Hypergraphs with Small Spectral Radius

Linyuan Lu

University of South Carolina

Collaborator: Shoudong Man

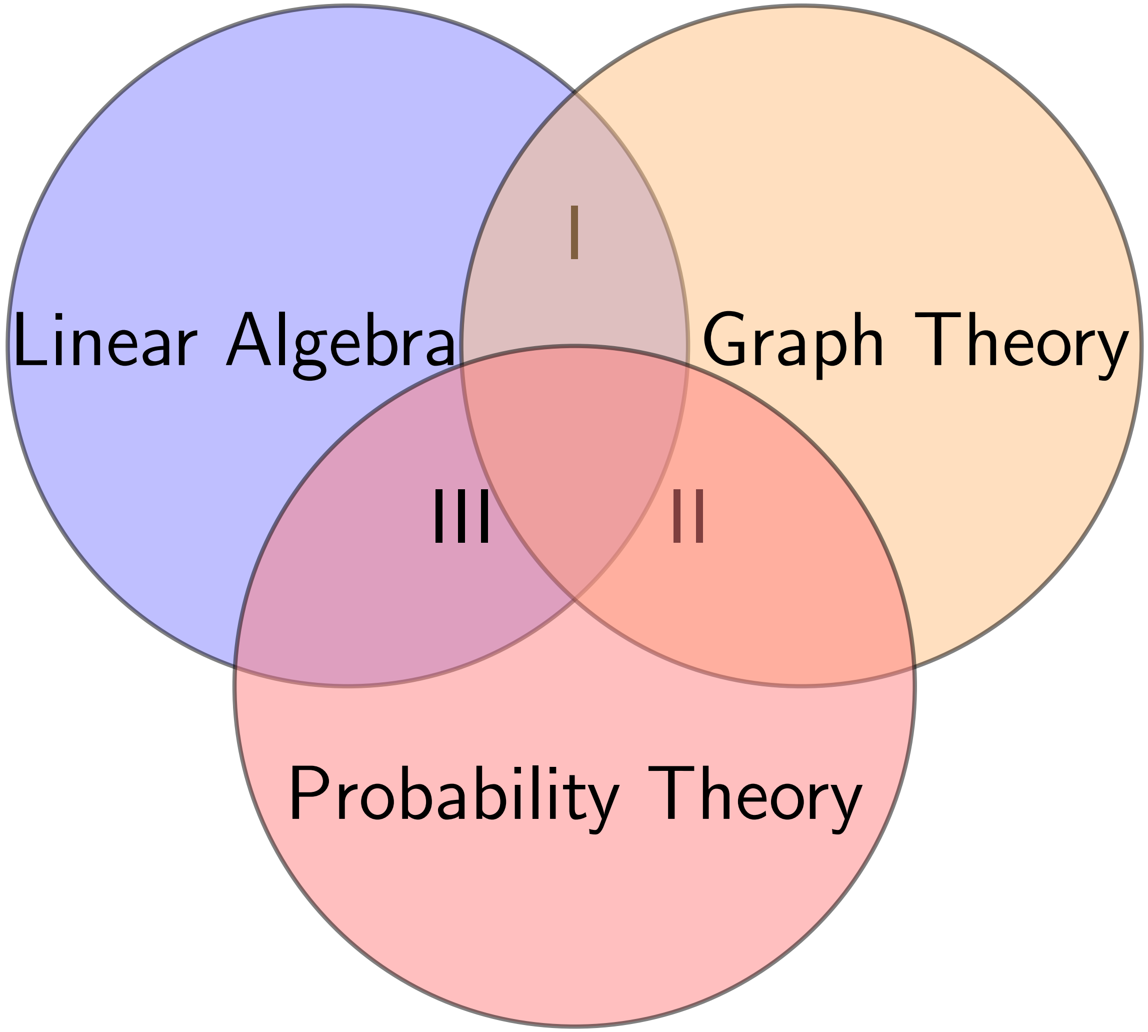
Selected Topics on Spectral Graph Theory (IV)
Nankai University, Tianjin, June 6, 2014

Five talks

Selected Topics on Spectral Graph Theory

1. Graphs with Small Spectral Radius
   Time: Friday (May 16) 4pm.-5:30p.m.
2. Laplacian and Random Walks on Graphs
   Time: Thursday (May 22) 4pm.-5:30p.m.
3. Spectra of Random Graphs
   Time: Thursday (May 29) 4pm.-5:30p.m.
4. Hypergraphs with Small Spectral Radius
   Time: Friday (June 6) 4pm.-5:30p.m.
5. Laplacian of Random Hypergraphs
   Time: Thursday (June 12) 4pm.-5:30p.m.

Backgrounds



I: Spectral Graph Theory   II: Random Graph Theory
III: Random Matrix Theory

Notations

- $G = (V, E)$: a simple connected graph on n vertices

Notations

- $G = (V, E)$: a simple connected graph on n vertices
- $A(G)$: the adjacency matrix

Notations

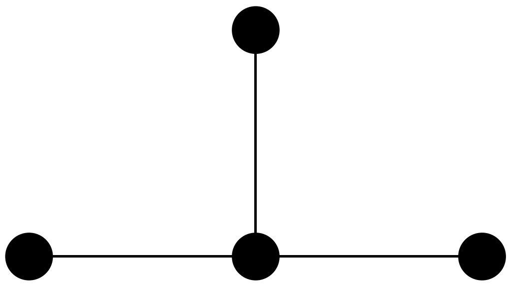
- $G = (V, E)$: a simple connected graph on n vertices
- $A(G)$: the adjacency matrix
- $\phi_G(\lambda) = \det(\lambda I - A(G))$: the characteristic polynomial

Notations

- $G = (V, E)$: a simple connected graph on n vertices
- $A(G)$: the adjacency matrix
- $\phi_G(\lambda) = \det(\lambda I - A(G))$: the characteristic polynomial
- $\rho(G)$ (spectral radius): the largest root of $\phi_G(\lambda)$

Notations

- $G = (V, E)$: a simple connected graph on n vertices
- $A(G)$: the adjacency matrix
- $\phi_G(\lambda) = \det(\lambda I - A(G))$: the characteristic polynomial
- $\rho(G)$ (spectral radius): the largest root of $\phi_G(\lambda)$

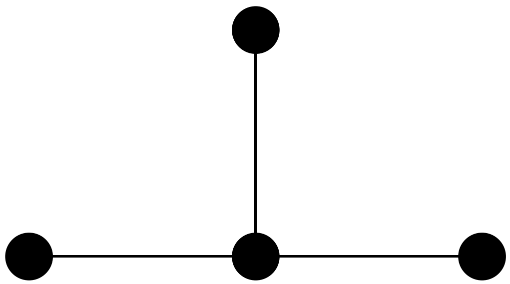


S_4

$$A(S_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Notations

- $G = (V, E)$: a simple connected graph on n vertices
- $A(G)$: the adjacency matrix
- $\phi_G(\lambda) = \det(\lambda I - A(G))$: the characteristic polynomial
- $\rho(G)$ (spectral radius): the largest root of $\phi_G(\lambda)$



S_4

$$A(S_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi_{S_4} = \lambda^4 - 3\lambda^2 \qquad \rho(S_4) = \sqrt{3}$$

Perron-Frobenius theorem

- $A = (a_{ij})$ is **non-negative** if $a_{ij} \geq 0$.
- A is **irreducible** if there exists a m such that A^m is positive.
- A is **aperiodic** if the greatest common divisor of all natural numbers m such that $(A^m)_{ii} > 0$ is 1.

Perron-Frobenius theorem

- $A = (a_{ij})$ is **non-negative** if $a_{ij} \geq 0$.
- A is **irreducible** if there exists a m such that A^m is positive.
- A is **aperiodic** if the greatest common divisor of all natural numbers m such that $(A^m)_{ii} > 0$ is 1.

Perron-Frobenius theorem: If A is an aperiodic irreducible non-negative matrix with spectral radius r, then r is the largest eigenvalue in absolute value of A, and A has an eigenvector α with eigenvalue r whose components are all positive.

Facts on $\rho(G)$

Apply Perron-Frobenius theorem to the adjacency matrix of a connected graph G.

- The eigenvector for $\rho(G)$ can be chosen so that all entries are positive.

Facts on $\rho(G)$

Apply Perron-Frobenius theorem to the adjacency matrix of a connected graph G.

- The eigenvector for $\rho(G)$ can be chosen so that all entries are positive.
- If α is a positive vector corresponding to the eigenvector λ, then $\rho(G) = \lambda$.

Facts on $\rho(G)$

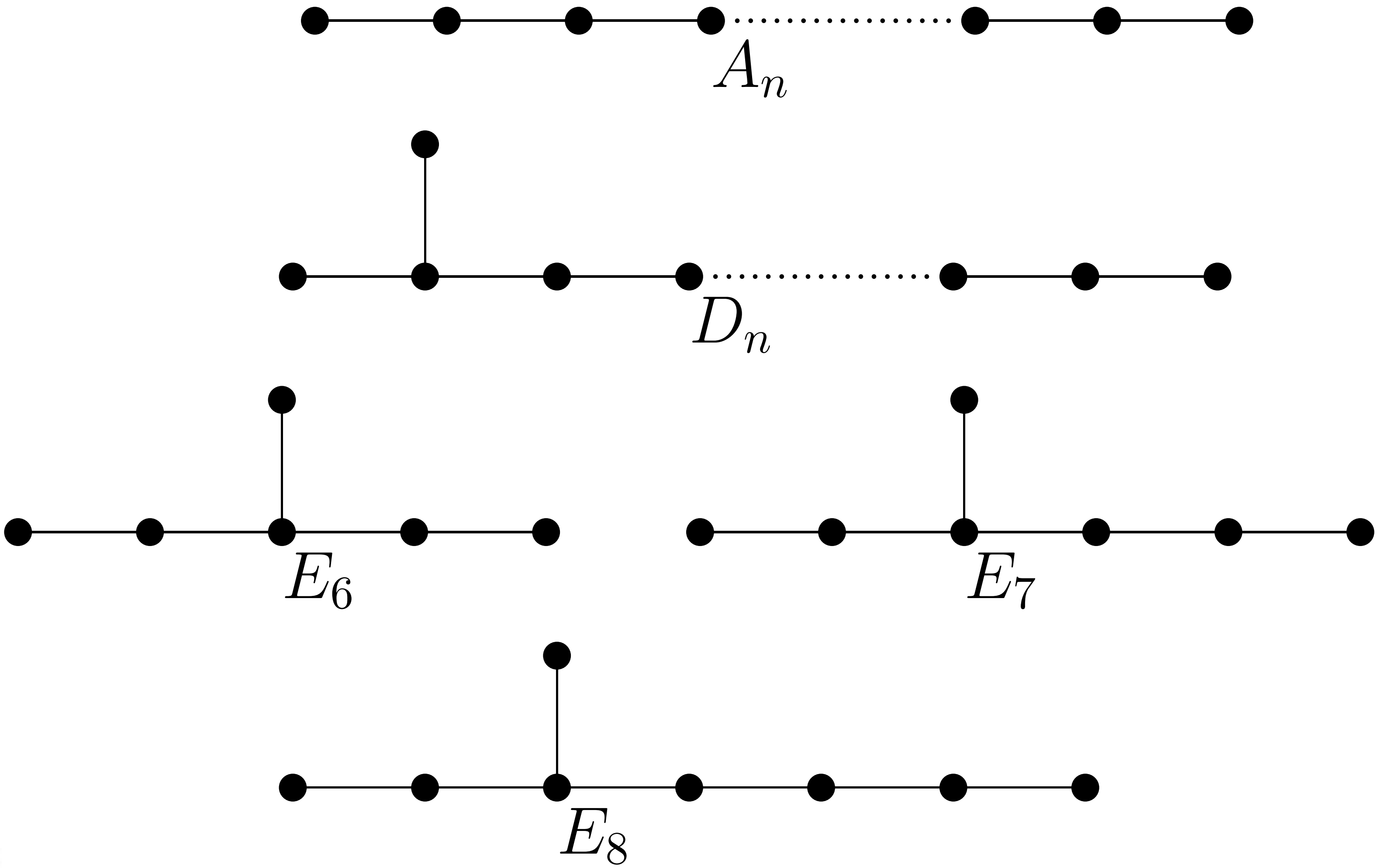
Apply Perron-Frobenius theorem to the adjacency matrix of a connected graph G.

- The eigenvector for $\rho(G)$ can be chosen so that all entries are positive.
- If α is a positive vector corresponding to the eigenvector λ, then $\rho(G) = \lambda$.
- For any proper subgraph H of G, we have

$$\rho(H) < \rho(G).$$

Graphs with $\rho(G) < 2$

Smith [1970]: $\rho(G) < 2$ if and only if G is a simple-laced Dynkin diagram.



Dynkin diagrams

- In the theory of Lie groups and Lie algebras, the simple Lie algebras are classified by Dynkin diagrams of their root systems.

Dynkin diagrams

- In the theory of Lie groups and Lie algebras, the simple Lie algebras are classified by Dynkin diagrams of their root systems.
- There are four infinite families (A_n, B_n, C_n, and D_n), and five exceptional cases (E_6, E_7, E_8, F_4, and G_2).

Dynkin diagrams

- In the theory of Lie groups and Lie algebras, the simple Lie algebras are classified by Dynkin diagrams of their root systems.
- There are four infinite families (A_n, B_n, C_n, and D_n), and five exceptional cases (E_6, E_7, E_8, F_4, and G_2).
- If all roots have the same length, then the root system is said to be simply laced; this occurs in the cases A, D and E.

