Spectra of Random Graphs

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University of South Carolina

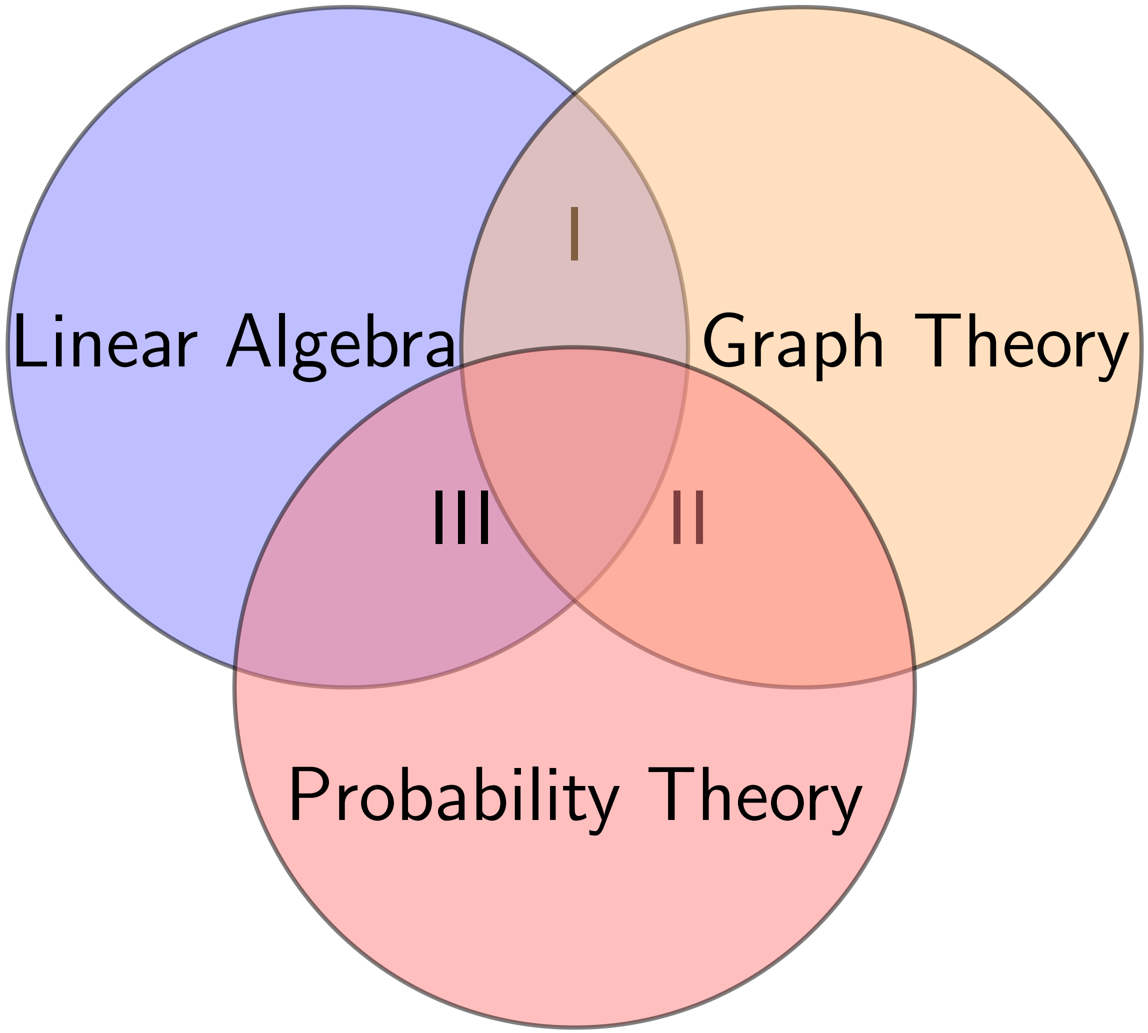
Selected Topics on Spectral Graph Theory (III)
Nankai University, Tianjin, May 29, 2014

Five talks

Selected Topics on Spectral Graph Theory

1. Graphs with Small Spectral Radius
   Time: Friday (May 16) 4pm.-5:30p.m.
2. Laplacian and Random Walks on Graphs
   Time: Thursday (May 22) 4pm.-5:30p.m.
3. Spectra of Random Graphs
   Time: Thursday (May 29) 4pm.-5:30p.m.
4. Hypergraphs with Small Spectral Radius
   Time: Friday (June 6) 4pm.-5:30p.m.
5. Laplacian of Random Hypergraphs
   Time: Thursday (June 12) 4pm.-5:30p.m.

Backgrounds



I: Spectral Graph Theory   II: Random Graph Theory
III: Random Matrix Theory

Outline

- Classical random theory: Erdős-Rényi model

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- Power law graphs

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- Classical random theory: Erdős-Rényi model
- Power law graphs
- Chung-Lu model
- Edge-independent random graphs

Preliminary

A graph consists of two sets V and E.

- V is the set of vertices (or nodes).
- E is the set of edges, where each edge is a pair of vertices.

The degree of a vertex is the number of edges, which are incident to that vertex.

Diameter: the maximum distance $d(u, v)$, where u and v are in the same connected component.

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Diameter: the maximum distance $d(u, v)$, where u and v are in the same connected component.

Average distance: the average among all distance $d(u, v)$ for pairs of u and v in the same connected component.

Random graphs

A random graph is a set of graphs together with a probability distribution on that set.

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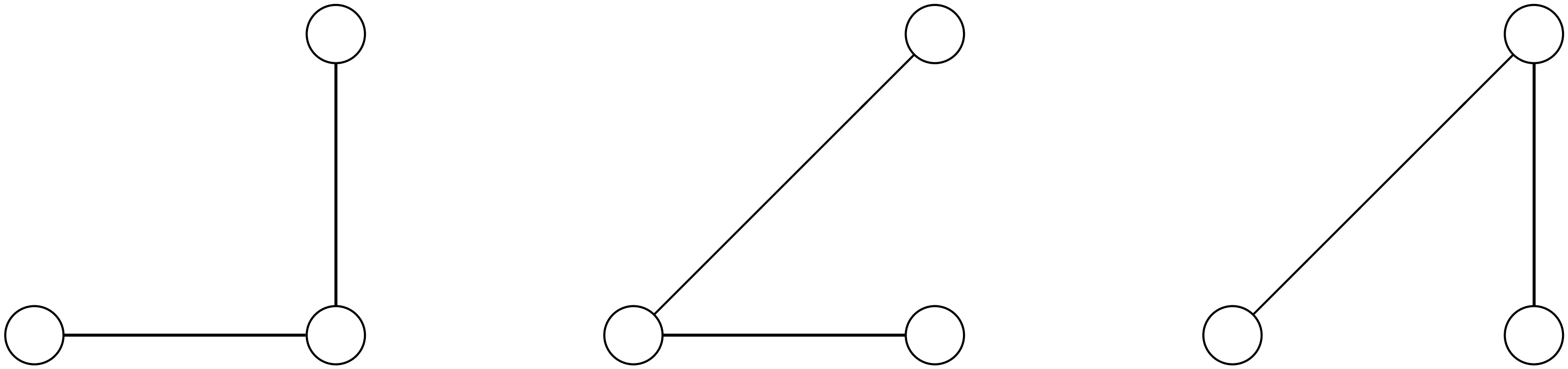
Example: A random graph on 3 vertices and 2 edges with the uniform distribution on it.



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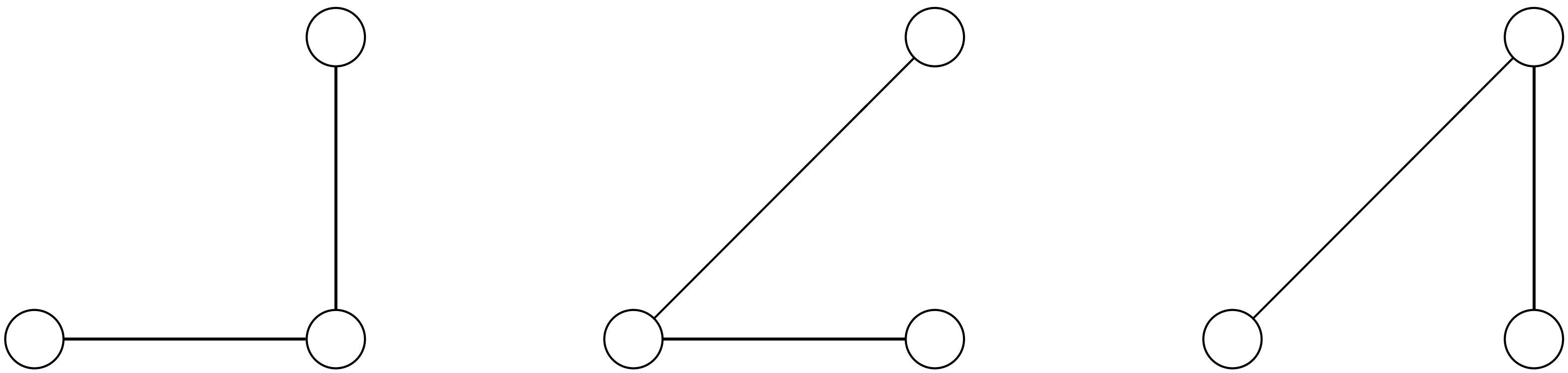

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A random graph G *almost surely* satisfies a property P, if

$$Pr(G \text{ satisfies } P) = 1 - o_n(1).$$

Erdős-Rényi model $G(n, p)$

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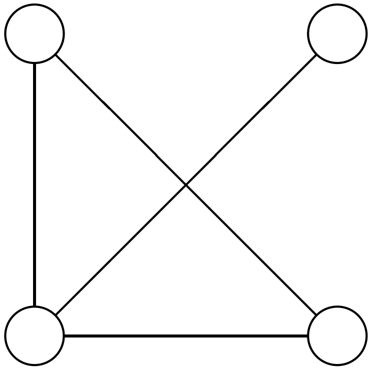

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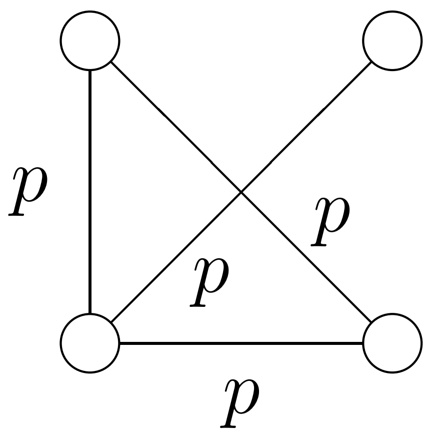

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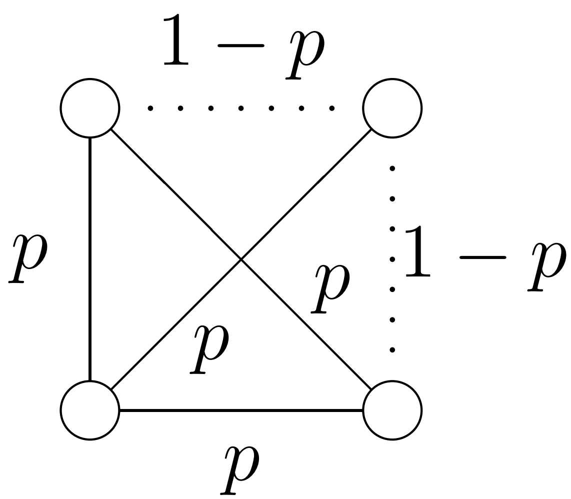

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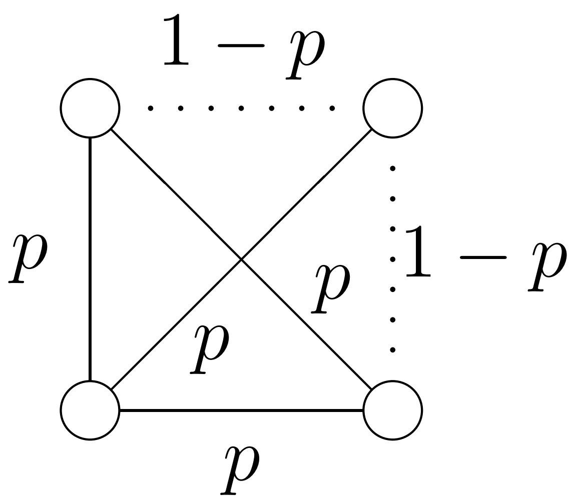

The probability of this graph is

$$p^4(1-p)^2.$$

Evolution of $G(n, p)$

Erdős-Rényi 1960s:

- $p \sim c/n$ for $0 < c < 1$: The largest connected component of $G_{n,p}$ is a tree and has about $\frac{1}{\alpha}(\log n - \frac{5}{2}\log\log n)$ vertices, where $\alpha = c - 1 - \log c$.

