walks $S_1 F_1 S_2 F_2 \cdots S_t F_t S_1$. Here $C^F_{ST} = X_F - \mathrm{E}(X_F)$ if $S \cap T = \emptyset$ and $S \cup T \subset F$; and $C^F_{ST} = 0$ otherwise.

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Different groups are mutually independent . In any non-zero product, every F appears at least twice.

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Different groups are mutually independent . In any non-zero product, every F appears at least twice. Those closed walks are called "good".

Counting good walks

For $1 \le i \le \lfloor \frac{t}{2} \rfloor$, let $\mathcal{G}_i^j$ be the set of good closed walks with exactly i distinct edges and j distinct vertices; and let $\mathcal{G}_i := \cup_j \mathcal{G}_i^j$.

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- If $w := S_1F_1S_2F_2 \cdots S_tF_tS_1 \in \mathcal{G}_i^j$, then

$$\mathrm{E}(c_{S_1S_2}^{F_1} c_{S_2S_3}^{F_2} \ldots c_{S_tS_1}^{F_t}) \le p^i(1-p)^i.$$

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- The maximum j such that $\mathcal{G}_i^j \neq \emptyset$ is $m_i := s + i(r-s)$.
- $|\mathcal{G}_i| = (1+o(1))|\mathcal{G}_i^{m_i}|$.

Counting $\mathcal{G}_i^{m_i}$

Mapping every walk in $\mathcal{G}_i^{m_i}$ into a triple $(\mathcal{S}, \mathrm{E}, \mathcal{C})$ where

- $\mathcal{S} := \{S_1, S_2, \ldots, S_i\}$, each S_l is a s-set.
- $\mathcal{E} := \{E_1, E_2, \ldots, E_{i-1}\}$, each E_l is a $r - 2s$-set.
- The sets in $\mathcal{S} \cup \mathcal{E}$ are pairwise disjoint.
- $\mathcal{C}$ is a valid string consists of i pairs of parentheses and $t - 2i$ $*$'s. For example,

$$((*()) * (*)*)$$

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$$|\mathcal{G}_i^{m_i}| \leq \frac{N!}{(N-m_i)!(s!)^r((r-2s)!)^{r-1}} \binom{t}{2i} \frac{1}{i+1} \binom{2i}{i} \left(i\binom{r-s}{s}\right)^{t-2i}.$$

Major term

When $t = 2k$, the major contribution is from the walks in $\mathcal{G}_k^{m_k}$ which can be encoded by $(\mathcal{S}, \mathrm{E}, \mathcal{C})$ where

- $\mathcal{S} := \{S_1, S_2, \ldots, S_k\}$, each S_l is a s-set.
- $\mathcal{E} := \{E_1, E_2, \ldots, E_{k-1}\}$, each E_l is a $r - 2s$-set.
- The sets in $\mathcal{S} \cup \mathcal{E}$ are pairwise disjoint.
- $\mathcal{C}$ is a valid string consists of k pairs of parentheses.

For example, $(())()$ is corresponding to the walk

$$S_1 F_1 S_2 F_2 S_3 F_2 S_2 F_1 S_1 F_3 S_4 F_3 S_1$$

where $F_1 = S_1 \cup S_2 \cup E_1$, $F_2 = S_2 \cup S_3 \cup E_2$, and $F_3 = S_4 \cup S_1 \cup E_3$.

Estimating $\mathrm{E}(\mathrm{Trace}(C^{2k}))$

Let $a_i = |\mathcal{G}_i^{m_i}| p^i (1-p)^i$.

$$\begin{aligned} \frac{a_i}{a_k} &\leq \frac{\binom{2k+1}{2i+1}\binom{2i+1}{i}}{\binom{2k+1}{k}} \left(\frac{i^2}{s!(r-2s)!n^{r-s}p(1-p)} \right)^{k-i} \\ &\leq \epsilon^{k-i}, \end{aligned}$$

where $\epsilon := \frac{9k^4}{s!(r-2s)!n^{r-s}p(1-p)} = o(1)$, since $n^{r-s}p(1-p) \gg k^4$.

$$\mathrm{E}(\mathrm{Trace}(C^{2k})) \approx \sum_{i=1}^{k} a_i = (1+o(1))a_k.$$

Done!

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Thank You