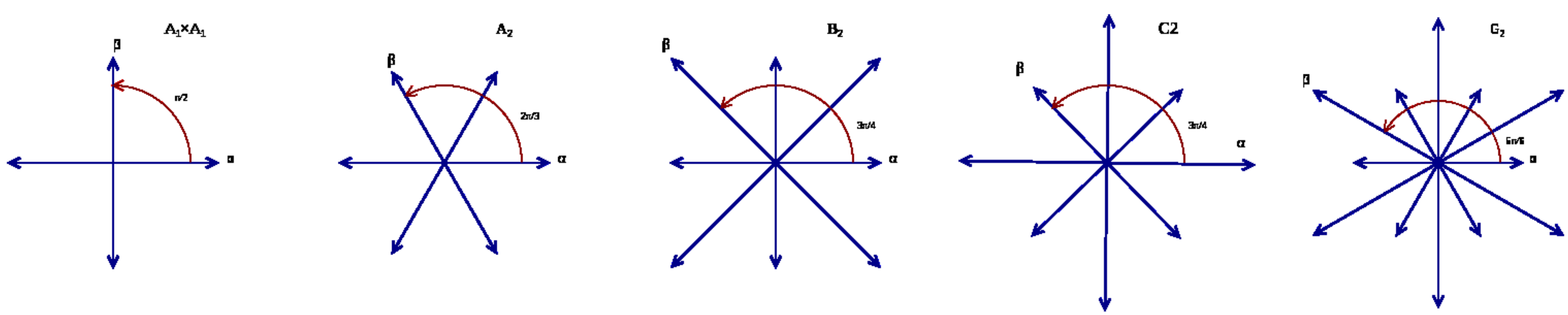
Dynkin diagrams

- In the theory of Lie groups and Lie algebras, the simple Lie algebras are classified by Dynkin diagrams of their root systems.
- There are four infinite families (A_n, B_n, C_n, and D_n), and five exceptional cases (E_6, E_7, E_8, F_4, and G_2).
- If all roots have the same length, then the root system is said to be simply laced; this occurs in the cases A, D and E.
- Smith's theorem gives an equivalent graph-theory definition for the simply-laced Dynkin diagrams.

Root system

A root system in $\mathbb{R}^n$ is a finite set Φ of non-zero vectors (called roots) that satisfy the following conditions:

- The roots span $\mathbb{R}^n$.
- The only scalar multiples of a root $x \in \Phi$ that belong to Φ are x itself and $-x$.
- For every root $x \in \Phi$, the set Φ is closed under reflection through the hyperplane perpendicular to x.
- If x and y are roots in Φ, then the projection of y onto the line through x is a half-integral multiple of x.



Connection

$$\rho(A) < 2$$

Connection

$$\rho(A) < 2 \Leftrightarrow$$

$I - \frac{1}{2}A$ is positive definite.

Connection

$$\rho(A) < 2 \Leftrightarrow$$

$I - \frac{1}{2}A$ is positive definite. $\Leftrightarrow$

Write $I - \frac{1}{2}A = BB'$.

Connection

$$\rho(A) < 2 \Leftrightarrow$$

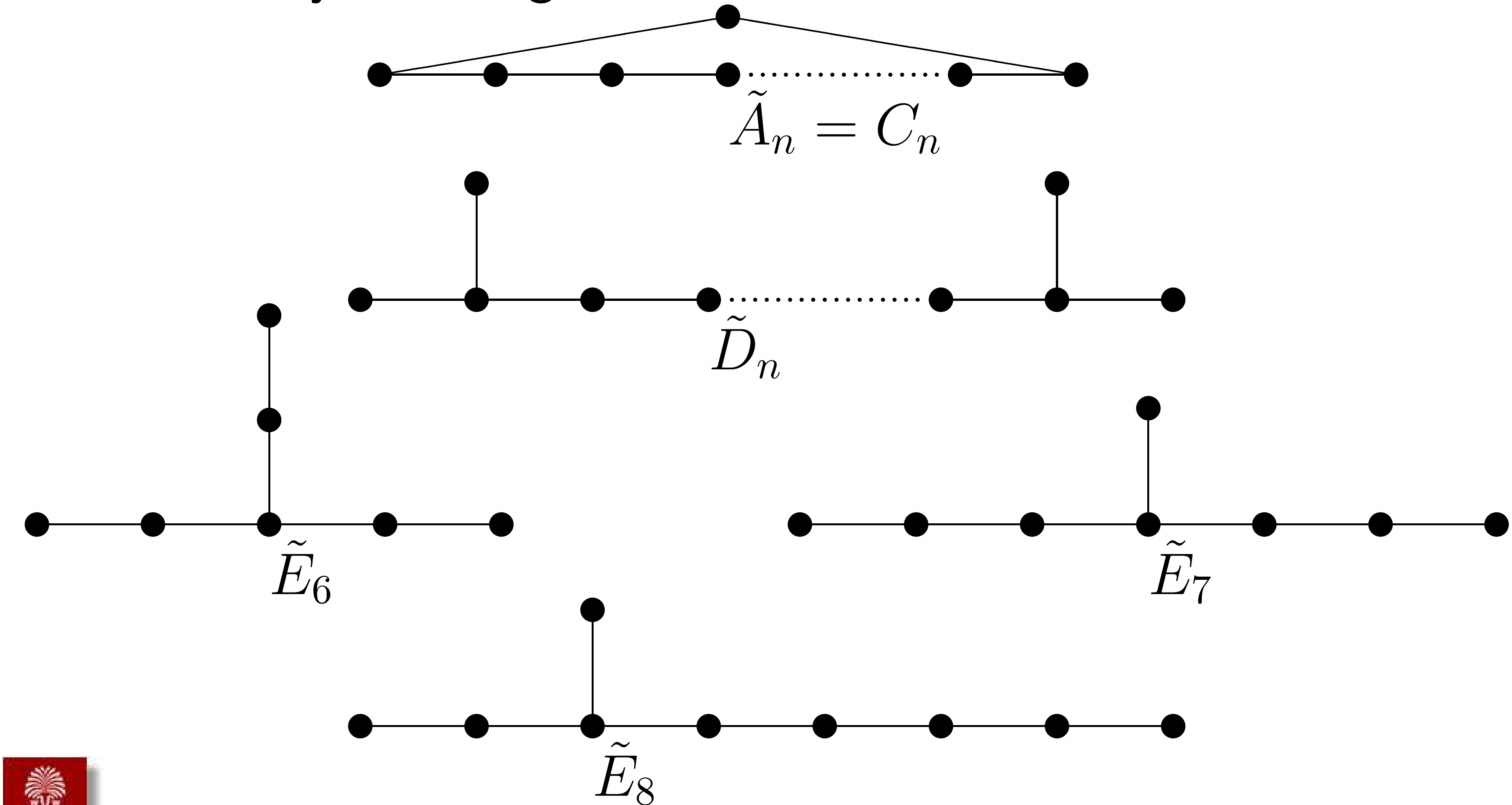
$I - \frac{1}{2}A$ is positive definite. $\Leftrightarrow$

Write $I - \frac{1}{2}A = BB'$. $\Leftrightarrow$

Let $\alpha_1, \ldots, \alpha_n$ be the column vector of B.
Then $\alpha_1, \ldots, \alpha_n$ forms a base of a root system.

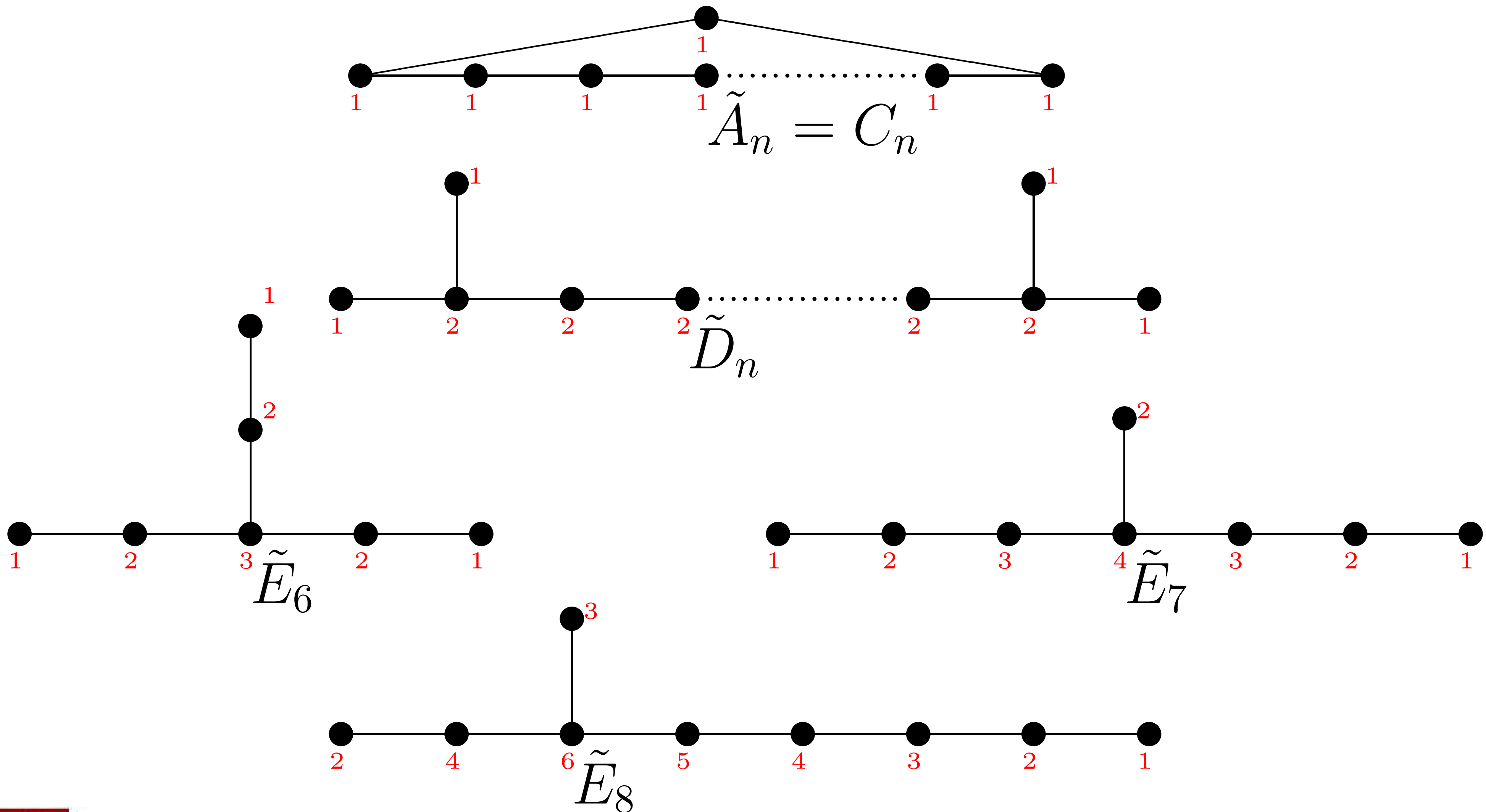
Graphs with $\rho(G) = 2$

Smith [1970]: $\rho(G) = 2$ if and only if G is a simple extended Dynkin diagram.



Proof of Smith's theorem

First, we show that $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ all have eigenvalue 2 with the positive eigenvectors below:



Continue

By Perron-Frobenius' theorem, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ all have spectral radius 2. Since A_n, D_n, E_6, E_7, and E_8 are proper subgraphs, their spectral radii are less than 2.

Continue

By Perron-Frobenius' theorem, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ all have spectral radius 2. Since A_n, D_n, E_6, E_7, and E_8 are proper subgraphs, their spectral radii are less than 2.

Now we show that the only connected graphs G with $\rho(G) \leq 2$ are in Smith's list.

Continue

By Perron-Frobenius' theorem, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ all have spectral radius 2. Since A_n, D_n, E_6, E_7, and E_8 are proper subgraphs, their spectral radii are less than 2.

Now we show that the only connected graphs G with $\rho(G) \leq 2$ are in Smith's list.

If G contains a cycle C, then $\rho(G) > 2$ unless $G = C$.

Continue

By Perron-Frobenius' theorem, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ all have spectral radius 2. Since A_n, D_n, E_6, E_7, and E_8 are proper subgraphs, their spectral radii are less than 2.

Now we show that the only connected graphs G with $\rho(G) \leq 2$ are in Smith's list.

If G contains a cycle C, then $\rho(G) > 2$ unless $G = C$. We can assume G is a tree.

Continue

By Perron-Frobenius' theorem, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ all have spectral radius 2. Since A_n, D_n, E_6, E_7, and E_8 are proper subgraphs, their spectral radii are less than 2.

Now we show that the only connected graphs G with $\rho(G) \leq 2$ are in Smith's list.

If G contains a cycle C, then $\rho(G) > 2$ unless $G = C$. We can assume G is a tree.

If there is a vertex of degree at least 4, then $\rho(G) > 2$, unless $G = D_5$.

Continue

By Perron-Frobenius' theorem, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ all have spectral radius 2. Since A_n, D_n, E_6, E_7, and E_8 are proper subgraphs, their spectral radii are less than 2.

Now we show that the only connected graphs G with $\rho(G) \leq 2$ are in Smith's list.

If G contains a cycle C, then $\rho(G) > 2$ unless $G = C$. We can assume G is a tree.

If there is a vertex of degree at least 4, then $\rho(G) > 2$, unless $G = D_5$. We can assume the degrees of G is at most 3.

Continue

By Perron-Frobenius' theorem, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ all have spectral radius 2. Since A_n, D_n, E_6, E_7, and E_8 are proper subgraphs, their spectral radii are less than 2.

Now we show that the only connected graphs G with $\rho(G) \leq 2$ are in Smith's list.

If G contains a cycle C, then $\rho(G) > 2$ unless $G = C$. We can assume G is a tree.

If there is a vertex of degree at least 4, then $\rho(G) > 2$, unless $G = D_5$. We can assume the degrees of G is at most 3.

If there are two vertices of degree 3, then G contains a subgraph $\tilde{D}_*$. Hence $\rho(G) > 2$, unless $G = \tilde{D}_n$.

Continue

If G has one vertex of degree 3, let i, j, k (say $i \leq j \leq k$) be the length of three paths attached to v. Write $G = E_{i,j,k}$.

- If $i \geq 2$, then $\rho(G) > \rho(E_{2,2,2}) = 2$ unless $G = E_{2,2,2} = \tilde{E}_6$.

Continue

If G has one vertex of degree 3, let i, j, k (say $i \leq j \leq k$) be the length of three paths attached to v. Write $G = E_{i,j,k}$.

- If $i \geq 2$, then $\rho(G) > \rho(E_{2,2,2}) = 2$ unless $G = E_{2,2,2} = \tilde{E}_6$. We can assume $i = 1$.

Continue

If G has one vertex of degree 3, let i, j, k (say $i \leq j \leq k$) be the length of three paths attached to v. Write $G = E_{i,j,k}$.

- If $i \geq 2$, then $\rho(G) > \rho(E_{2,2,2}) = 2$ unless $G = E_{2,2,2} = \tilde{E}_6$. We can assume $i = 1$.
- If $i = 1$ and $j \geq 3$, then $\rho(G) > \rho(E_{1,3,3}) = 2$ unless $G = E_{1,3,3} = \tilde{E}_7$.

Continue

If G has one vertex of degree 3, let i, j, k (say $i \leq j \leq k$) be the length of three paths attached to v. Write $G = E_{i,j,k}$.