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- $p \sim 1/n + c/n^{4/3}$, the largest connected component is $\Theta(n^{2/3})$. Double jump: $\Theta(\log n) \to \Theta(n^{2/3}) \to \Theta(n)$.
- $p \sim c/n$ for $c > 1$: Except for one "giant" component, all the other components are relatively small. The giant component has approximately $f(c)n$ vertices, where

$$f(c) = 1 - \frac{1}{c}\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!}(ce^{-c})^k.$$

Diameter of $G(n,p)$

Bollobás (1985): (denser graph)

$$diam(G(n,p)) = \left\lfloor \frac{\log n}{\log np} \right\rfloor \text{ or } \left\lceil \frac{\log n}{\log np} \right\rceil \text{ if } np \gg \log n.$$

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Chung Lu, (2000) (Sparser graph)

$$diam(G(n,p)) = \begin{cases} (1+o(1))\frac{\log n}{\log np} & \text{if } np \to \infty \\ \Theta(\frac{\log n}{\log np}) & \text{if } \infty > np > 1. \end{cases}$$

Wigner's semicircle law

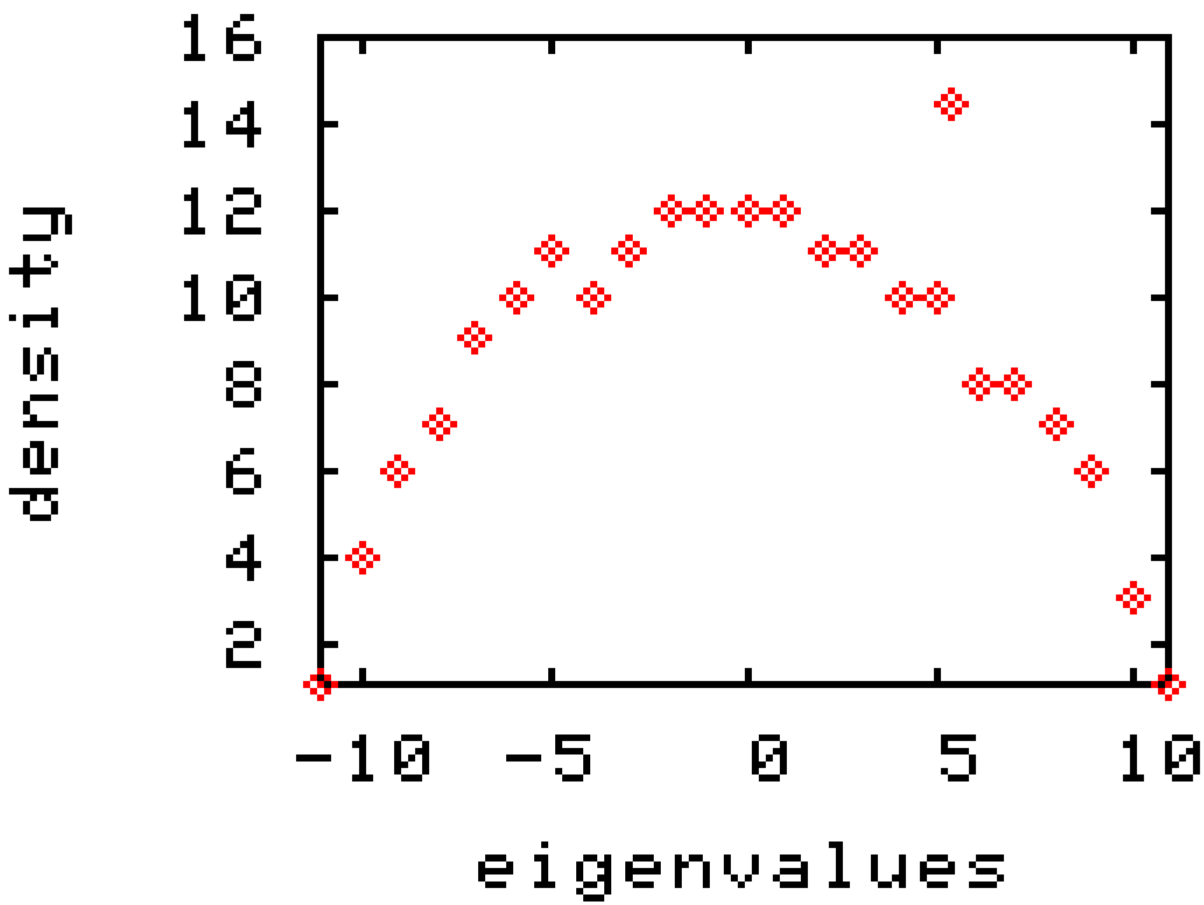
(Wigner, 1958)

- A is a real symmetric $n \times n$ matrix.
- Entries a_{ij} are independent random variables.
- $E(a_{ij}^{2k+1}) = 0$.
- $E(a_{ij}^2) = m^2$.
- $E(a_{ij}^{2k}) < M$.

The distribution of eigenvalues of A converges into a semicircle distribution of radius $2m\sqrt{n}$.

Spectra of $G(n, p)$

The eigenvalues of an Erdős-Rényi random graph follow the semicircle law. (Füredi and Komlós, 1981)



Laplacian eigenvalues also follow the semicircle law.

Challenge

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- How to model these complex graphs by random graphs?

Challenge

Erdős-Rényi model $G(n, p)$ is classical, simple, beautiful..., but not suitable to model complex graphs.

- What are complex graphs?
- How to model these complex graphs by random graphs?
- How to deduce the graph properties of these general random graph models?

Examples of complex graphs

WWW Graphs

Call Graphs

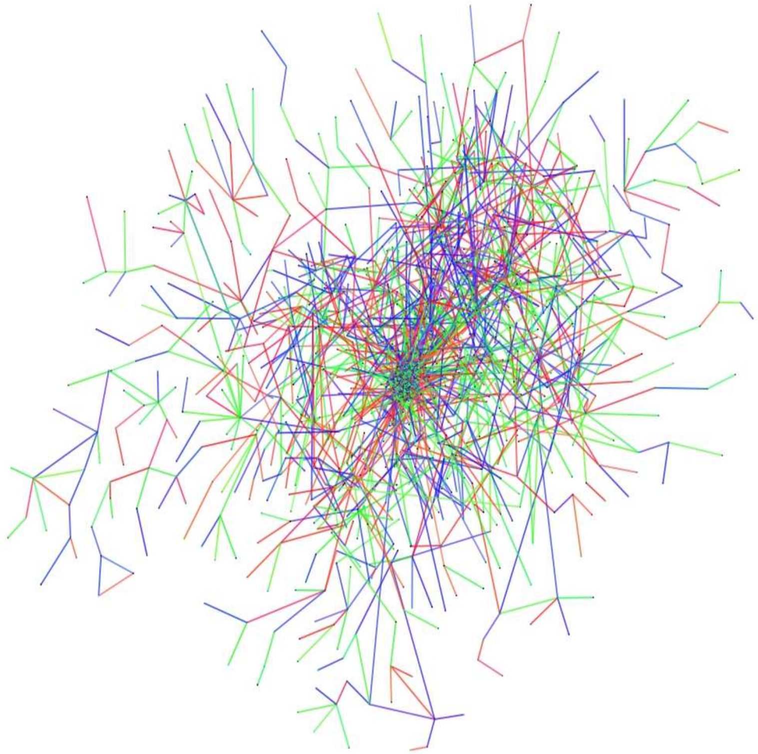
Collaboration Graphs

Gene Regulatory Graphs

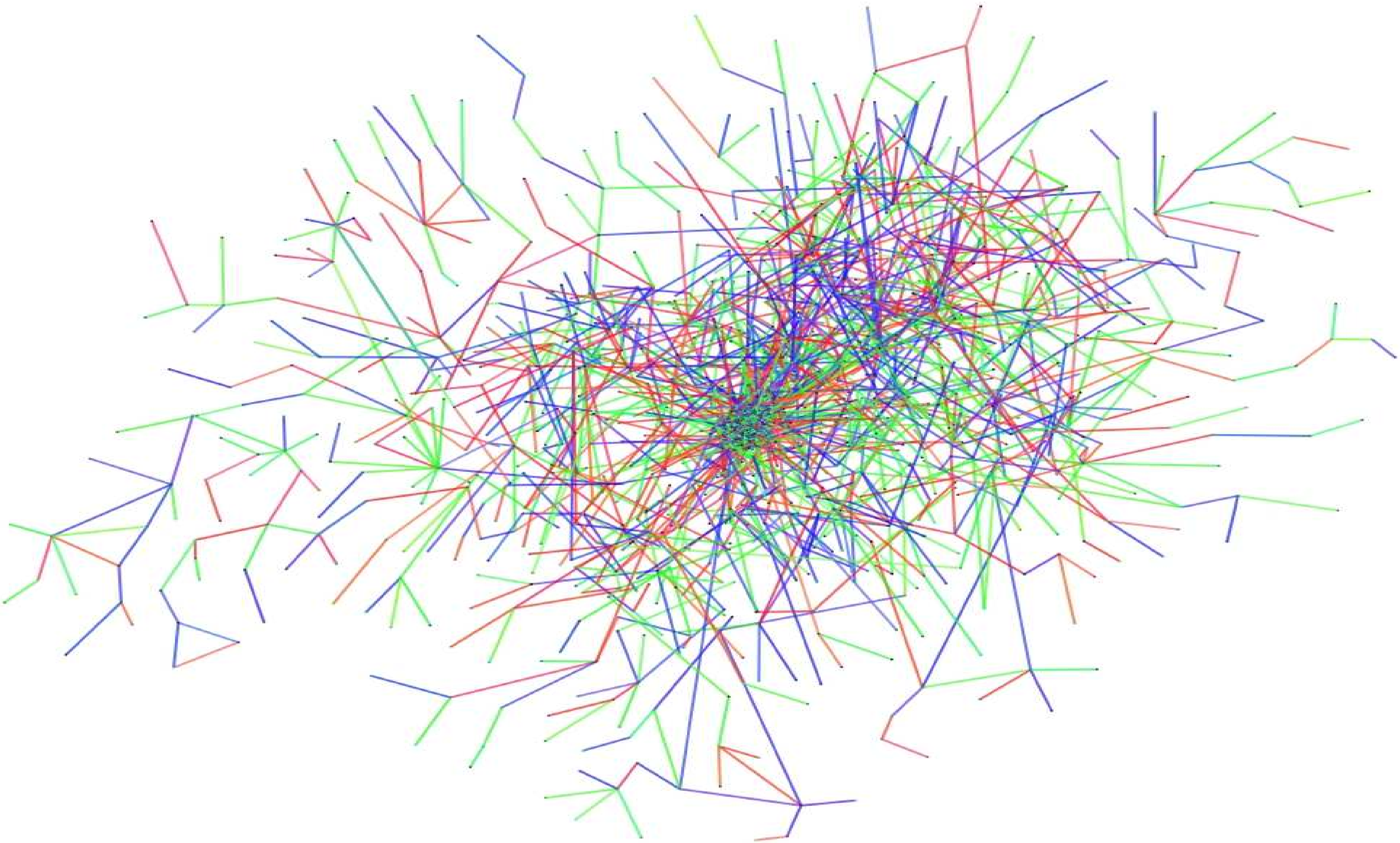
Graph of U.S. Power Grid

Costars Graph of Actors

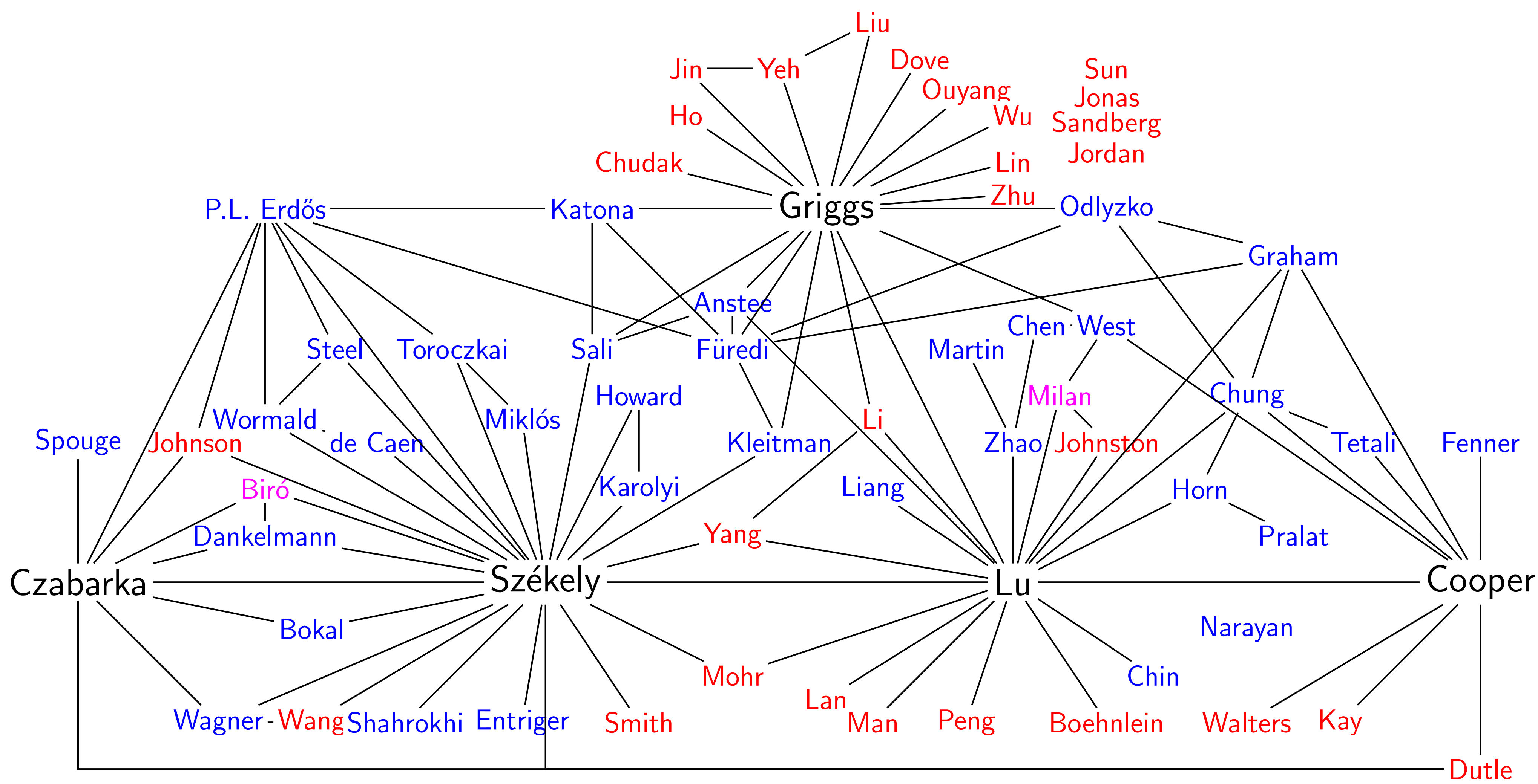
⋮



A subgraph of the Collaboration Graph

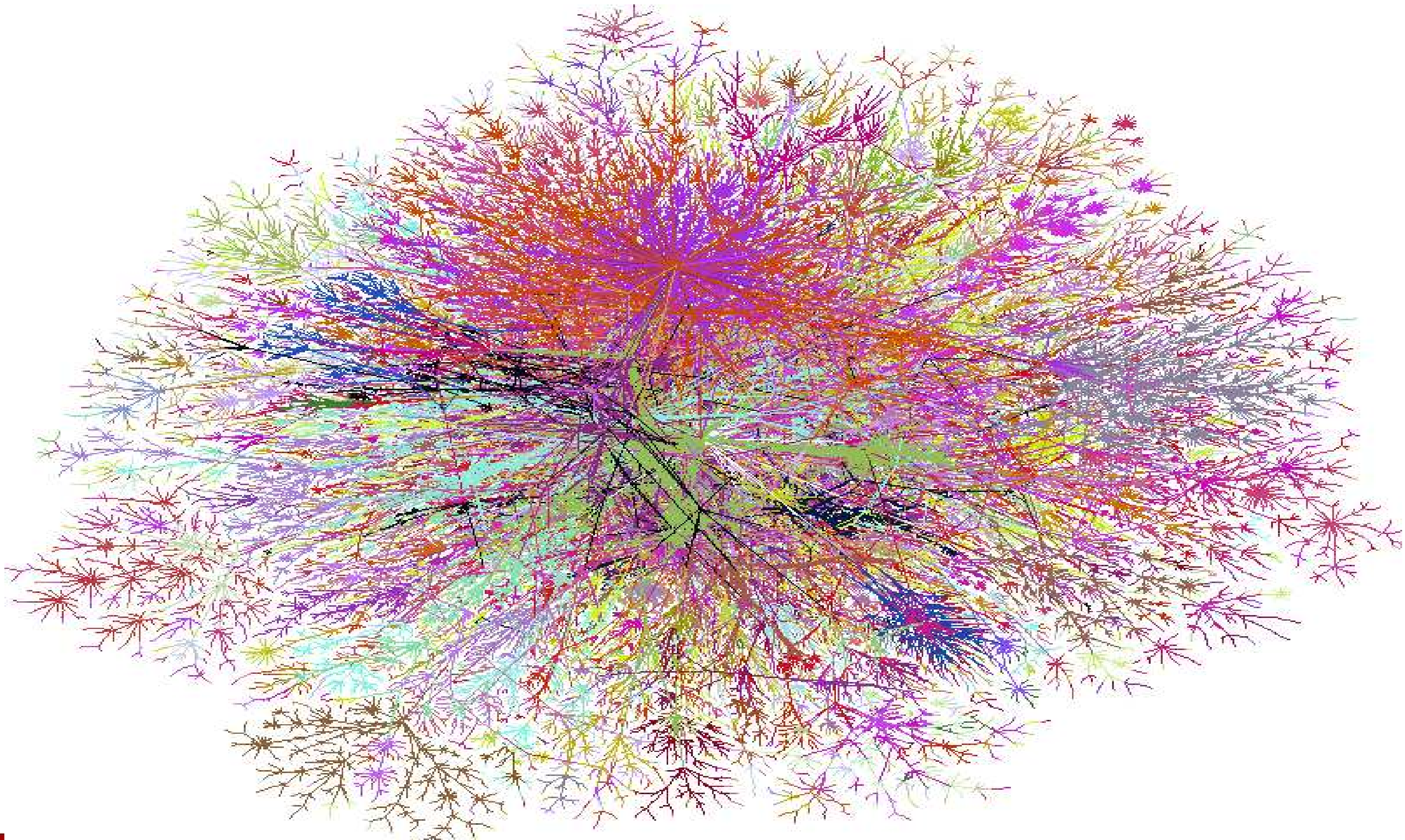


Collaboration Graph at USC



Faculty, Ph.D. students, Postdocs, and visitors to the Combinatorics Group at the University of South Carolina.

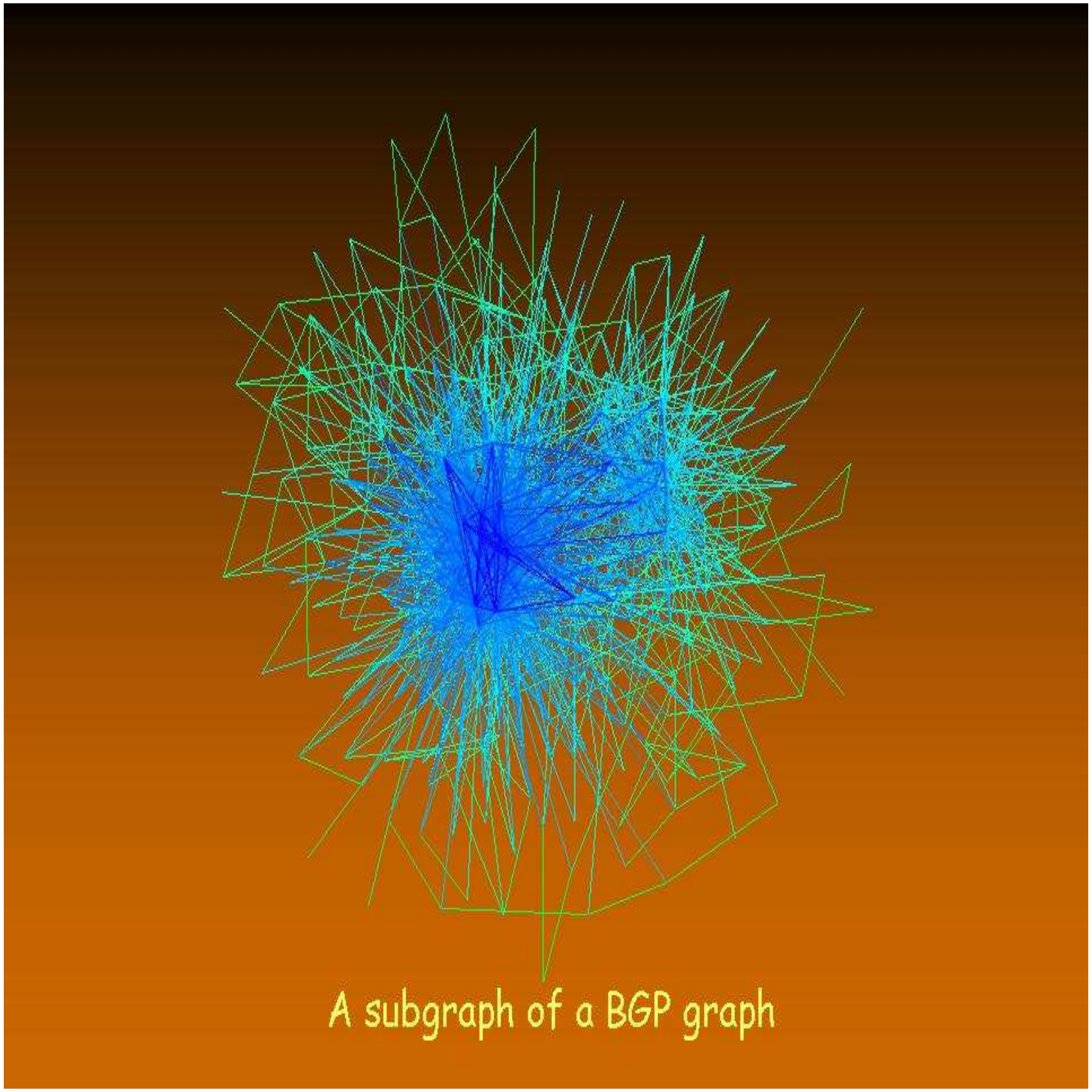
An IP Graph (by Bill Cheswick)