- If $i \geq 2$, then $\rho(G) > \rho(E_{2,2,2}) = 2$ unless $G = E_{2,2,2} = \tilde{E}_6$. We can assume $i = 1$.
- If $i = 1$ and $j \geq 3$, then $\rho(G) > \rho(E_{1,3,3}) = 2$ unless $G = E_{1,3,3} = \tilde{E}_7$. We can assume $i = 1$ and $j = 1$ or 2.

Continue

If G has one vertex of degree 3, let i, j, k (say $i \leq j \leq k$) be the length of three paths attached to v. Write $G = E_{i,j,k}$.

- If $i \geq 2$, then $\rho(G) > \rho(E_{2,2,2}) = 2$ unless $G = E_{2,2,2} = \tilde{E}_6$. We can assume $i = 1$.
- If $i = 1$ and $j \geq 3$, then $\rho(G) > \rho(E_{1,3,3}) = 2$ unless $G = E_{1,3,3} = \tilde{E}_7$. We can assume $i = 1$ and $j = 1$ or 2.
- If $i = 1$ and $j = 1$, then $G = D_n$.

Continue

If G has one vertex of degree 3, let i, j, k (say $i \leq j \leq k$) be the length of three paths attached to v. Write $G = E_{i,j,k}$.

- If $i \geq 2$, then $\rho(G) > \rho(E_{2,2,2}) = 2$ unless $G = E_{2,2,2} = \tilde{E}_6$. We can assume $i = 1$.
- If $i = 1$ and $j \geq 3$, then $\rho(G) > \rho(E_{1,3,3}) = 2$ unless $G = E_{1,3,3} = \tilde{E}_7$. We can assume $i = 1$ and $j = 1$ or 2.
- If $i = 1$ and $j = 1$, then $G = D_n$.
- If $i = 1$, $j = 2$, and $k \geq 5$, then $\rho(G) > \rho(E_{1,2,5}) = 2$ unless $G = E_{1,2,5} = \tilde{E}_8$.

Continue

If G has one vertex of degree 3, let i, j, k (say $i \leq j \leq k$) be the length of three paths attached to v. Write $G = E_{i,j,k}$.

- If $i \geq 2$, then $\rho(G) > \rho(E_{2,2,2}) = 2$ unless $G = E_{2,2,2} = \tilde{E}_6$. We can assume $i = 1$.
- If $i = 1$ and $j \geq 3$, then $\rho(G) > \rho(E_{1,3,3}) = 2$ unless $G = E_{1,3,3} = \tilde{E}_7$. We can assume $i = 1$ and $j = 1$ or 2.
- If $i = 1$ and $j = 1$, then $G = D_n$.
- If $i = 1$, $j = 2$, and $k \geq 5$, then $\rho(G) > \rho(E_{1,2,5}) = 2$ unless $G = E_{1,2,5} = \tilde{E}_8$.
- If $i = 1$, $j = 2$, and $k = 2, 3, 4$, then $G = E_6$, E_7, and E_8.

Continue

If G has one vertex of degree 3, let i, j, k (say $i \leq j \leq k$) be the length of three paths attached to v. Write $G = E_{i,j,k}$.

- If $i \geq 2$, then $\rho(G) > \rho(E_{2,2,2}) = 2$ unless $G = E_{2,2,2} = \tilde{E}_6$. We can assume $i = 1$.
- If $i = 1$ and $j \geq 3$, then $\rho(G) > \rho(E_{1,3,3}) = 2$ unless $G = E_{1,3,3} = \tilde{E}_7$. We can assume $i = 1$ and $j = 1$ or 2.
- If $i = 1$ and $j = 1$, then $G = D_n$.
- If $i = 1$, $j = 2$, and $k \geq 5$, then $\rho(G) > \rho(E_{1,2,5}) = 2$ unless $G = E_{1,2,5} = \tilde{E}_8$.
- If $i = 1$, $j = 2$, and $k = 2, 3, 4$, then $G = E_6$, E_7, and E_8.

If all degrees of G are at most 2, then $G = A_n$. $\square$

Rayleigh quotient

For a real symmetric matrix $A = (a_{ij})$, consider the following optimization problem.

Rayleigh quotient

For a real symmetric matrix $A = (a_{ij})$, consider the following optimization problem.

maximize $\sum_{i,j=1}^{n} a_{ij} x_i x_j$
subject to $\sum_{i=1}^{n} x_i^2 = 1.$

Rayleigh quotient

For a real symmetric matrix $A = (a_{ij})$, consider the following optimization problem.

maximize $\sum_{i,j=1}^{n} a_{ij}x_i x_j$
subject to $\sum_{i=1}^{n} x_i^2 = 1$.

Suppose the maximum value λ is achieved at x^*. Then

- $\sum_{j=1}^{n} a_{ij}x_j^* = \lambda x_i^*$ for each i. I.e., $Ax^* = \lambda x^*$.

Rayleigh quotient

For a real symmetric matrix $A = (a_{ij})$, consider the following optimization problem.

maximize $\sum_{i,j=1}^{n} a_{ij} x_i x_j$
subject to $\sum_{i=1}^{n} x_i^2 = 1$.

Suppose the maximum value λ is achieved at x^*. Then

- $\sum_{j=1}^{n} a_{ij} x_j^* = \lambda x_i^*$ for each i. I.e., $Ax^* = \lambda x^*$.
- $\lambda = \rho(A)$.

Hypermatrix

A real non-negative hypermatrix $A = (a_{i_1 i_2 \cdots i_r})$ is called symmetric if it $a_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_r)} = a_{i_1 i_2 \cdots i_r}$ for any permutation σ of indices.

Consider the following optimization problem.

Hypermatrix

A real non-negative hypermatrix $A = (a_{i_1 i_2 \cdots i_r})$ is called symmetric if it $a_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_r)} = a_{i_1 i_2 \cdots i_r}$ for any permutation σ of indices.
Consider the following optimization problem.

maximize $\sum_{i_1,\ldots,i_r=1}^{n} a_{i_1 i_2 \cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$
subject to $\sum_{i=1}^{n} x_i^p = 1.$

Hypermatrix

A real non-negative hypermatrix $A = (a_{i_1 i_2 \cdots i_r})$ is called symmetric if it $a_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_r)} = a_{i_1 i_2 \cdots i_r}$ for any permutation σ of indices.

Consider the following optimization problem.

maximize $\sum_{i_1,\ldots,i_r=1}^{n} a_{i_1 i_2 \cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$
subject to $\sum_{i=1}^{n} x_i^p = 1.$

Suppose the maximum value λ is achieved at x^*. Then

- $\sum_{i_2,\ldots,i_r=1}^{n} a_{i i_2 \cdots i_r} x^*_{i_2} \cdots x^*_{i_r} = \lambda r {x_i^*}^{r-1}$ for each i such that $x_i \neq 0$.

Hypermatrix

A real non-negative hypermatrix $A = (a_{i_1 i_2 \cdots i_r})$ is called symmetric if it $a_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_r)} = a_{i_1 i_2 \cdots i_r}$ for any permutation σ of indices.

Consider the following optimization problem.

maximize $\sum_{i_1,\ldots,i_r=1}^{n} a_{i_1 i_2 \cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$
subject to $\sum_{i=1}^{n} x_i^p = 1.$

Suppose the maximum value λ is achieved at x^*. Then

- $\sum_{i_2,\ldots,i_r=1}^{n} a_{i i_2 \cdots i_r} x^*_{i_2} \cdots x^*_{i_r} = \lambda r x_i^{*\,r-1}$ for each i such that $x_i \neq 0$.
- λ is called (the largest) p-spectrum of A.

Part II: hypergraphs

The r-uniform hypergraph (or r-graph) $H = (V, E)$:

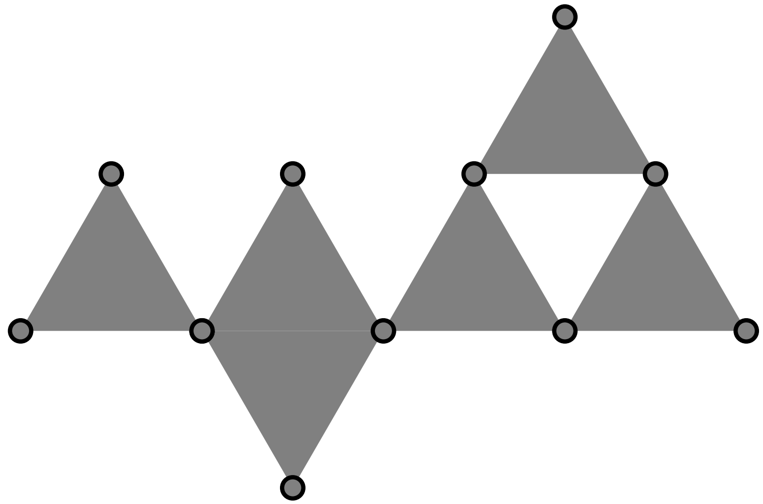
- V: the vertex set.
- $E \subset \binom{V}{r}$: the set of (hyper)edges.

Part II: hypergraphs

The r-uniform hypergraph (or r-graph) $H = (V, E)$:

- V: the vertex set.
- $E \subset \binom{V}{r}$: the set of (hyper)edges.

An example of 3-graph:



H has 11 vertices and 6 edges.

Basic concepts

- A walk: $v_0e_1v_1e_2v_2\cdots, e_lv_l$ where $v_{i-1}, v_i \in e_i$ for $1 \leq i \leq l$.
- A path: a walk so that all v_i's e_i's are distinct.
- A closed walk: a walk with $v_0 = v_l$.
- A cycle: a path with $v_0 = v_l$.

Basic concepts

- A walk: $v_0 e_1 v_1 e_2 v_2 \cdots, e_l v_l$ where $v_{i-1}, v_i \in e_i$ for $1 \leq i \leq l$.
- A path: a walk so that all v_i's e_i's are distinct.
- A closed walk: a walk with $v_0 = v_l$.
- A cycle: a path with $v_0 = v_l$.

H is connected if for any two vertices u, v, there is a uv-path: $v_0 e_1 v_1 e_2 v_2 \cdots, e_l v_l$ so that $v_0 = u$ and $v_l = v$.

Basic concepts

- A walk: $v_0e_1v_1e_2v_2\cdots,e_lv_l$ where $v_{i-1},v_i\in e_i$ for $1\le i\le l$.
- A path: a walk so that all v_i's e_i's are distinct.
- A closed walk: a walk with $v_0=v_l$.
- A cycle: a path with $v_0=v_l$.

H is connected if for any two vertices u,v, there is a uv-path: $v_0e_1v_1e_2v_2\cdots,e_lv_l$ so that $v_0=u$ and $v_l=v$.

H is simple if $|e\cap e'|\le 1$ for any $e,e'\in E$.

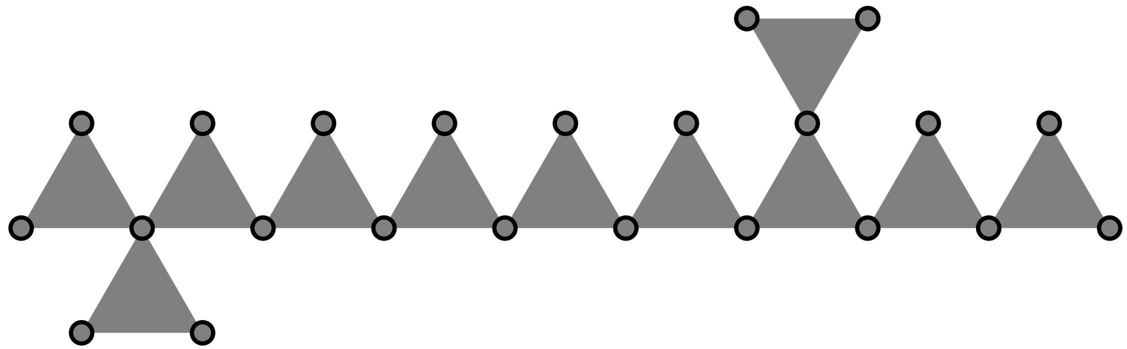
Basic concepts

- A walk: $v_0e_1v_1e_2v_2\cdots,e_lv_l$ where $v_{i-1},v_i\in e_i$ for $1\le i\le l$.
- A path: a walk so that all v_i's e_i's are distinct.
- A closed walk: a walk with $v_0=v_l$.
- A cycle: a path with $v_0=v_l$.

H is connected if for any two vertices u,v, there is a uv-path: $v_0e_1v_1e_2v_2\cdots,e_lv_l$ so that $v_0=u$ and $v_l=v$.

H is simple if $|e\cap e'|\le 1$ for any $e,e'\in E$.

A hypertree is an acyclic connected hypergraph.



Spectral radius of H

The spectral radius of H is

$$\rho(H) = r! \max_{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n-0} \frac{\sum_{\{i_1,i_2,\ldots,i_r\}\in E} x_{i_1}x_{i_2}\cdots x_{i_r}}{\sum_{v\in V} x_v^r}$$

$$= r! \max_{\|\mathbf{x}\|_r=1} \sum_{\{i_1,i_2,\ldots,i_r\}\in E} x_{i_1}x_{i_2}\cdots x_{i_r}.$$

Spectral radius of H

The spectral radius of H is

$$\rho(H) = r! \max_{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n - 0} \frac{\sum_{\{i_1,i_2,\ldots,i_r\}\in E} x_{i_1}x_{i_2}\cdots x_{i_r}}{\sum_{v\in V} x_v^r}$$
$$= r! \max_{\|\mathbf{x}\|_r=1} \sum_{\{i_1,i_2,\ldots,i_r\}\in E} x_{i_1}x_{i_2}\cdots x_{i_r}.$$

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called an eigenvector of $\rho(H)$ if the above maximum reaches at $\mathbf{x}$.

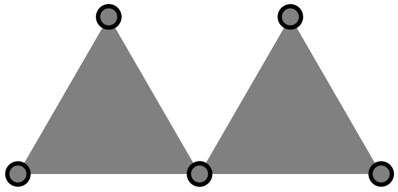
Spectral radius of H

The spectral radius of H is

$$\rho(H) = r! \max_{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n-0} \frac{\sum_{\{i_1,i_2,\ldots,i_r\}\in E} x_{i_1}x_{i_2}\cdots x_{i_r}}{\sum_{v\in V} x_v^r}$$

$$= r! \max_{\|\mathbf{x}\|_r=1} \sum_{\{i_1,i_2,\ldots,i_r\}\in E} x_{i_1}x_{i_2}\cdots x_{i_r}.$$

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called an eigenvector of $\rho(H)$ if the above maximum reaches at $\mathbf{x}$.