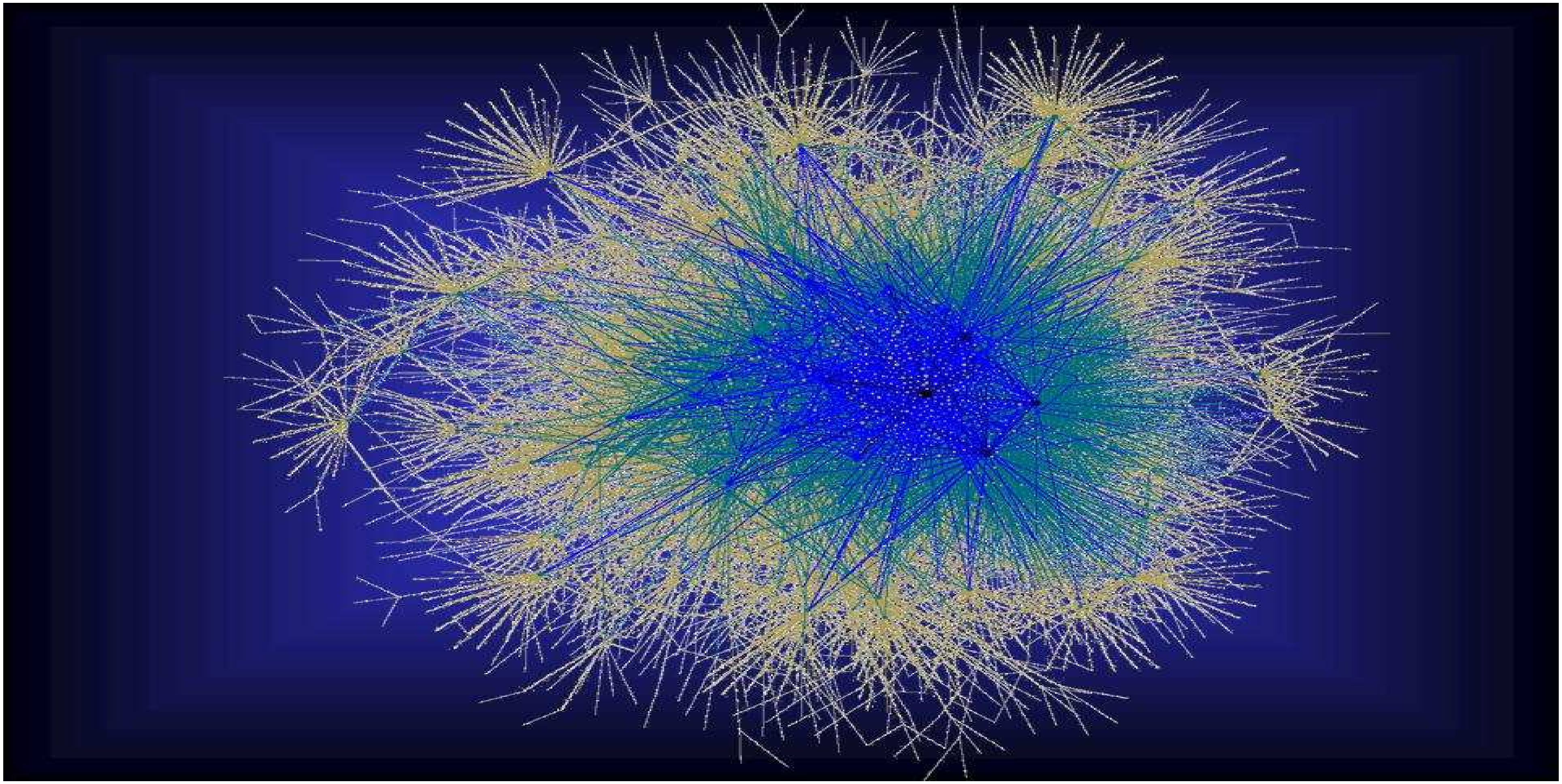
BGP Graph

Vertex: AS (autonomous system)

Edges: AS pairs in BGP routing table.



Large BGP subgraph



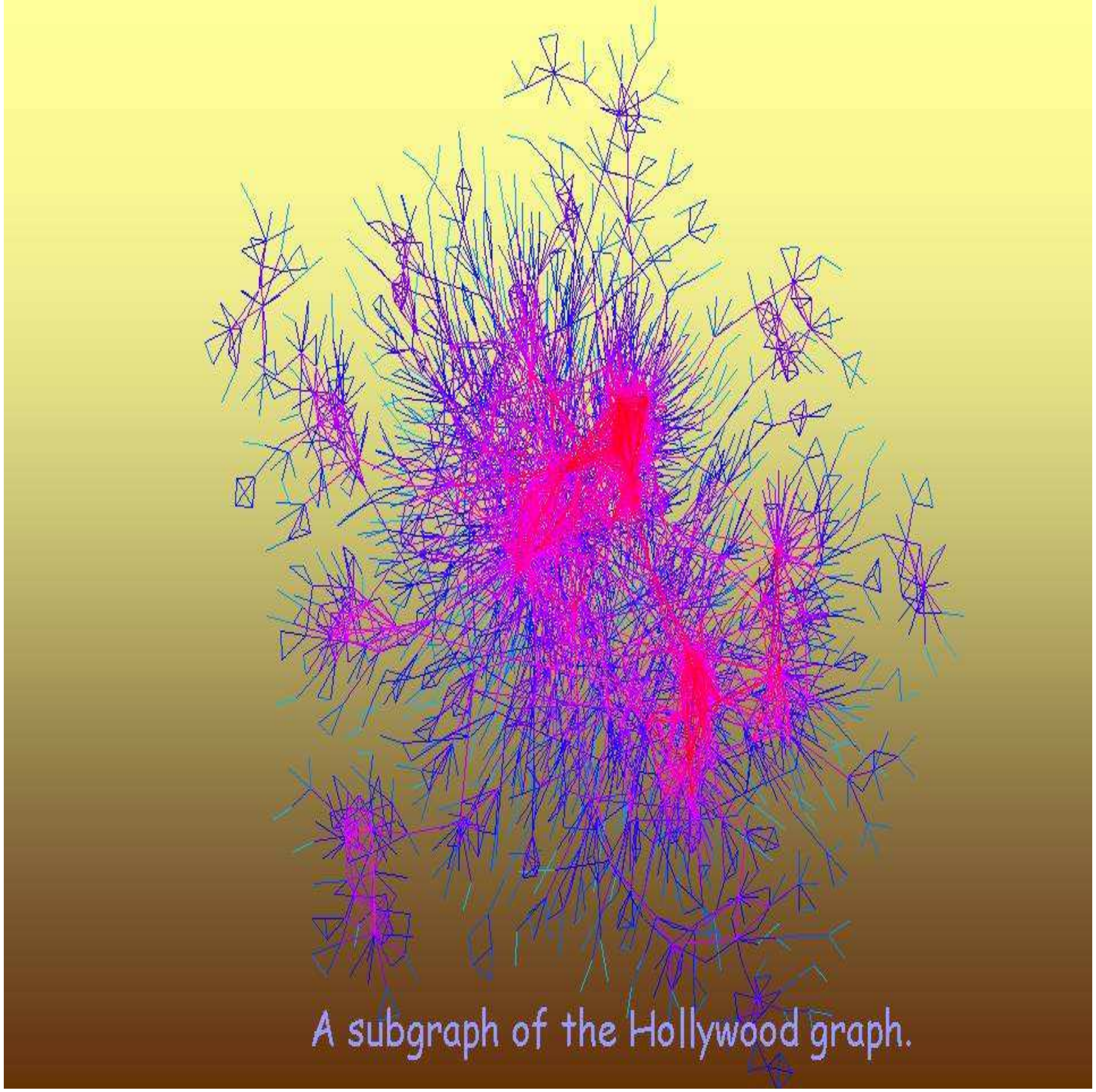
Only a portion of 6400 vertices and 13000 edges is drawn.

Hollywood Graph

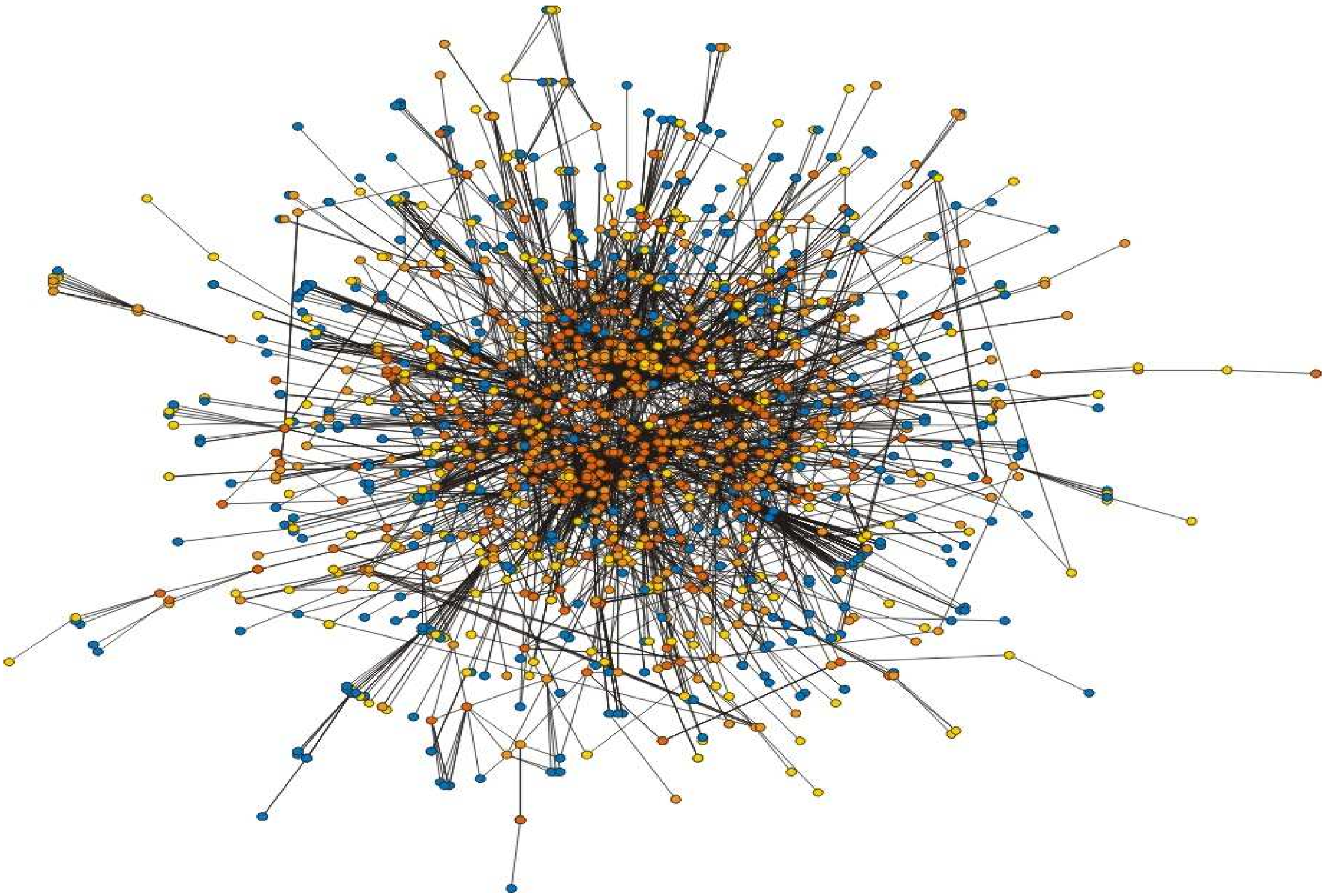
Vertex: actors and actress

Edges: co-playing in the same movie

Only 10,000 out of 225,000 are shown.