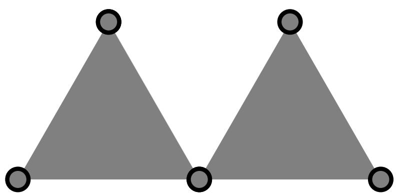
$\rho(H)$ maximizes $3!(x_1x_2x_3 + x_3x_4x_5)$ subject to $x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 1$.

Spectral radius of H

The spectral radius of H is

$$\rho(H) = r! \max_{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n-0} \frac{\sum_{\{i_1,i_2,\ldots,i_r\}\in E} x_{i_1}x_{i_2}\cdots x_{i_r}}{\sum_{v\in V} x_v^r}$$
$$= r! \max_{\|\mathbf{x}\|_r=1} \sum_{\{i_1,i_2,\ldots,i_r\}\in E} x_{i_1}x_{i_2}\cdots x_{i_r}.$$

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called an eigenvector of $\rho(H)$ if the above maximum reaches at $\mathbf{x}$.



$\rho(H)$ maximizes $3!(x_1x_2x_3 + x_3x_4x_5)$ subject to $x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 1$.

$$\rho(H) = 2\sqrt[3]{2}, \text{ eigenvector } \mathbf{x} = (1, 1, \sqrt[3]{2}, 1, 1).$$

Spectra of hypergraphs

- Spectrum of real symmetric hypermatrix
  - Qi [2005]
  - Chang-Pearson-Zhang [2008]
  - Fridland-Gaubert [2010]
  - Friedland-Gaubert-Han [2013]
- Spectrum of adjacency tensor of hypergraphs
  - Cooper and Dutle [2012]
  - Keevash-Lenz-Mubayi [2013+]
  - Nikiforov [2013+]
- Laplacian of hypergraphs
  - Chung [1993]
  - Rodríguz [2009]
  - Lu-Peng [2012]

Perron-Frobenius theory

Perron-Frobenius theorem for graphs: Let A be the adjacency matrix of a connected graph G. Then

- A has a unique (up to a scale) positive eigenvector α.
- The eigenvector corresponds to the largest eigenvalue of A.
- Any nonnegative eigenvector must be positive.

Perron-Frobenius theory

Perron-Frobenius theorem for graphs: Let A be the adjacency matrix of a connected graph G. Then

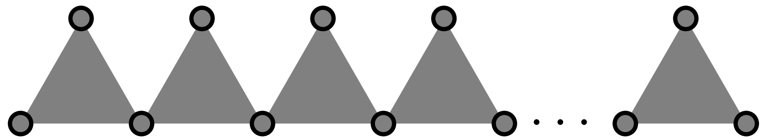
- A has a unique (up to a scale) positive eigenvector α.
- The eigenvector corresponds to the largest eigenvalue of A.
- Any nonnegative eigenvector must be positive.

Perron-Frobenius theorem for hypergraphs [Cooper-Dutle 2012,Fridland-Gaubert-2010, Nikiforov 2013+] If H is connected, then there is a unique positive eigenvector (up to a scale) for $\rho(H)$.

Limit point of spectral radius

Let $A_n^{(r)}$ denote the simple r-uniform path on n edges and

$$\rho_r := \lim_{n\to\infty} \rho(A_n^{(r)}).$$



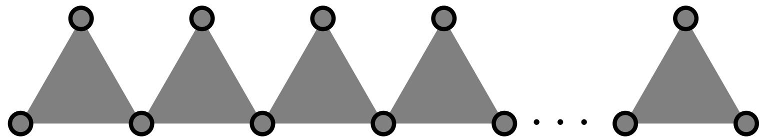
$A_n^{(3)}$

It is not hard to show $\rho_r = (r-1)!\sqrt[r]{4}$.

Limit point of spectral radius

Let $A_n^{(r)}$ denote the simple r-uniform path on n edges and

$$\rho_r := \lim_{n\to\infty} \rho(A_n^{(r)}).$$



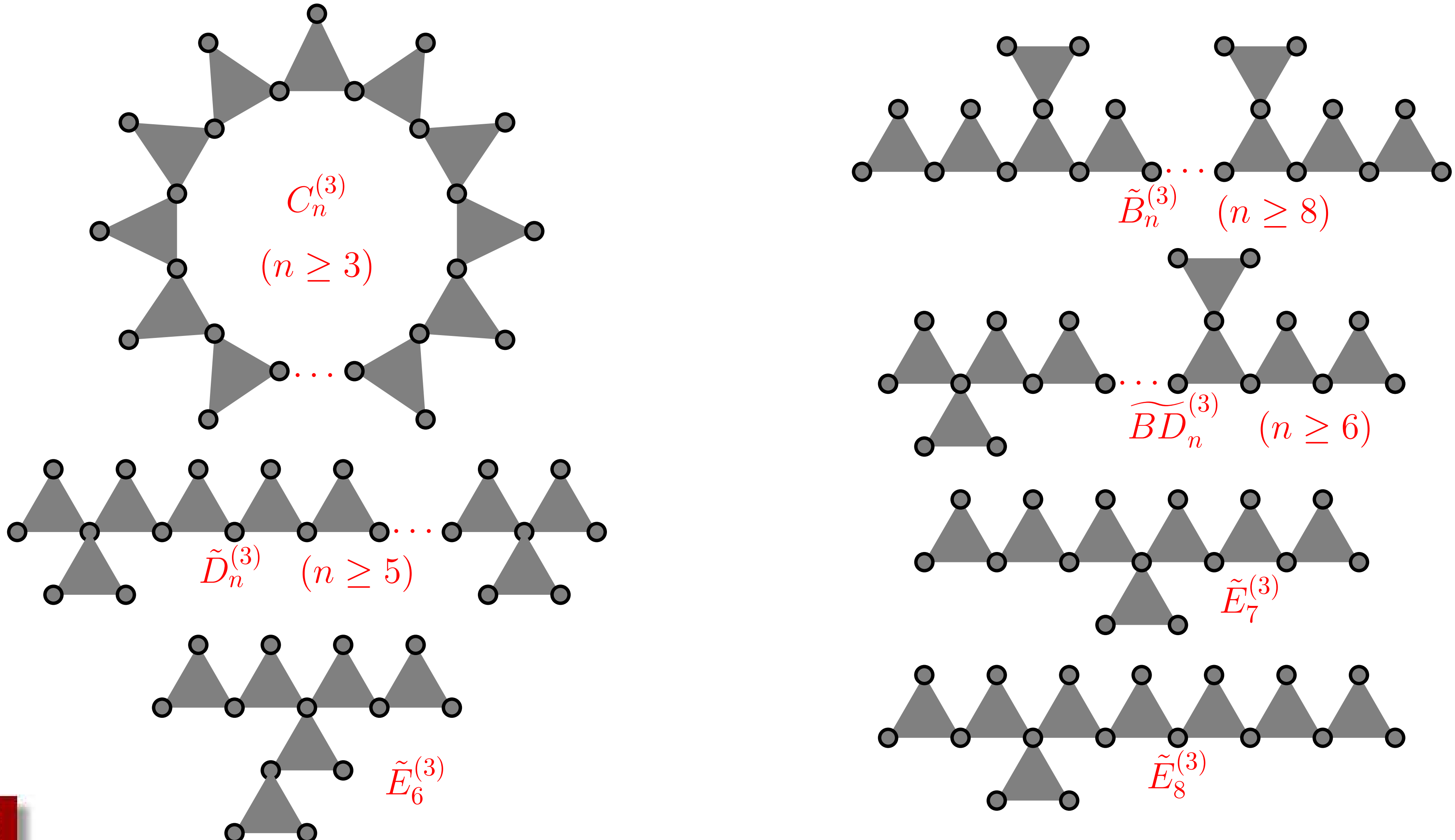
$A_n^{(3)}$

It is not hard to show $\rho_r = (r-1)!\sqrt[r]{4}$.

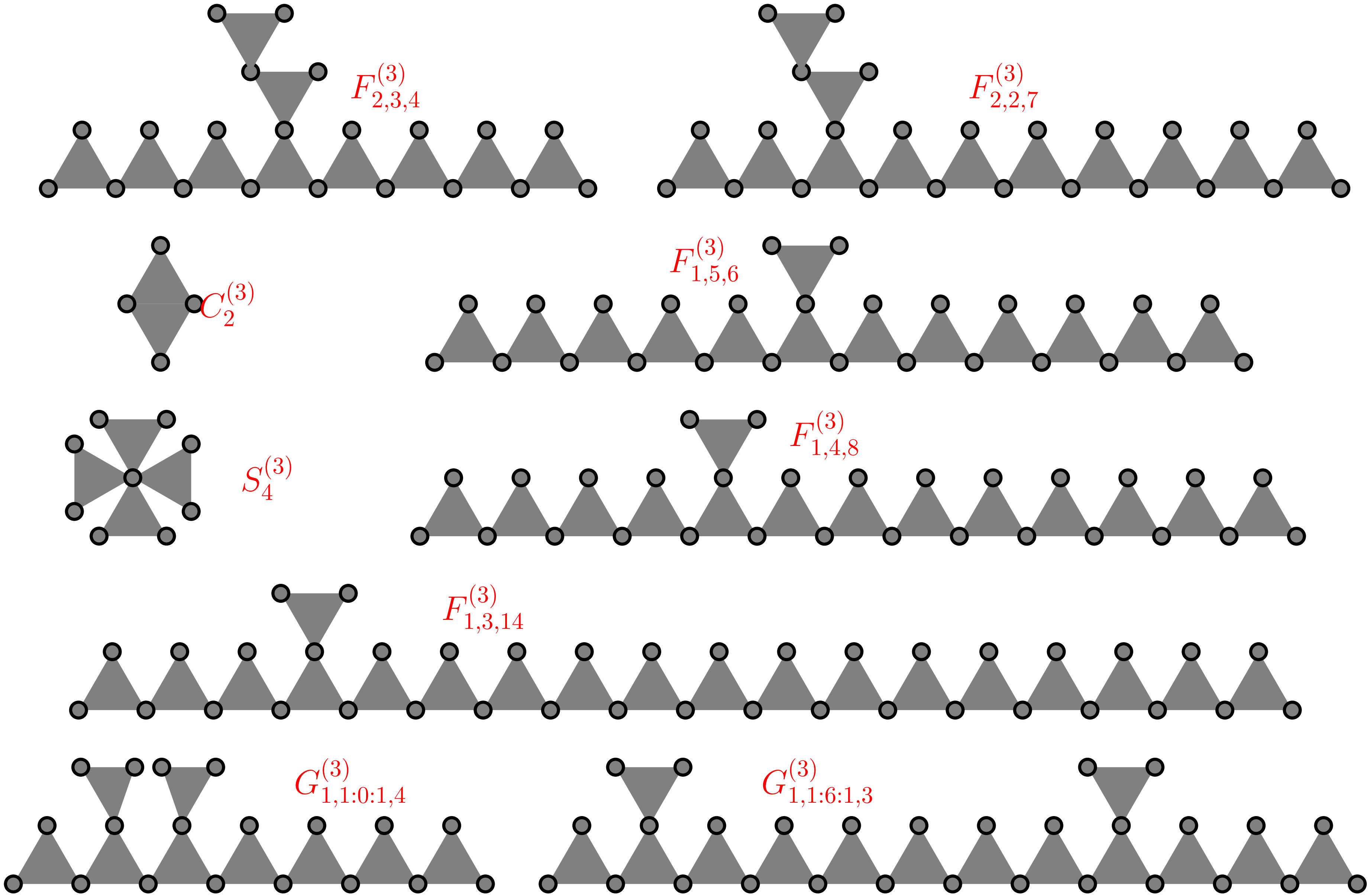
Question: Can we classify all r-uniform hypergraphs H with $\rho(H) \leq \rho_r$?

Classification for $\rho(H) = \rho_3$

Theorem [Lu-Man, 2013+] The complete list of all connected 3-uniform hypergraph H with $\rho(H) = \rho_3$ consists of 4 families and 12 exceptional hypergraphs.

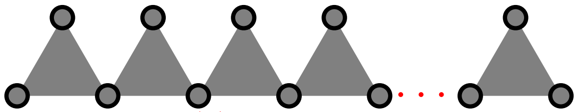


Continue

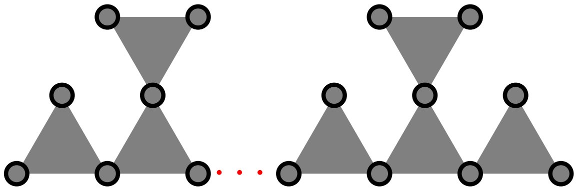


Classification for $\rho(H) < \rho_3$

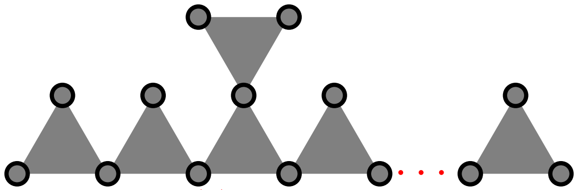
Theorem [Lu-Man, 2013+] The complete list of all connected 3-uniform hypergraph H with $\rho(H) < \rho_3$ consists of 7 families and 31 exceptional hypergraphs.