Protein-interaction network



Snel, Bork & Huynen, PNAS 99, 5890 (2002)

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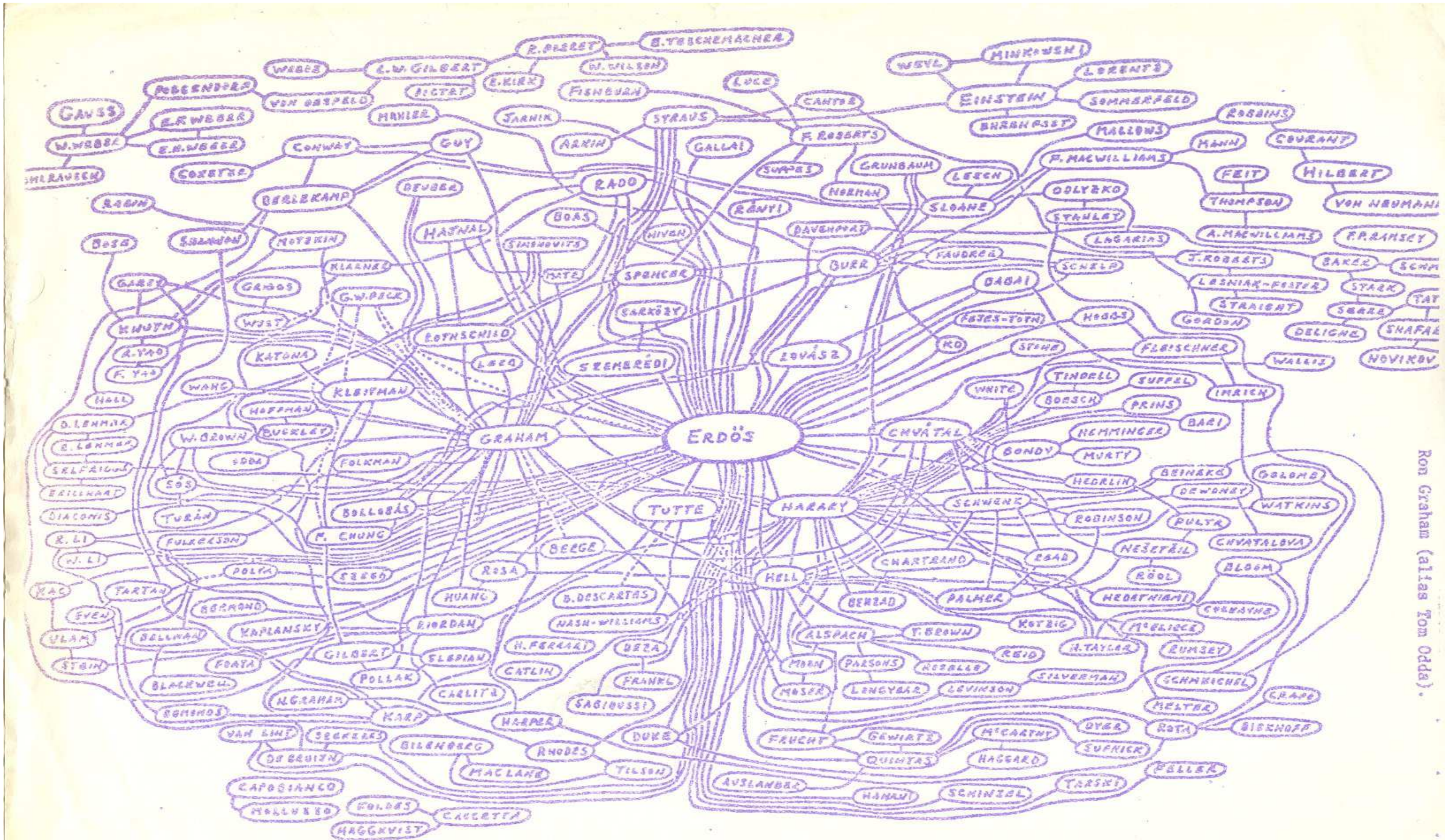

Figure 1
To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

Folklore of Erdős numbers

- Erdős has Erdős number 0.
- Erdős' coauthor has Erdős number 1.
- Erdős' coauthor's coauthor has Erdős number 2.

$$\vdots$$

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My Erdős number is 2.

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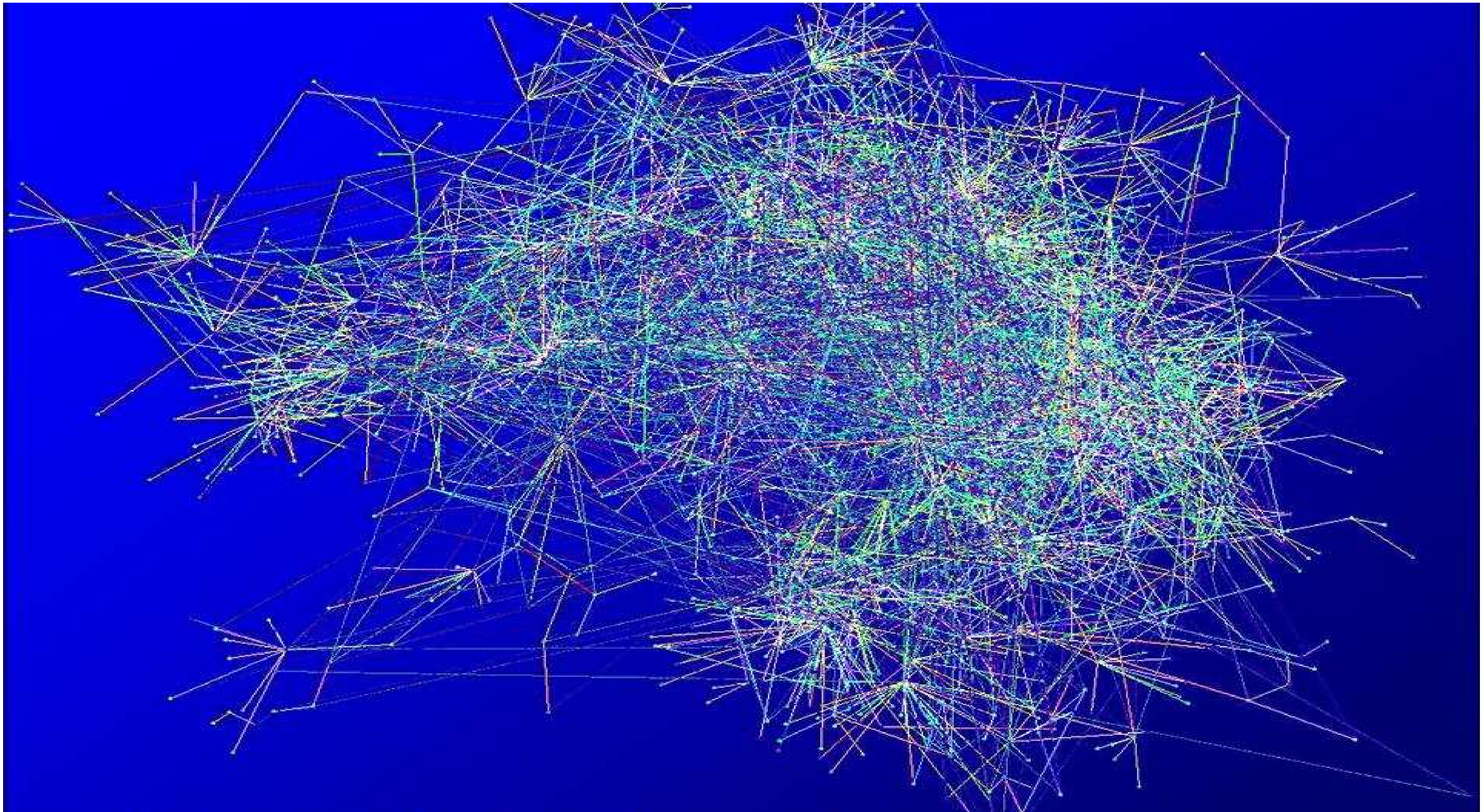
My Erdős number is 2.

Erdős number is the graph distance to Erdős in the Collaboration graph.

Collaboration Graph



An induced subgraph of the collaboration graph (with Erdos number at most 2).
Made by Fan Chung Graham and Lincoln Lu in 2002.

Characteristics

- Large

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- Large
- Sparse

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Characteristics

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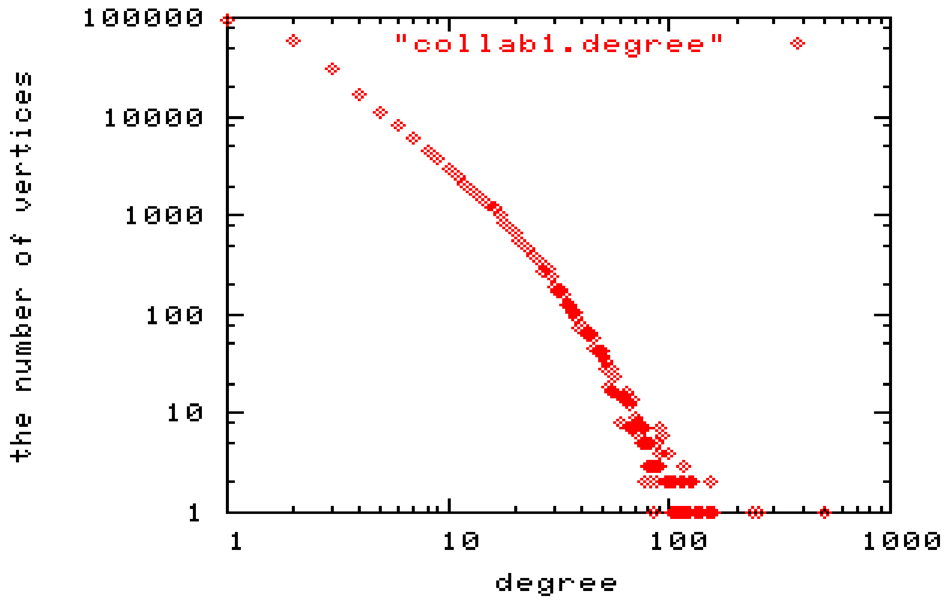
- Small world phenomenon

The power law

The number of vertices of degree k is approximately proportional to $k^{-\beta}$ for some positive β.



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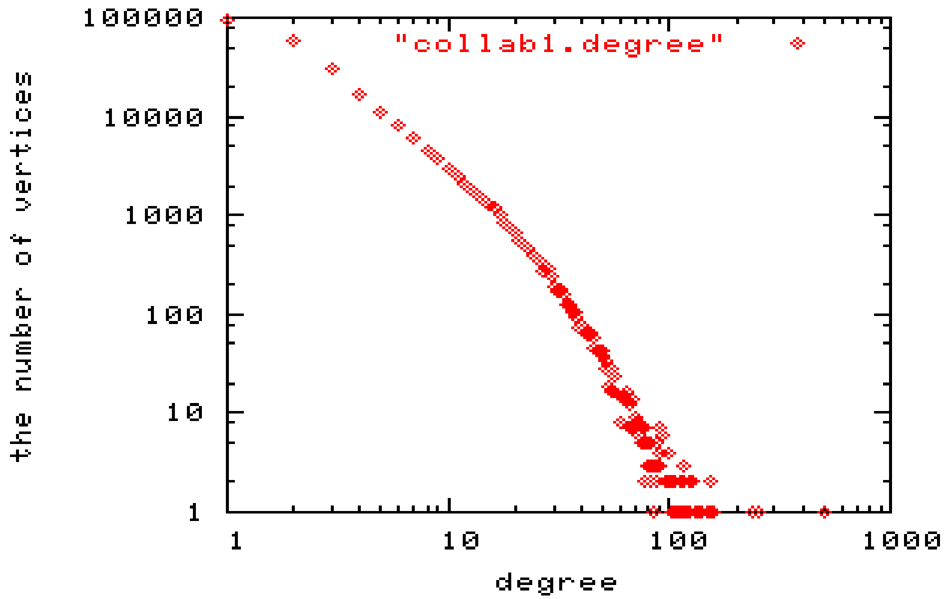
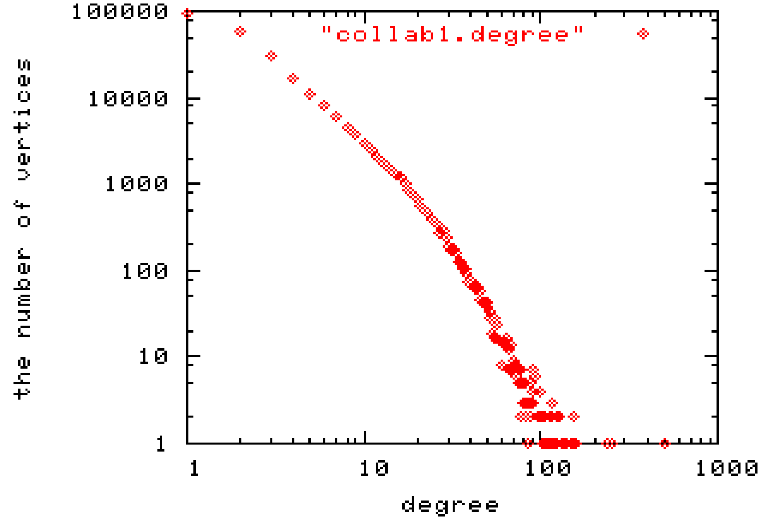


A power law graph is a graph whose degree sequence satisfies the power law.

Power law distribution



Left: The collaboration graph follows the power law degree distribution with exponent $\beta \approx 3.0$

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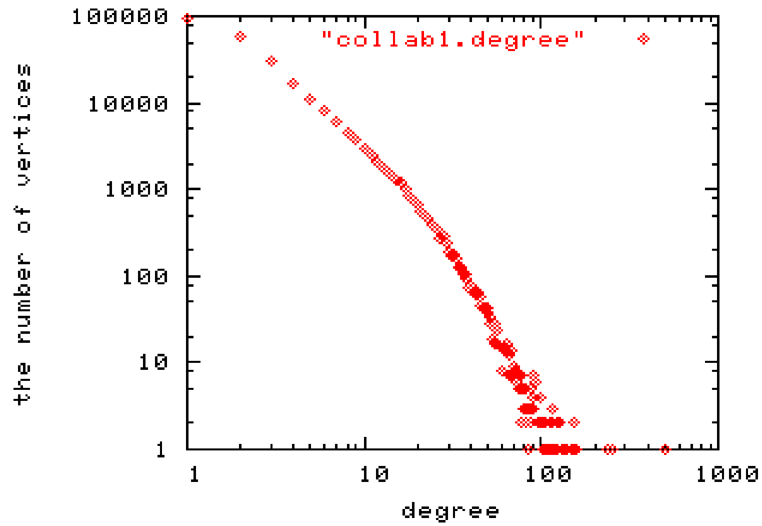
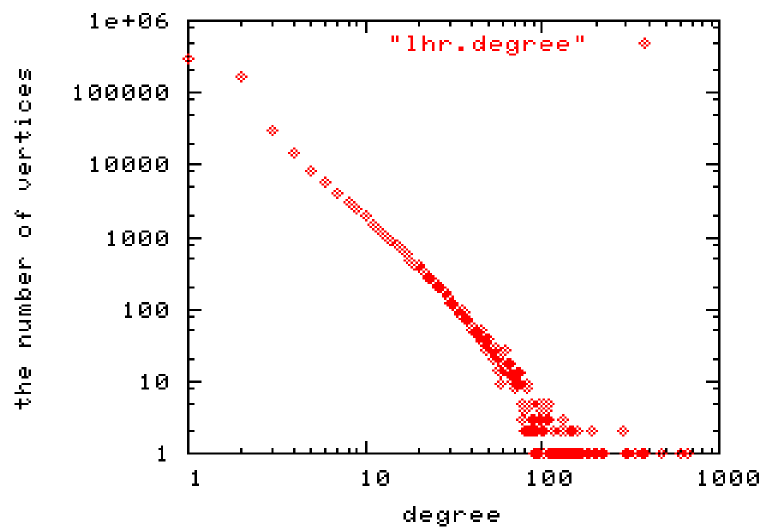
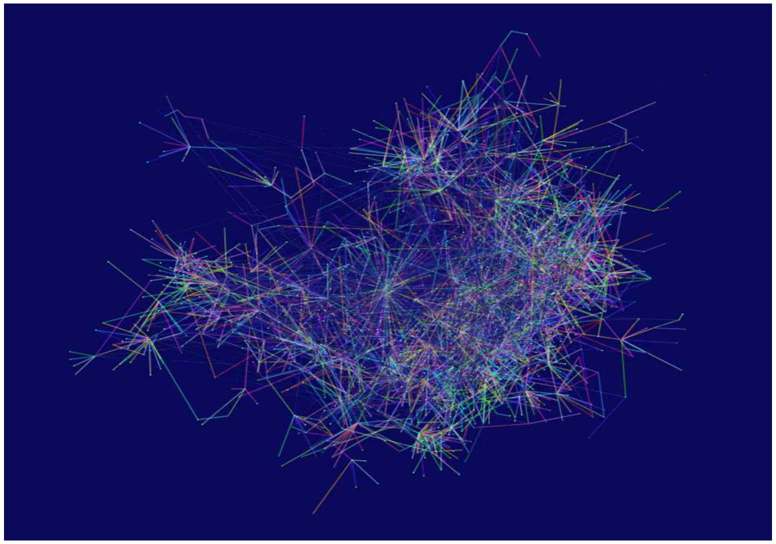


Left: The collaboration graph follows the power law degree distribution with exponent $\beta \approx 3.0$

Right: An IP graph follows the power law degree distribution with exponent $\beta \approx 2.4$

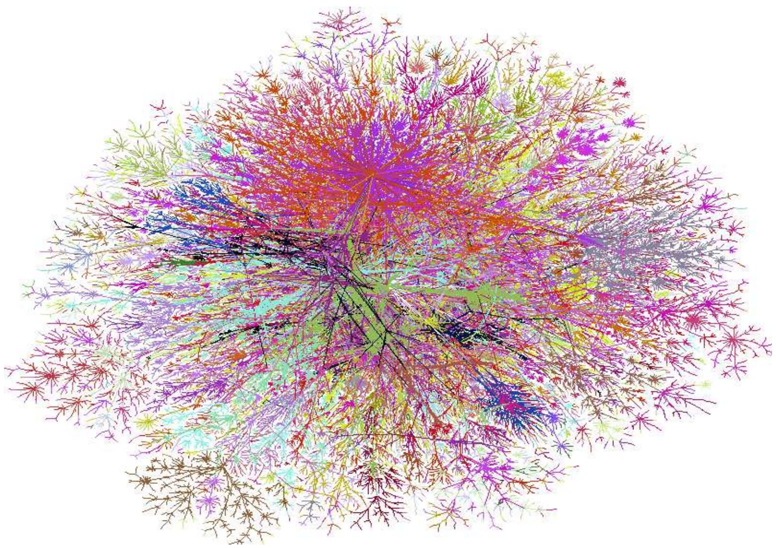


Power law graphs

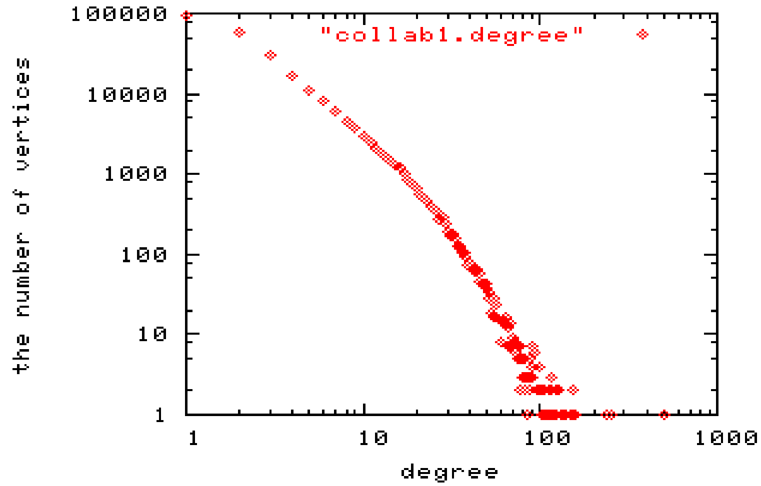
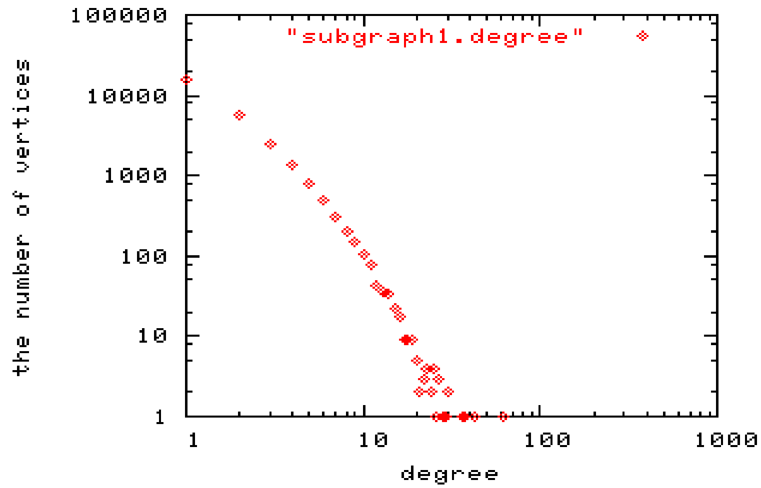


Left: Part of the collaboration graph (authors with Erdős number 2)

Right: An IP graph (by Bill Cheswick)



Robustness of Power Law

size	degree distribution
25,3339	
52,186	

Basic questions

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- What graph properties can be derived from the model?

Model $G(w_1, w_2, \ldots, w_n)$

Random graph model with given expected degree sequence (Chung-Lu model)

- n nodes with weights $w_1, w_2, \ldots, w_n$.

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$$\prod_{ij \in E(H)} p_{ij} \prod_{ij \notin E(H)} (1 - p_{ij}).$$

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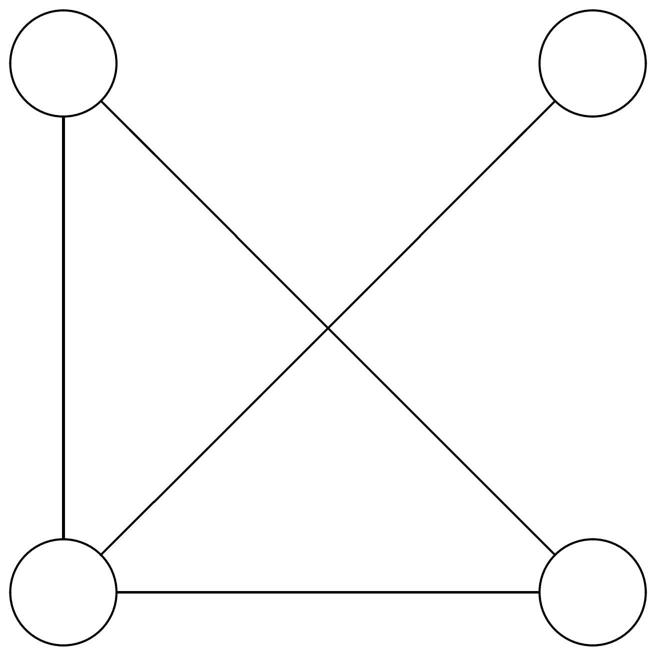
- The expected degree of vertex i is w_i.