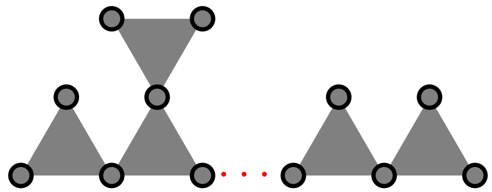
$A_n^{(3)}$ $(n \geq 1)$



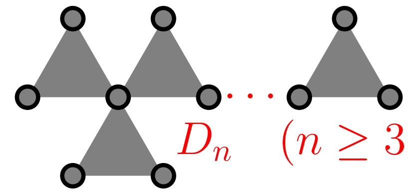
$B'^{(3)}_n$ $(n \geq 6)$



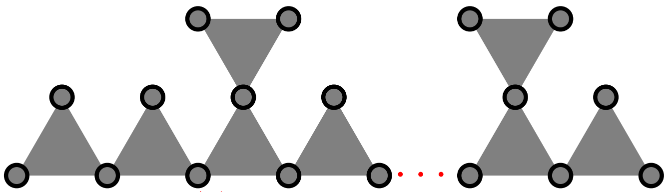
$B_n^{(3)}$ $(n \geq 5)$



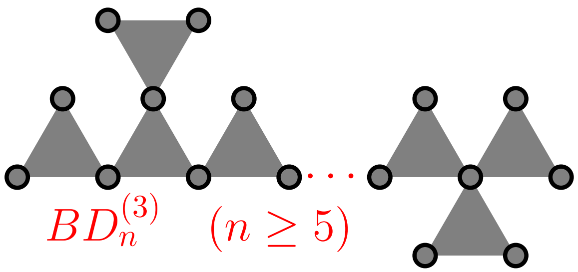
$D'^{(3)}_n$ $(n \geq 4)$



D_n $(n \geq 3)$



$\bar{B}_n^{(3)}$ $(n \geq 7)$

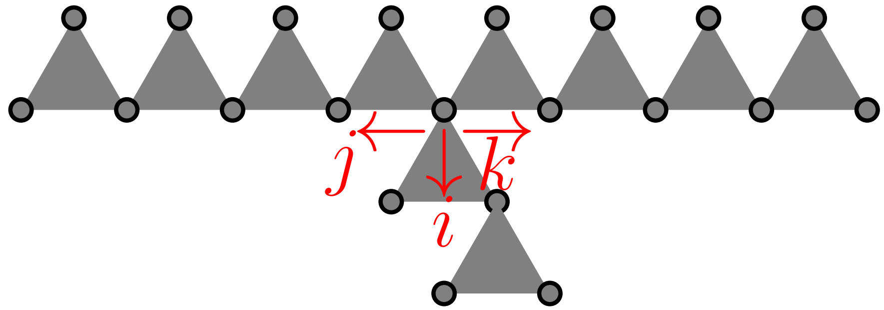


$BD_n^{(3)}$ $(n \geq 5)$

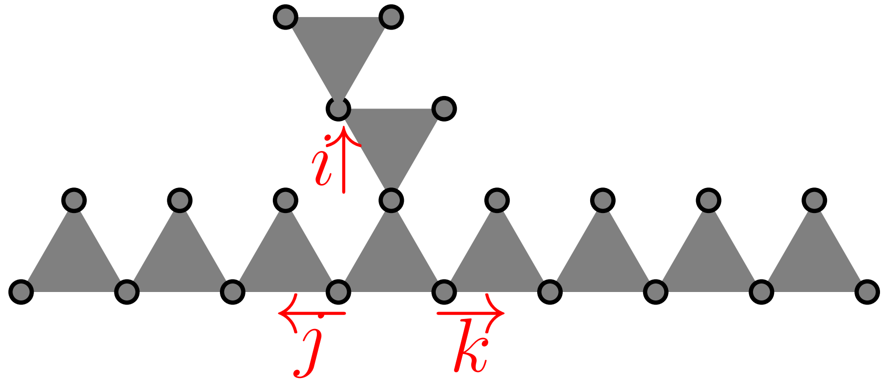
- $E_6^{(3)}$, $E_7^{(3)}$, $E_8^{(3)}$, $F_{2,3,3}^{(3)}$, $F_{1,5,5}^{(3)}$.
- $F_{2,2,k}^{(3)}$ (for $2 \leq k \leq 6$)
- $F_{1,3,k}^{(3)}$ (for $3 \leq k \leq 13$)
- $F_{1,4,k}^{(3)}$ (for $4 \leq k \leq 7$).
- $G_{1,1:k:1,3}^{(3)}$ (for $0 \leq k \leq 5$).

Three families

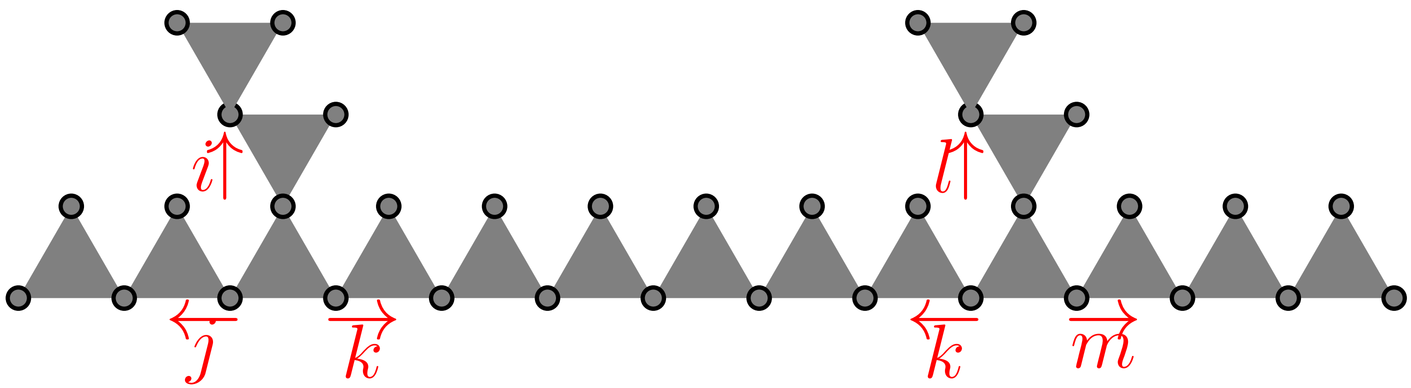
$E^{(3)}_{i,j,k}$:



$F^{(3)}_{i,j,k}$:



$G^{(3)}_{i,j:k:l,m}$:



Our method

Lemma [Lu-Man 2013+] An r-uniform hypergraph H has spectral radius $\rho(H) = (r-1)!\alpha^{-1/r}$ if and only if H has a consistently α-normal labeling.

Our method

Lemma [Lu-Man 2013+] An r-uniform hypergraph H has spectral radius $\rho(H) = (r-1)!\alpha^{-1/r}$ if and only if H has a consistently α-normal labeling.

A α-**normal** labeling assigns a positive number to each incidence relation (v, e) a value $B(v, e)$ satisfying

- $\prod_{v\colon v \in e} B(v, e) = \alpha$ for any edge e.
- $\sum_{e\colon v \in e} B(v, e) = 1$ for any vertex v.

Our method

Lemma [Lu-Man 2013+] An r-uniform hypergraph H has spectral radius $\rho(H) = (r-1)!\alpha^{-1/r}$ if and only if H has a consistently α-normal labeling.

A α-**normal** labeling assigns a positive number to each incidence relation (v, e) a value $B(v, e)$ satisfying

- $\prod_{v\colon v \in e} B(v, e) = \alpha$ for any edge e.
- $\sum_{e\colon v \in e} B(v, e) = 1$ for any vertex v.

B is called consistent if for any cycle $v_0 e_1 v_1 e_2 \ldots v_l$ $(v_l = v_0)$

$$\prod_{i=1}^{l} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1.$$

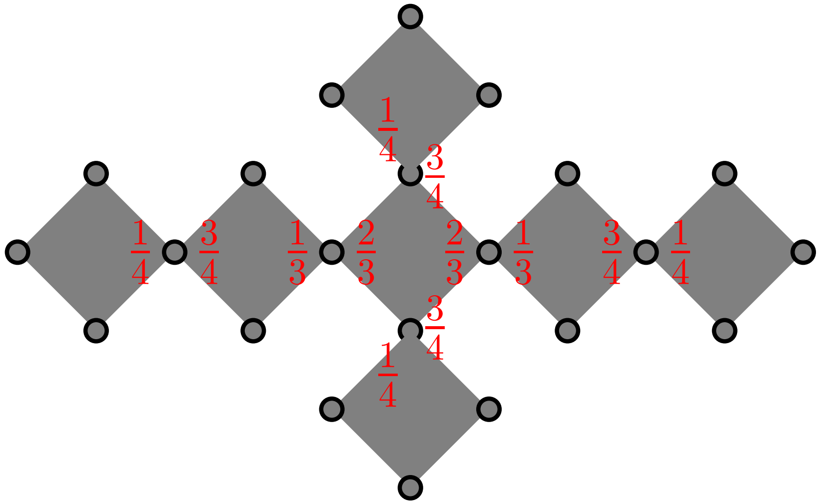
If H is a hypertree, then any α-normal labeling is automatically consistent.

$\frac{1}{4}$-labeling

Corollary: An r-uniform hypergraph H has spectral radius $\rho(H) = \rho_r$ if and only if H has a consistent $\frac{1}{4}$-normal labeling.

$\frac{1}{4}$-labeling

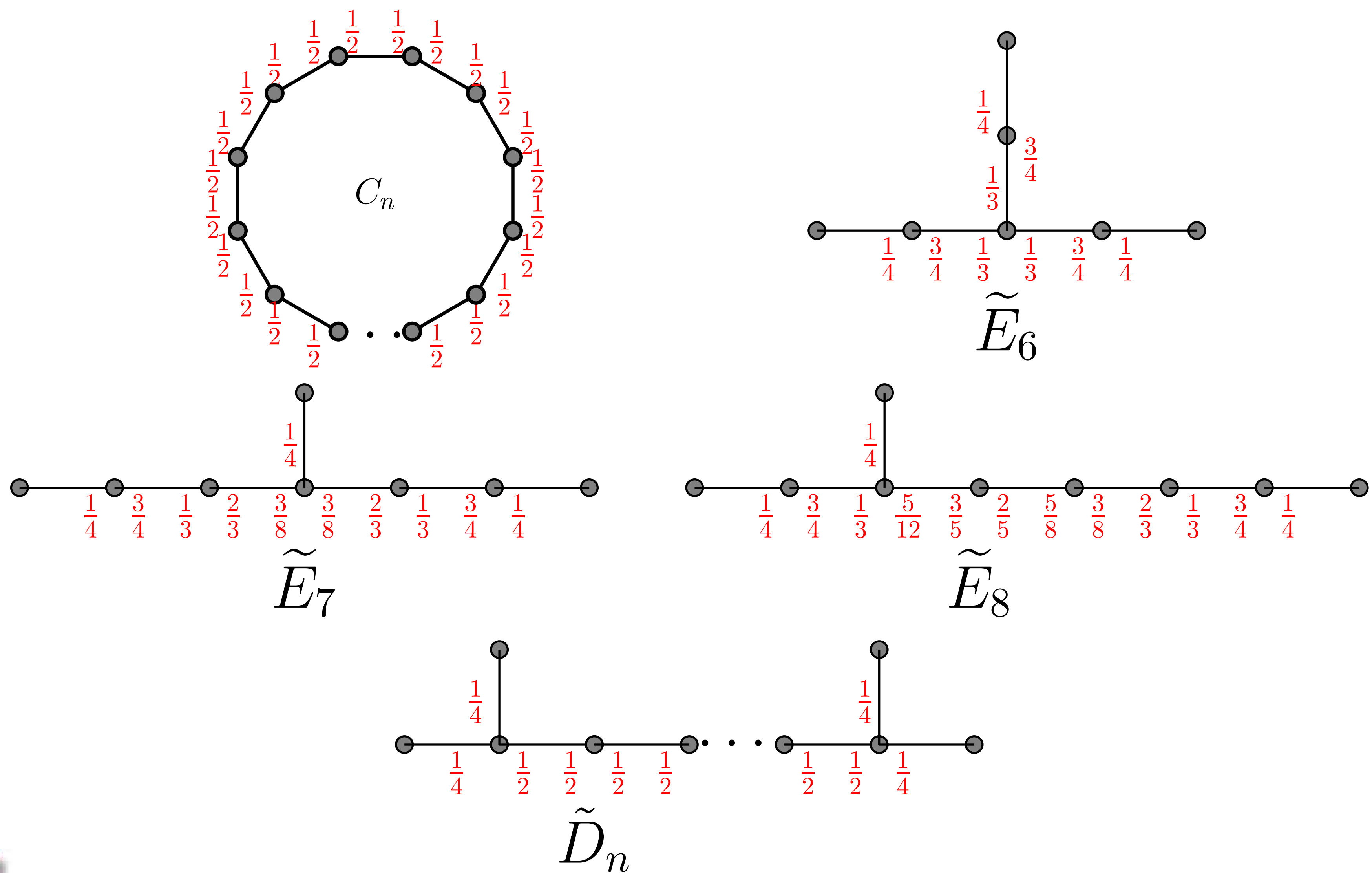
Corollary: An r-uniform hypergraph H has spectral radius $\rho(H) = \rho_r$ if and only if H has a consistent $\frac{1}{4}$-normal labeling.



An example of $\frac{1}{4}$-labeling. Here all leaf vertices are labeled by 1.

Graphs with $\rho(G) = 2$

$\tilde{A}_n$, $\tilde{D}_n$, E_6, E_7, and E_8, all have $\frac{1}{4}$-Labeling.



α-supernormal

A hypergraph H is called α-supernormal if there exists a weighted incidence matrix B satisfying

1. $\sum_{e:\, v\in e} B(v,e) \geq 1$, for any $v \in V(H)$.
2. $\prod_{v\in e} B(v,e) \leq \alpha$, for any $e \in E(H)$.

α-supernormal

A hypergraph H is called α-supernormal if there exists a weighted incidence matrix B satisfying

1. $\sum_{e:\, v\in e} B(v,e) \geq 1$, for any $v \in V(H)$.
2. $\prod_{v\in e} B(v,e) \leq \alpha$, for any $e \in E(H)$.

Lemma: Let H be an r-uniform hypergraph. If H is strictly and consistently α-super normal, then the spectral radius of H satisfies

$$\rho(H) > (r-1)!\alpha^{-\frac{1}{r}}.$$

α-subnormal

A hypergraph H is called α-subnormal if there exists a weighted incidence matrix B satisfying

1. $\sum_{e:\, v\in e} B(v,e) \leq 1$, for any $v \in V(H)$.
2. $\prod_{v\in e} B(v,e) \geq \alpha$, for any $e \in E(H)$.

α-subnormal

A hypergraph H is called α-subnormal if there exists a weighted incidence matrix B satisfying

1. $\sum_{e\colon v\in e} B(v,e) \leq 1$, for any $v \in V(H)$.
2. $\prod_{v\in e} B(v,e) \geq \alpha$, for any $e \in E(H)$.

Lemma: Let H be an r-uniform hypergraph. If H is α-subnormal, then the spectral radius of H satisfies

$$\rho(H) \leq (r-1)!\alpha^{-\frac{1}{r}}.$$

Moreover, if H is strictly α-subnormal then $\rho(H) < (r-1)!\alpha^{-\frac{1}{r}}$.

Proof of Main Lemma

"$\Longleftarrow$" Let $x := (x_1, ..., x_n)$ be the Perron-Frobenis eigenvector of H. Define B as follows:

$$B(v,e) = \begin{cases} \frac{(r-1)! \prod_{u\in e} x_u}{\rho(H) x_v^r} & \text{if } v \in e \\ 0 & \text{otherwise.} \end{cases}$$

From this definition, for any edge e, we have

$$\prod_{v\in e} B(v,e) = \prod_{v\in e} \frac{(r-1)! \prod_{u\in e} x_u}{\rho(H) x_v^r} = \left(\frac{(r-1)!}{\rho(H)} \right)^r = \alpha.$$

Continue

For any v,

$$\sum_e B(v,e) = \sum_{\{v,i_2,\cdots,i_r\}\in E(H)} \frac{(r-1)!\prod_{u\in e} x_u}{\rho(H)x_v^r} = \frac{\rho(H)}{\rho(H)} = 1.$$

Continue

For any v,

$$\sum_{e} B(v,e) = \sum_{\{v,i_2,\cdots,i_r\}\in E(H)} \frac{(r-1)!\prod_{u\in e} x_u}{\rho(H)x_v^r} = \frac{\rho(H)}{\rho(H)} = 1.$$

To show that B is consistent, for any cycle $v_0e_1v_1e_2\dots v_l$ ($v_l = v_0$), we have

$$\prod_{i=1}^{l} \frac{B(v_i,e_i)}{B(v_{i-1},e_i)} = \prod_{i=1}^{l} \frac{x_{v_{i-1}}^r}{x_{v_i}^r} = 1.$$

Continue

"$\Longrightarrow$" Let B be consistently α-normal. For any non-zero vector $\mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$, we have

$$\begin{aligned} r! \sum_{\{x_{v_1}, x_{v_2}, \ldots, x_{v_r}\} \in E(H)} x_{v_1} x_{v_2} \cdots x_{v_r} &= \frac{r!}{\alpha^{\frac{1}{r}}} \sum_{e \in E(H)} \prod_{v \in e} (B^{\frac{1}{r}}(v,e) x_v) \\ &\leq \frac{r!}{\alpha^{\frac{1}{r}}} \sum_{e \in E(H)} \frac{\sum_{v \in e} (B(v,e) x_v^r)}{r} \\ &= \frac{(r-1)!}{\alpha^{\frac{1}{r}}} \|x\|_r^r. \qquad (1) \end{aligned}$$

This inequality implies $\rho(H) \leq \frac{(r-1)!}{\alpha^{\frac{1}{r}}}$.

Continue

The equality holds if H is α-normal and there is a non-zero solution $\{x_i\}$ for the system of the following homogeneous linear equations: $e = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \in E(H)$

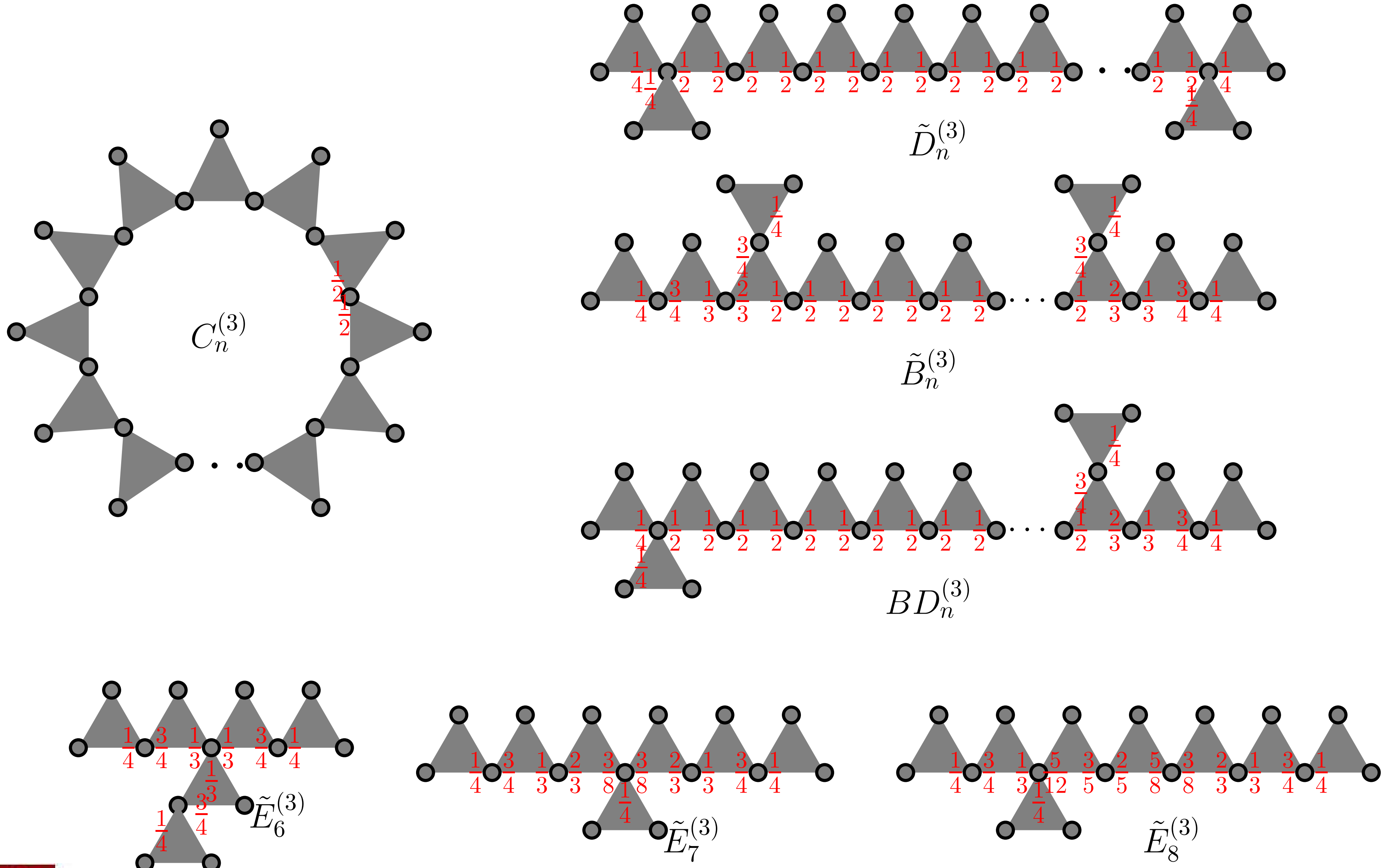
$$B(v_{i_1}, e)^{1/r} \cdot x_{i_1} = B(v_{i_2}, e)^{1/r} \cdot x_{i_2} = \cdots = B(v_{i_r}, e)^{1/r} \cdot x_{i_r}. \tag{2}$$

Picking any vertex v_0 and setting $x^*_{v_0} = 1$, define $x^*_u = \left(\prod_{i=1}^{l} \frac{B(v_{i-1}, e_i)}{B(v_i, e_i)}\right)^{1/r}$ if there is a path $v_0 e_1 v_1 e_2 \cdots v_l (= u)$ connecting v_0 and u. Since H is connected, such path must exist. The consistent condition guarantees that x^*_u is independent of the choice of the path. It is easy to check that $(x^*_1, \ldots, x^*_n)$ is a solution of (2).

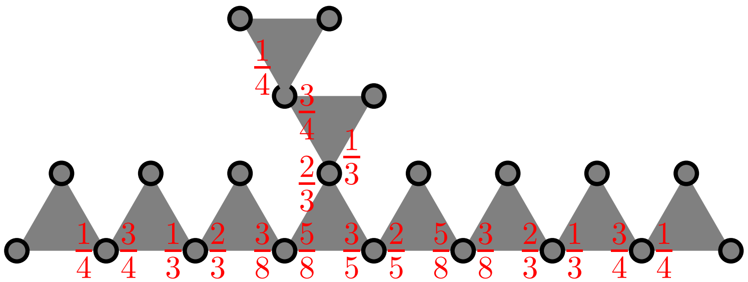
Thus, $\rho(H) = \frac{(r-1)!}{\alpha^{\frac{1}{r}}}$.

$\frac{1}{4}$-normal labelings for $r = 3$

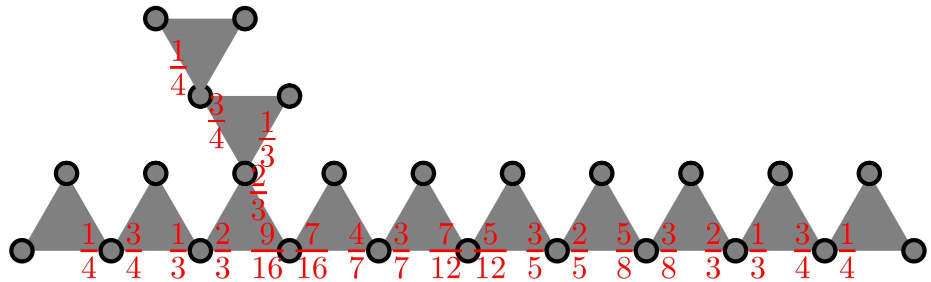
When $r = 3$ and $\alpha = \frac{1}{4}$, then $\rho(H) = \rho_3$.



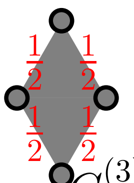
Continue



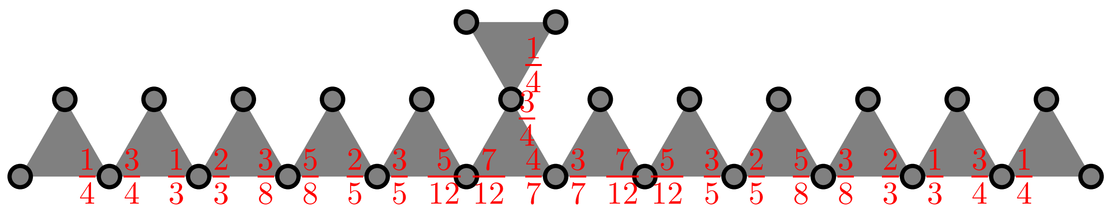
$F^{(3)}_{2,3,4}$



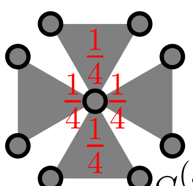
$F^{(3)}_{2,2,7}$



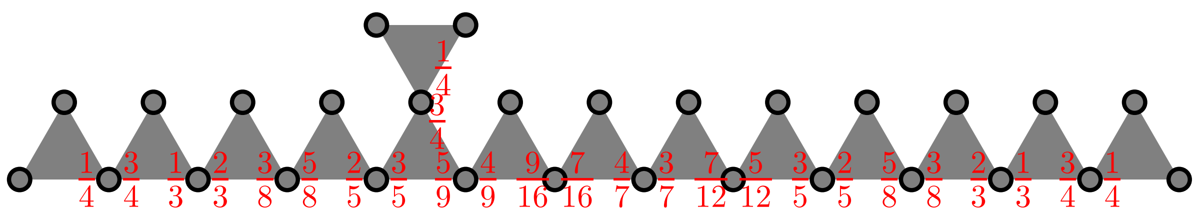
$C^{(3)}_2$



$F^{(3)}_{1,5,6}$

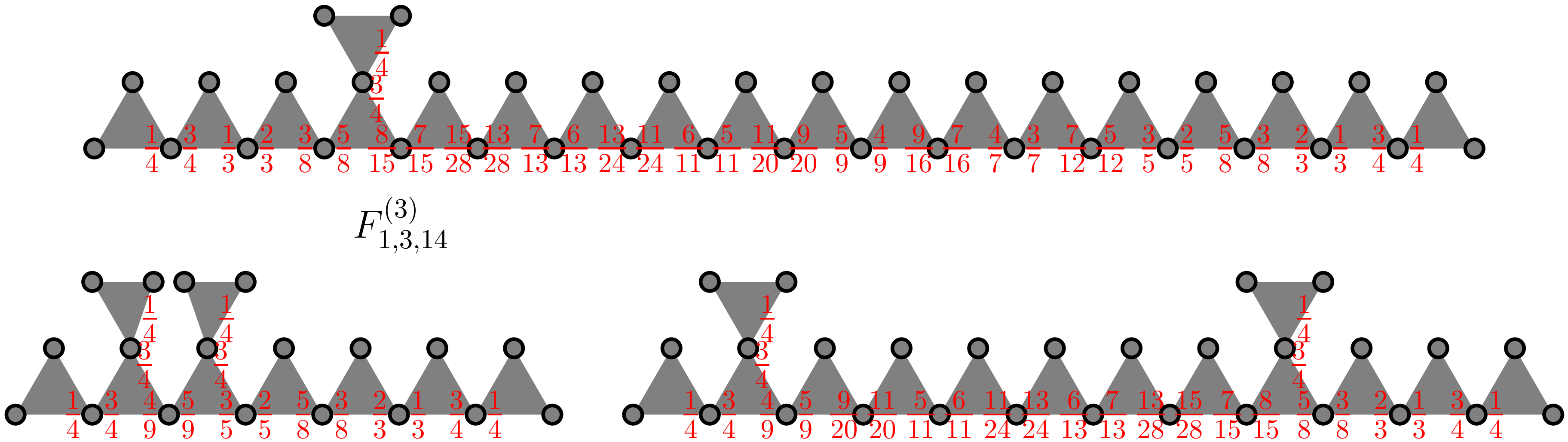


$S^{(3)}_4$



$F^{(3)}_{1,4,8}$

Continue



$F^{(3)}_{1,3,14}$

$G^{(3)}_{1,1:0:1,4}$ $G^{(3)}_{1,1:6:1,3}$

All these 3-graphs have spectral radius ρ_3.

Proof of classification

The hypergraphs listed in Theorem 1 except for $G^{(3)}_{1,1:k:1,3}$ (for $0 \leq k \leq 5$) are proper subgraphs of some hypergraphs in the list of Theorem 2. These hypergraphs have the spectral radius less than ρ_3. We can also show $\rho(G^{(3)}_{1,1:k:1,3}) < \rho_3$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

Proof of classification

The hypergraphs listed in Theorem 1 except for $G^{(3)}_{1,1:k:1,3}$ (for $0 \leq k \leq 5$) are proper subgraphs of some hypergraphs in the list of Theorem 2. These hypergraphs have the spectral radius less than ρ_3. We can also show $\rho(G^{(3)}_{1,1:k:1,3}) < \rho_3$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

Now assume that H is a connected 3-graph with $\rho(H) \leq \rho_3$.

Proof of classification

The hypergraphs listed in Theorem 1 except for $G^{(3)}_{1,1:k:1,3}$ (for $0 \leq k \leq 5$) are proper subgraphs of some hypergraphs in the list of Theorem 2. These hypergraphs have the spectral radius less than ρ_3. We can also show $\rho(G^{(3)}_{1,1:k:1,3}) < \rho_3$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

Now assume that H is a connected 3-graph with $\rho(H) \leq \rho_3$.