An example: $G(w_1, w_2, w_3, w_4)$



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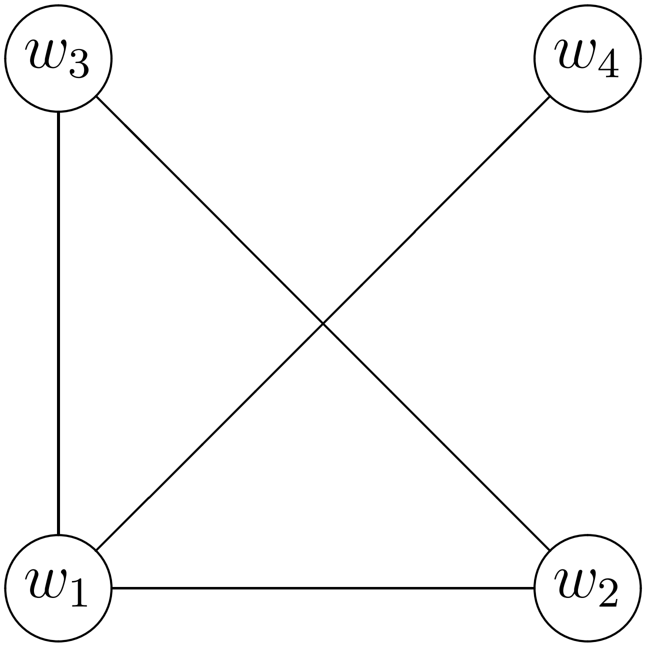

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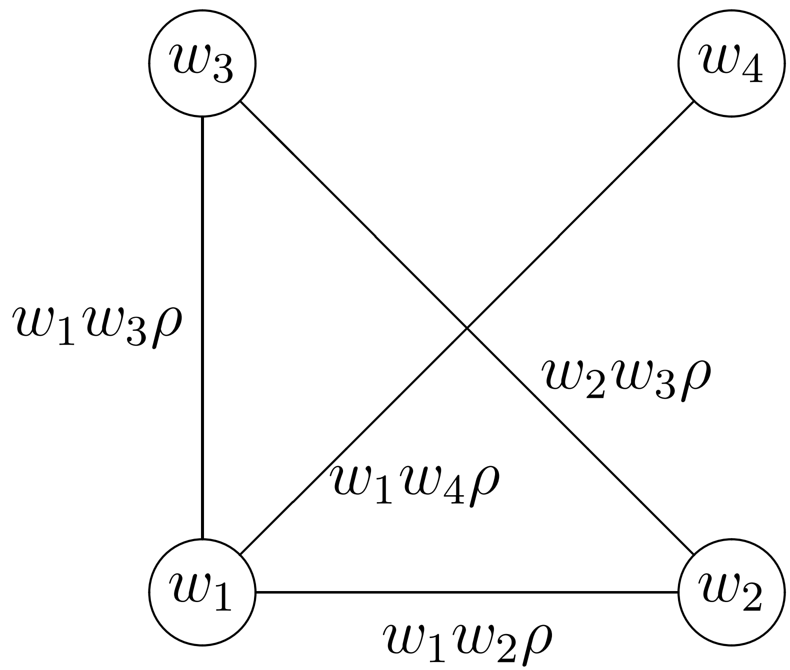

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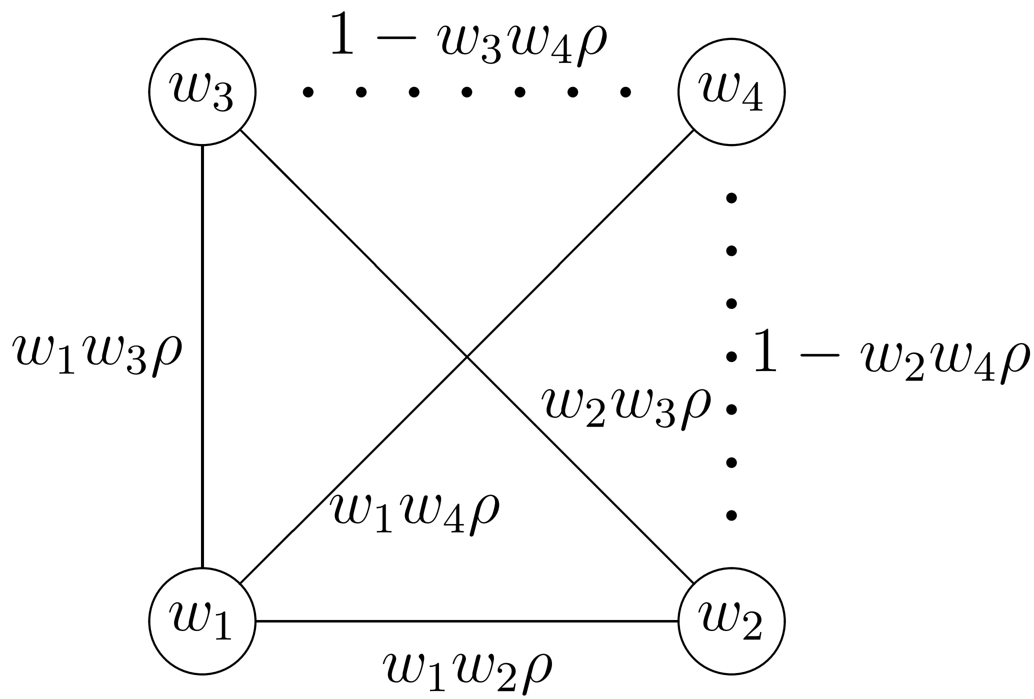

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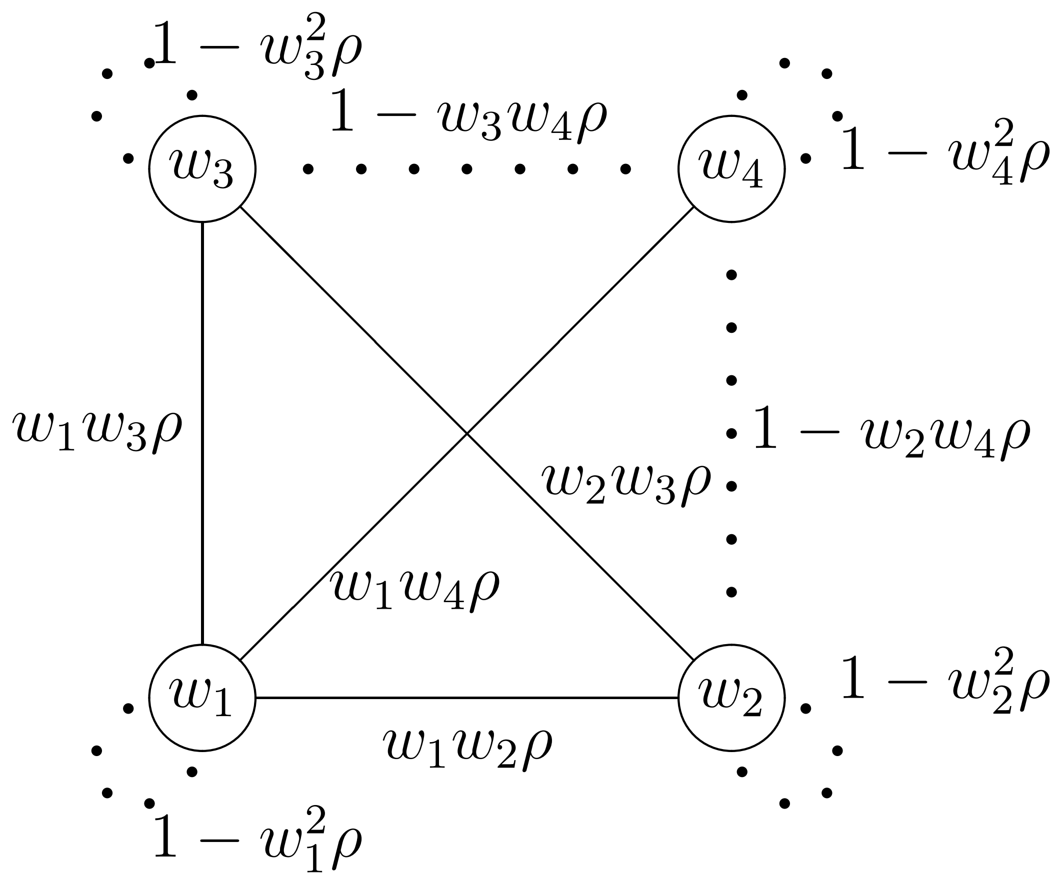

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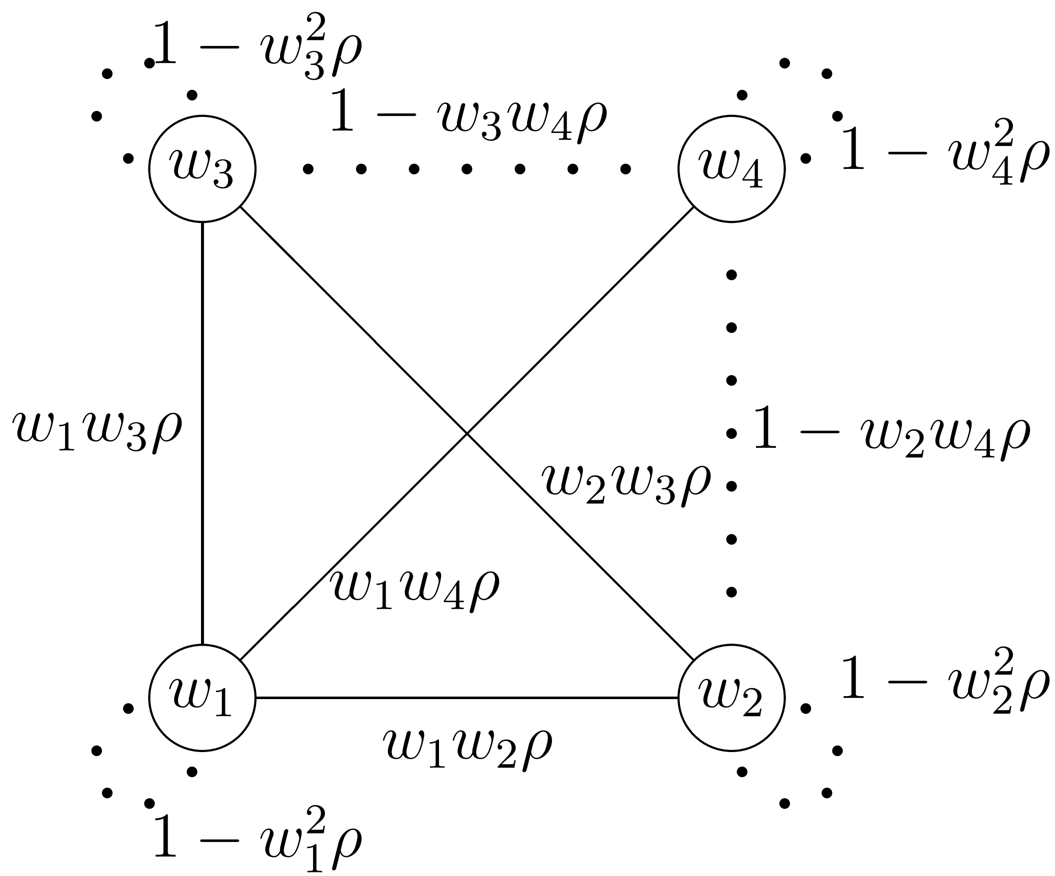

The probability of the graph is

$$w_1^3 w_2^2 w_3^2 w_4 \rho^4 (1 - w_2w_4\rho) \times (1 - w_3w_4\rho) \prod_{i=1}^{4} (1 - w_i^2\rho).$$

Chung-Lu model

For $G = G(w_1, \ldots, w_n)$, let

- $d = \frac{1}{n} \sum_{i=1}^{n} w_i$
- $\tilde{d} = \frac{\sum_{i=1}^{n} w_i^2}{\sum_{i=1}^{n} w_i}$.
- The volume of S: $\mathrm{Vol}(S) = \sum_{i \in S} w_i$.