If H is not simple, then $\rho(H) > \rho_3$ unless $H = C_2^{(3)}$.

Proof of classification

The hypergraphs listed in Theorem 1 except for $G^{(3)}_{1,1:k:1,3}$ (for $0 \leq k \leq 5$) are proper subgraphs of some hypergraphs in the list of Theorem 2. These hypergraphs have the spectral radius less than ρ_3. We can also show $\rho(G^{(3)}_{1,1:k:1,3}) < \rho_3$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

Now assume that H is a connected 3-graph with $\rho(H) \leq \rho_3$.

If H is not simple, then $\rho(H) > \rho_3$ unless $H = C_2^{(3)}$. We can assume that H is a simple 3-graph.

Proof of classification

The hypergraphs listed in Theorem 1 except for $G^{(3)}_{1,1:k:1,3}$ (for $0 \le k \le 5$) are proper subgraphs of some hypergraphs in the list of Theorem 2. These hypergraphs have the spectral radius less than ρ_3. We can also show $\rho(G^{(3)}_{1,1:k:1,3}) < \rho_3$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

Now assume that H is a connected 3-graph with $\rho(H) \le \rho_3$.

If H is not simple, then $\rho(H) > \rho_3$ unless $H = C^{(3)}_2$. We can assume that H is a simple 3-graph.

If H contains a cycle $C^{(3)}_k$, then $\rho(H) > \rho_3$ unless $H = C^{(3)}_k$.

Proof of classification

The hypergraphs listed in Theorem 1 except for $G^{(3)}_{1,1:k:1,3}$ (for $0 \leq k \leq 5$) are proper subgraphs of some hypergraphs in the list of Theorem 2. These hypergraphs have the spectral radius less than ρ_3. We can also show $\rho(G^{(3)}_{1,1:k:1,3}) < \rho_3$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

Now assume that H is a connected 3-graph with $\rho(H) \leq \rho_3$.

If H is not simple, then $\rho(H) > \rho_3$ unless $H = C^{(3)}_2$. We can assume that H is a simple 3-graph.

If H contains a cycle $C^{(3)}_k$, then $\rho(H) > \rho_3$ unless $H = C^{(3)}_k$. We can assume that H is a hypertree.

Proof of classification

The hypergraphs listed in Theorem 1 except for $G^{(3)}_{1,1:k:1,3}$ (for $0 \leq k \leq 5$) are proper subgraphs of some hypergraphs in the list of Theorem 2. These hypergraphs have the spectral radius less than ρ_3. We can also show $\rho(G^{(3)}_{1,1:k:1,3}) < \rho_3$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

Now assume that H is a connected 3-graph with $\rho(H) \leq \rho_3$.

If H is not simple, then $\rho(H) > \rho_3$ unless $H = C^{(3)}_2$. We can assume that H is a simple 3-graph.

If H contains a cycle $C^{(3)}_k$, then $\rho(H) > \rho_3$ unless $H = C^{(3)}_k$. We can assume that H is a hypertree.

If H contains a vertex of degree ≥ 4, then $\rho(H) > \rho_3$ unless $H = S^{(3)}_4$.

Proof of classification

The hypergraphs listed in Theorem 1 except for $G^{(3)}_{1,1:k:1,3}$ (for $0 \le k \le 5$) are proper subgraphs of some hypergraphs in the list of Theorem 2. These hypergraphs have the spectral radius less than ρ_3. We can also show $\rho(G^{(3)}_{1,1:k:1,3}) < \rho_3$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

Now assume that H is a connected 3-graph with $\rho(H) \le \rho_3$.

If H is not simple, then $\rho(H) > \rho_3$ unless $H = C^{(3)}_2$. We can assume that H is a simple 3-graph.

If H contains a cycle $C^{(3)}_k$, then $\rho(H) > \rho_3$ unless $H = C^{(3)}_k$. We can assume that H is a hypertree.

If H contains a vertex of degree ≥ 4, then $\rho(H) > \rho_3$ unless $H = S^{(3)}_4$. We can assume all vertex degrees are at most 3.

Continue

If there exists two vertexes u and v with $d_u = d_v = 3$, then H contains $\tilde{D}_k^{(3)}$ as a subgraph. We have $\rho(H) > \rho_3$ unless $H = \tilde{D}_n^{(3)}$.

Continue

If there exists two vertexes u and v with $d_u = d_v = 3$, then H contains $\tilde{D}_k^{(3)}$ as a subgraph. We have $\rho(H) > \rho_3$ unless $H = \tilde{D}_n^{(3)}$.

Suppose that v is the unique vertex with degree 3 and all other vertices have degree at most 2. Consider the three branches attached to v.

- If every branch has at least two edges, then H contains $\tilde{E}_6^{(3)}$ as a subgraph. We have $\rho(H) > \rho(\tilde{E}_6^{(3)}) = \rho_3$. unless $H = \tilde{E}_6^{(3)}$.

Continue

If there exists two vertexes u and v with $d_u = d_v = 3$, then H contains $\tilde{D}_k^{(3)}$ as a subgraph. We have $\rho(H) > \rho_3$ unless $H = \tilde{D}_n^{(3)}$.

Suppose that v is the unique vertex with degree 3 and all other vertices have degree at most 2. Consider the three branches attached to v.

- If every branch has at least two edges, then H contains $\tilde{E}_6^{(3)}$ as a subgraph. We have $\rho(H) > \rho(\tilde{E}_6^{(3)}) = \rho_3$. unless $H = \tilde{E}_6^{(3)}$.

Continue

If there exists two vertexes u and v with $d_u = d_v = 3$, then H contains $\tilde{D}_k^{(3)}$ as a subgraph. We have $\rho(H) > \rho_3$ unless $H = \tilde{D}_n^{(3)}$.

Suppose that v is the unique vertex with degree 3 and all other vertices have degree at most 2. Consider the three branches attached to v.

- If every branch has at least two edges, then H contains $\tilde{E}_6^{(3)}$ as a subgraph. We have $\rho(H) > \rho(\tilde{E}_6^{(3)}) = \rho_3$. unless $H = \tilde{E}_6^{(3)}$.
  Thus we can assume that the first branch consists of only one edge.

Continue

If there exists two vertexes u and v with $d_u = d_v = 3$, then H contains $\tilde{D}_k^{(3)}$ as a subgraph. We have $\rho(H) > \rho_3$ unless $H = \tilde{D}_n^{(3)}$.

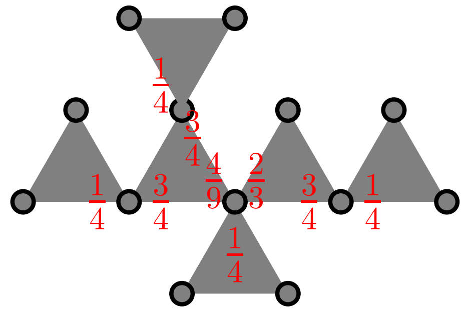
Suppose that v is the unique vertex with degree 3 and all other vertices have degree at most 2. Consider the three branches attached to v.

- If every branch has at least two edges, then H contains $\tilde{E}_6^{(3)}$ as a subgraph. We have $\rho(H) > \rho(\tilde{E}_6^{(3)}) = \rho_3$. unless $H = \tilde{E}_6^{(3)}$.
  Thus we can assume that the first branch consists of only one edge.

  An edge e is called a *branching edge* if every vertex of e is not a leaf vertex.

Continue

- If the second branch has at least two edges and the third branch consist of a branching edge, then H consists of a subgraph G', which can be eventually contracted to G shown below.



G has a $\frac{1}{4}$-supernormal labeling

Note that the sum of the labelings of G at the center vertex is $\frac{4}{9} + \frac{1}{3} + \frac{1}{4} > 1$. Thus G is strictly $\frac{1}{4}$-supernormal and $\rho(G) > \rho_3$. We have $\rho(H) > \rho(G') > \rho_3$.

Continue

- The first and second branch each consist of one edge and the third branch consists of at least one branching edge. Since $\rho(\widetilde{BD}_n^{(3)}) = \rho_3$, H can not contain $\widetilde{BD}_n^{(3)}$ as a proper subgraph. Thus the only possible hypergraphs are $\widetilde{BD}_n^{(3)}$ and $BD_n^{(3)}$.

Continue

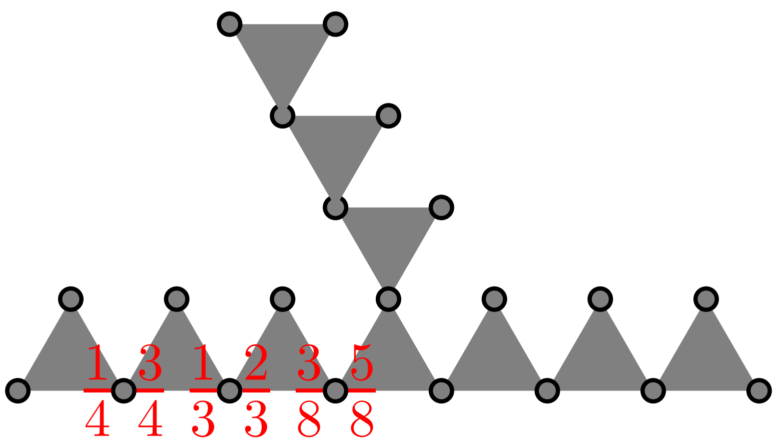
- The first and second branch each consist of one edge and the third branch consists of at least one branching edge. Since $\rho(\widetilde{BD}_n^{(3)}) = \rho_3$, H can not contain $\widetilde{BD}_n^{(3)}$ as a proper subgraph. Thus the only possible hypergraphs are $\widetilde{BD}_n^{(3)}$ and $BD_n^{(3)}$.
- There is no branching edge in H. Let i, j, k $(i \leq j \leq k)$ be the length of three branches of the vertex v and denote this graph by $E_{i,j,k}^{(3)}$. We have shown that $i = 1$. Note that $E_{1,3,3}^{(3)} = \tilde{E}_7^{(3)}$ and $E_{1,2,5}^{(3)} = \tilde{E}_8^{(3)}$ have spectral radius ρ_3. So (j, k) can only have the following choices: $(2, 5), (2, 4), (3, 3), (2, 3), (2, 2)$ and $(1, k), k \geq 1$. The corresponding graphs are $\tilde{E}_8^{(3)}$, $E_8^{(3)}$, $\tilde{E}_7^{(3)}$, $E_7^{(3)}$, $E_6^{(3)}$, and $D_n^{(3)}$.

Continue

Now we can assume that all degrees of vertices in H have degrees at most 2.

- If H has no branching edge, then $H = A_n$ (a path).
- If H has exactly one branching edge, then $H = F^{(3)}_{i,j,k}$.

  We will first show that $\rho(F^{(3)}_{3,3,3}) > \rho_3$ (see below).

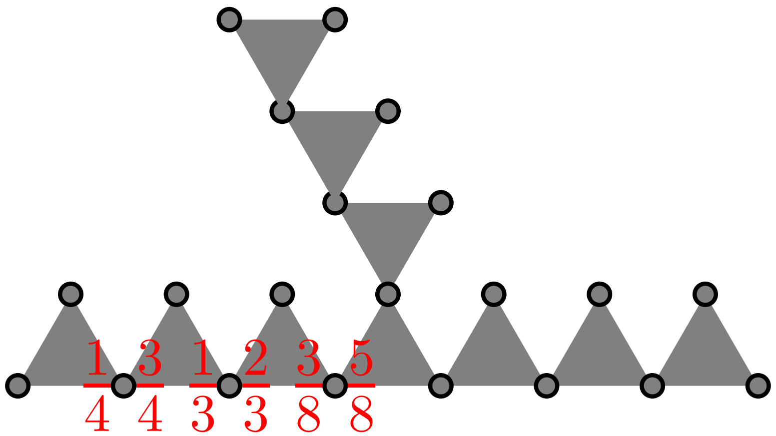


Continue

Now we can assume that all degrees of vertices in H have degrees at most 2.

- If H has no branching edge, then $H = A_n$ (a path).
- If H has exactly one branching edge, then $H = F^{(3)}_{i,j,k}$.

  We will first show that $\rho(F^{(3)}_{3,3,3}) > \rho_3$ (see below).



So H must not contain the subgraph $F^{(3)}_{3,3,3}$. Since $i \leq j \leq k$, we must have $i = 1$ or 2.

Continue

When $i = 2$ and $j = 3$, as $\rho(F^{(3)}_{2,3,4}) = \rho_3$, there are only two possible hypergraphs: $F^{(3)}_{2,3,3}$ and $F^{(3)}_{2,3,4}$.

Continue

When $i = 2$ and $j = 3$, as $\rho(F^{(3)}_{2,3,4}) = \rho_3$, there are only two possible hypergraphs: $F^{(3)}_{2,3,3}$ and $F^{(3)}_{2,3,4}$.

When $i = 2$ and $j = 2$, as $\rho(F^{(3)}_{2,2,7}) = \rho_3$, we must have $2 \le k \le 7$.

Continue

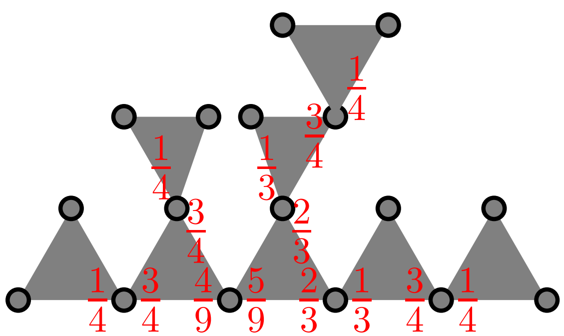
When $i = 2$ and $j = 3$, as $\rho(F^{(3)}_{2,3,4}) = \rho_3$, there are only two possible hypergraphs: $F^{(3)}_{2,3,3}$ and $F^{(3)}_{2,3,4}$.

When $i = 2$ and $j = 2$, as $\rho(F^{(3)}_{2,2,7}) = \rho_3$, we must have $2 \leq k \leq 7$.

When $i = 1$, as $\rho(F^{(3)}_{1,5,6}) = \rho_3$, we must have $j \leq 5$. When $j = 5$, we have two possible hypergraphs: $F^{(3)}_{1,5,5}$ and $F^{(3)}_{1,5,6}$. When $j = 4$, as $\rho(F^{(3)}_{1,4,8}) = \rho_3$, we have 5 possible hypergraphs: $F^{(3)}_{1,4,k}$ for $4 \leq k \leq 8$. When $j = 3$, as $\rho(F^{(3)}_{1,3,14}) = \rho_3$, we have 12 possible hypergraphs: $F^{(3)}_{1,3,k}$ for $3 \leq k \leq 14$. When $j = 2$, all the values of k are possible, and we get the family $B^{(3)}_n$. When $j = 1$, all the values of k are possible, and we get the family $D'^{(3)}_n$.

Continue

- If H has exactly two branching edges, then $H = G^{(3)}_{i,j:k:l,m}$ $(i \le j, l \le m)$. If $i + j \ge 3$ and $l + m \ge 3$, then H contains a subgraph $G^{(3)}_{1,2:k:1,2} = \tilde{B}^{(3)}_{k+8}$. Since the family $\tilde{B}^{(3)}_n$ have the spectral radius equal to ρ_3, we conclude H must be $\tilde{B}^{(3)}_n$ itself. For the remaining cases, we can assume $i = j = 1$. We have $\rho(H) \ge \rho(G^{(3)}_{1,1:0:2,2}) > \rho_3$ (see the labeling below.)



a $\frac{1}{4}$-supernormal labeling of $G^{(3)}_{1,1:0:2,2}$

Continue

In particular, there is no such hypergraph with $m \geq 5$.

Continue

In particular, there is no such hypergraph with $m \geq 5$.

If $m = 4$, then we only get one hypergraph $G^{(3)}_{1,1:0:1,4}$.

Continue

In particular, there is no such hypergraph with $m \geq 5$.

If $m = 4$, then we only get one hypergraph $G^{(3)}_{1,1:0:1,4}$.

If $m = 3$, as $\rho(G^{(3)}_{1,1:6:1,3}) = \rho_3$, we get 7 hypergraphs: $\rho(G^{(3)}_{1,1:k:1,3})$ for $0 \leq k \leq 6$.

Continue

In particular, there is no such hypergraph with $m \geq 5$.

If $m = 4$, then we only get one hypergraph $G^{(3)}_{1,1:0:1,4}$.

If $m = 3$, as $\rho(G^{(3)}_{1,1:6:1,3}) = \rho_3$, we get 7 hypergraphs: $\rho(G^{(3)}_{1,1:k:1,3})$ for $0 \leq k \leq 6$.

If $m = 2$, then any k works. We get the family $\bar{B}^{(3)}_n$.

Continue

In particular, there is no such hypergraph with $m \geq 5$.

If $m = 4$, then we only get one hypergraph $G^{(3)}_{1,1:0:1,4}$.

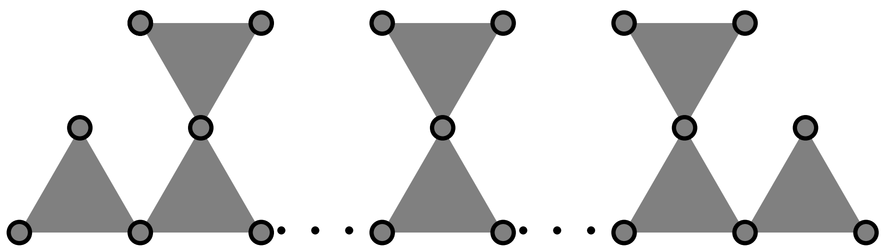
If $m = 3$, as $\rho(G^{(3)}_{1,1:6:1,3}) = \rho_3$, we get 7 hypergraphs: $\rho(G^{(3)}_{1,1:k:1,3})$ for $0 \leq k \leq 6$.

If $m = 2$, then any k works. We get the family $\bar{B}^{(3)}_n$.

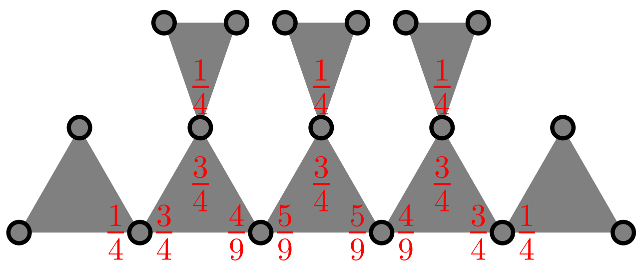
If $m = 1$, then any k works. We get the family $B'^{(3)}_n$.

Continue

- H contains at least three branching edges. Since all degrees of vertices are at most 2, any branching edges lie in a path. Thus, H contains a subgraph M' in the following figure. By contracting the middle edges connecting the branching edges, we get a hypergraph M. We can see that M admits the following $\frac{1}{4}$-supernormal labeling.



a subgraph M'



after contraction: M

We have $\rho(H) \geq \rho(M') \geq \rho(M) > \rho_3$. $\square$

Continue

Reduction and Extension

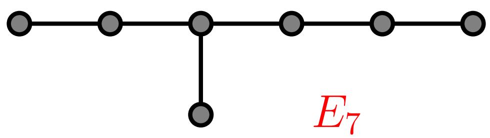
- An r-uniform hypergraph $H = (V, E)$ is called reducible if every edge e contains a leaf vertex v_e.

Reduction and Extension

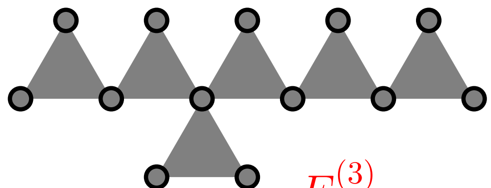
- An r-uniform hypergraph $H = (V, E)$ is called reducible if every edge e contains a leaf vertex v_e.
- Removing v_e from each e, we get a $H' = (V', E')$, i.e., $V' = V \setminus \{v_e \colon e \in E\}$ and $E' = \{e - v_e \colon e \in E\}$.

Reduction and Extension

- An r-uniform hypergraph $H = (V, E)$ is called reducible if every edge e contains a leaf vertex v_e.
- Removing v_e from each e, we get a $H' = (V', E')$, i.e., $V' = V \setminus \{v_e : e \in E\}$ and $E' = \{e - v_e : e \in E\}$.
- We say that H' is reduced from H and H extends H'.



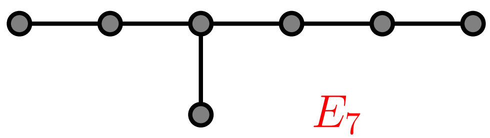
E_7



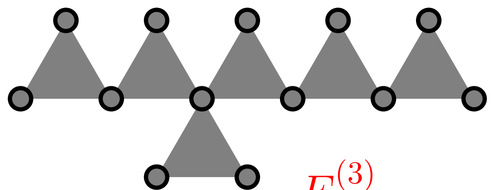
$E_7^{(3)}$

Reduction and Extension

- An r-uniform hypergraph $H = (V, E)$ is called reducible if every edge e contains a leaf vertex v_e.
- Removing v_e from each e, we get a $H' = (V', E')$, i.e., $V' = V \setminus \{v_e : e \in E\}$ and $E' = \{e - v_e : e \in E\}$.
- We say that H' is reduced from H and H extends H'.



E_7



$E_7^{(3)}$

Lemma If H exends H', then

$$\rho(H) \leq \rho_r \text{ iff } \rho(H') \leq \rho_{r-1}.$$

Proof

$\rho(H) \leq \rho_r$ implies that H has a consistently α-normal labeling with $\alpha \leq \frac{1}{4}$. Since the labeling near every leaf vertex is 1, this labeling induces an α-normal labeling of H'. Thus, $\rho(H') \leq \rho_{r-1}$.

Proof

$\rho(H) \le \rho_r$ implies that H has a consistently α-normal labeling with $\alpha \le \frac{1}{4}$. Since the labeling near every leaf vertex is 1, this labeling induces an α-normal labeling of H'. Thus, $\rho(H') \le \rho_{r-1}$.

Every step can be reversed.

Proof

$\rho(H) \le \rho_r$ implies that H has a consistently α-normal labeling with $\alpha \le \frac{1}{4}$. Since the labeling near every leaf vertex is 1, this labeling induces an α-normal labeling of H'. Thus, $\rho(H') \le \rho_{r-1}$.

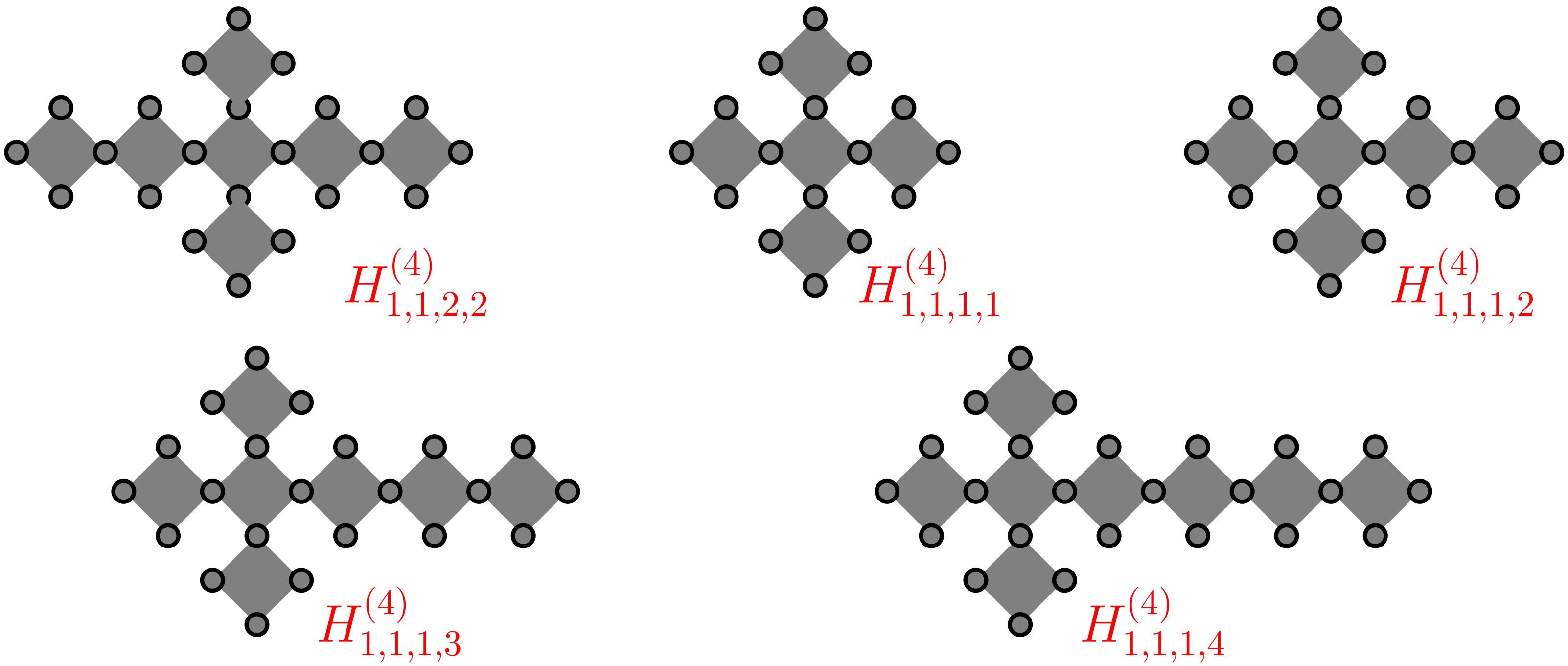
Every step can be reversed.

"$\rho(H) = \rho_r$ iff $\rho(H') = \rho_{r-1}$" can be proved in a similar way.

The case $r \geq 4$

Theorem [Lu, Man, 2013+]

- For $r \geq 5$, every r-uniform hypergraph with $\rho(H) \leq \rho_r$ is reducible.
- For $r = 4$, there is only one irreducible r-uniform hypergraph with $\rho(H) = \rho_r$ and four irreducible r-uniform hypergraph with $\rho(H) < \rho_r$



Classification for $r \geq 4$

Theorem [Lu-Man 2013+] Let $r \geq 4$ and $\rho_r = (r-1)!\sqrt[r]{4}$. If the spectral radius of a connected r-uniform hypergraph H is equal to ρ_r, then H must be one of the following graphs:

1. Extended from 3-graphs: $C_n^{(r)}$, $\tilde{D}_n^{(r)}$, $\tilde{B}_n^{(r)}$, $\widetilde{BD}_n^{(r)}$, $C_2^{(r)}$, $S_4^{(r)}$, $\tilde{E}_6^{(r)}$, $\tilde{E}_7^{(r)}$, $\tilde{E}_8^{(r)}$, $F_{2,3,4}^{(r)}$, $F_{2,2,7}^{(r)}$, $F_{1,5,6}^{(r)}$, $F_{1,4,8}^{(r)}$, $F_{1,3,14}^{(r)}$, $G_{1,1:0:1,4}^{(r)}$, and $G_{1,1:6:1,3}^{(r)}$.
2. Extended from the 4-graph: $H_{1,1,2,2}^{(r)}$.

Classification for $r \geq 4$

Theorem [Lu-Man 2013+] Let $r \geq 4$ and $\rho_r = (r-1)!\sqrt[r]{4}$. If the spectral radius of a connected r-uniform hypergraph H is less than ρ_r, then H must be one of the following graphs:

1. Extended from 3-graphs: $A_n^{(r)}$, $D_n^{(r)}$, $D'^{(r)}_n$, $B_n^{(r)}$, $B'^{(r)}_n$, $\bar{B}_n^{(r)}$, $BD_n^{(r)}$, $E_6^{(r)}$, $E_7^{(r)}$, $E_8^{(r)}$, $F_{2,3,3}^{(r)}$, $F_{2,2,j}^{(r)}$ (for $2 \leq j \leq 6$), $F_{1,3,j}^{(r)}$ (for $3 \leq j \leq 13$), $F_{1,4,j}^{(r)}$ (for $4 \leq j \leq 7$), $F_{1,5,5}^{(r)}$, and $G_{1,1:j:1,3}^{(r)}$ (for $0 \leq j \leq 5$).
2. Extended from 4-graphs: $H_{1,1,1,1}^{(r)}$, $H_{1,1,1,2}^{(r)}$, $H_{1,1,1,3}^{(r)}$, $H_{1,1,1,4}^{(r)}$.

Open problem

Are these r-uniform hypergraphs with $\rho(H) \leq \rho_r$ associated to some algebraic or geometric structures as the ADE system does?

Open problem

Are these r-uniform hypergraphs with $\rho(H) \leq \rho_r$ associated to some algebraic or geometric structures as the ADE system does?

Reference: Linyuan Lu and Shoudong Man, Connected Hypergraphs with Small Spectral Radius http://arxiv.org/pdf/1402.5402

Homepage: http://www.math.sc.edu/$\sim$ lu/

Thank You