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"$=$" holds if and only if $w_1 = \cdots = w_n$.

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"$=$" holds if and only if $w_1 = \cdots = w_n$.

A connected component S is called a giant component if

$$\mathrm{vol}(S) = \Theta(\mathrm{vol}(G)).$$

Connected components

Chung and Lu (2001) For $G = G(w_1, \ldots, w_n)$,

- If $\tilde{d} < 1 - \epsilon$, then almost surely, all components have volume at most $O(\sqrt{n}\log n)$.

Connected components

Chung and Lu (2001) For $G = G(w_1, \ldots, w_n)$,

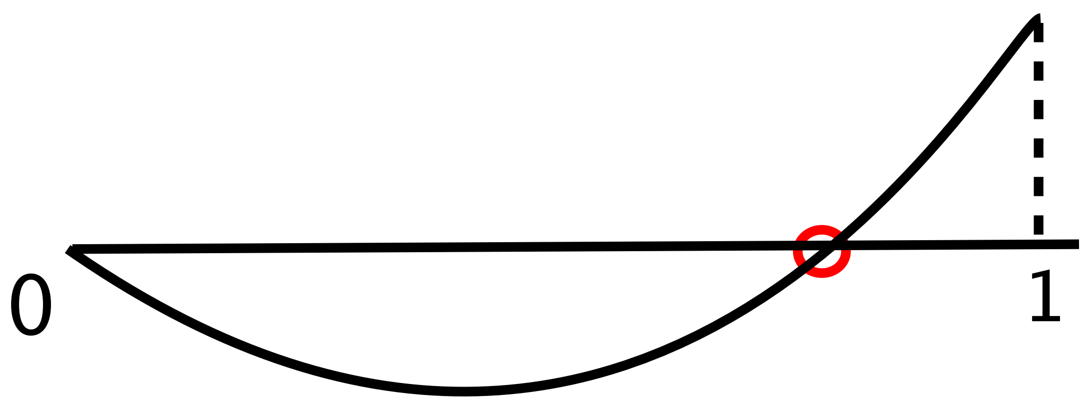
- If $\tilde{d} < 1 - \epsilon$, then almost surely, all components have volume at most $O(\sqrt{n} \log n)$.
- If $d > 1 + \epsilon$, then almost surely there is a unique giant component of volume $\Theta(\mathrm{Vol}(G))$. All other components have size at most

$$\begin{cases} \frac{\log n}{d-1-\log d-\epsilon d} & \text{if } \frac{1}{1-\epsilon} < d < \frac{2}{1-\epsilon} \\ \frac{\log n}{1+\log d-\log 4+2\log(1-\epsilon)} & \text{if } d > \frac{4}{e(1-\epsilon)^2}. \end{cases}$$

Volume of Giant Component

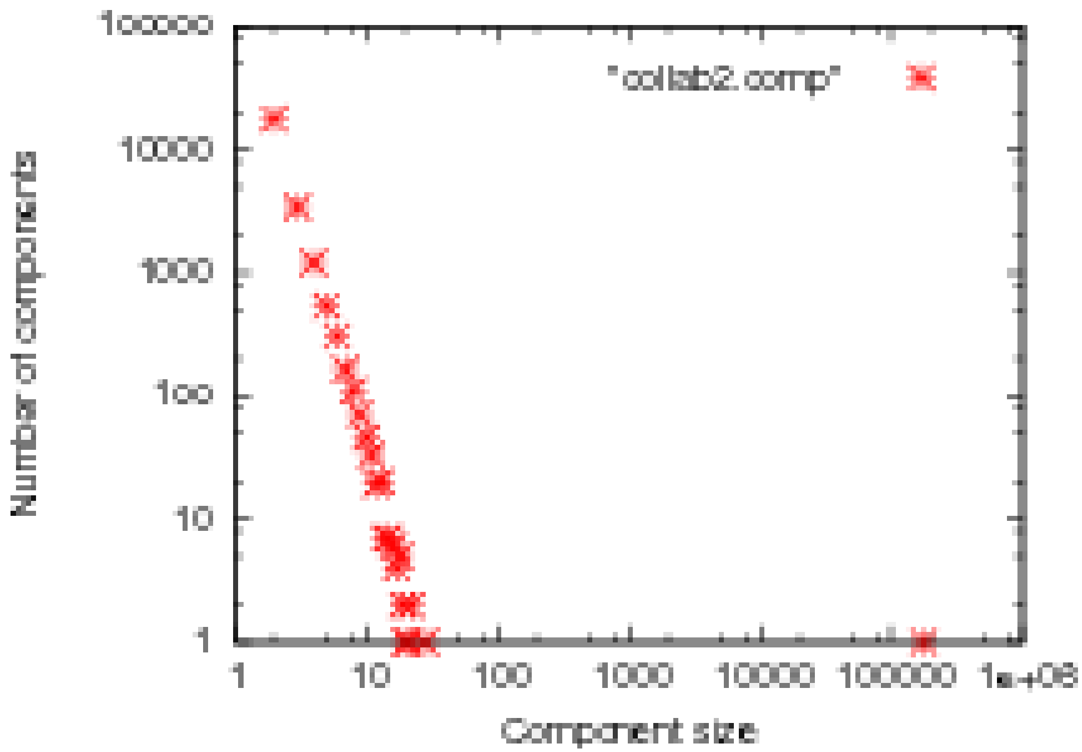
Chung and Lu (2004)
If the average degree is strictly greater than 1, then almost surely the giant component in a graph G in $G(\mathbf{w})$ has volume $(\lambda_0 + O(\sqrt{n}\frac{\log^{3.5} n}{\mathrm{Vol}(G)}))\mathrm{Vol}(G)$, where λ_0 is the unique positive root of the following equation:

$$\sum_{i=1}^{n} w_i e^{-w_i\lambda} = (1-\lambda)\sum_{i=1}^{n} w_i.$$



A real application

Apply to the Collaboration Graph (2002 data):
The size of giant component is predicted to be about $177,400$ by our theory. This is rather close to the actual value $176,000$, within an error bound of less than 1%.



$G(n,p)$ versus $G(w_1,\ldots,w_n)$

Question: Does the random graph with equal expected degrees generates the smallest giant component among all possible degree distribution with the same volume?

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Question: Does the random graph with equal expected degrees generates the smallest giant component among all possible degree distribution with the same volume?
Chung Lu (2004)

- Yes, for $1 < d \leq \frac{e}{e-1}$.
- No, for sufficiently large d.
- When $d \geq \frac{4}{e}$, almost surely the giant component of $G(w_1,\ldots,w_n)$ has volume at least

$$\left(\frac{1}{2}(1+\sqrt{1-\frac{4}{de}}) + o(1)\right)\mathrm{Vol}(G).$$

This is asymptotically best possible.

Diameter of $G(w_1, \ldots, w_n)$

Chung Lu (2002)

- For a random graph G with admissible expected degree sequence $(w_1, \ldots, w_n)$, the average distance is almost surely $(1 + o(1))\frac{\log n}{\log \tilde{d}}$.

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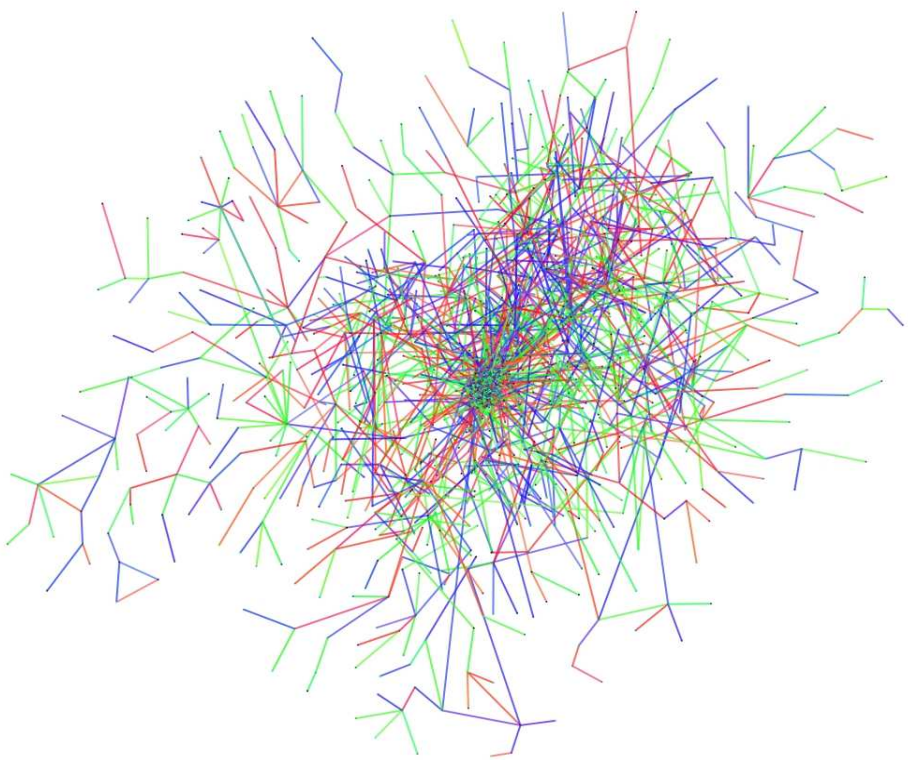
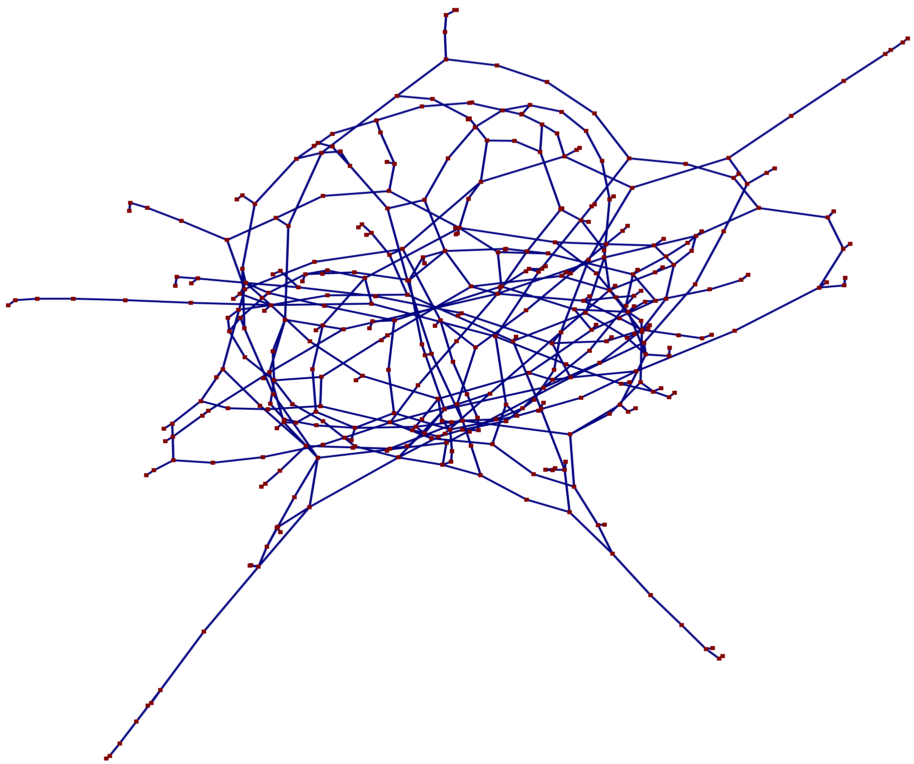
These results apply to $G(n, p)$ and random power law graph with $\beta > 3$.

Non-admissible graph versus admissible graph



A random subgraph of the Collaboration Graph.



A Connected component of $G(n, p)$ with $n = 500$ and $p = 0.002$.

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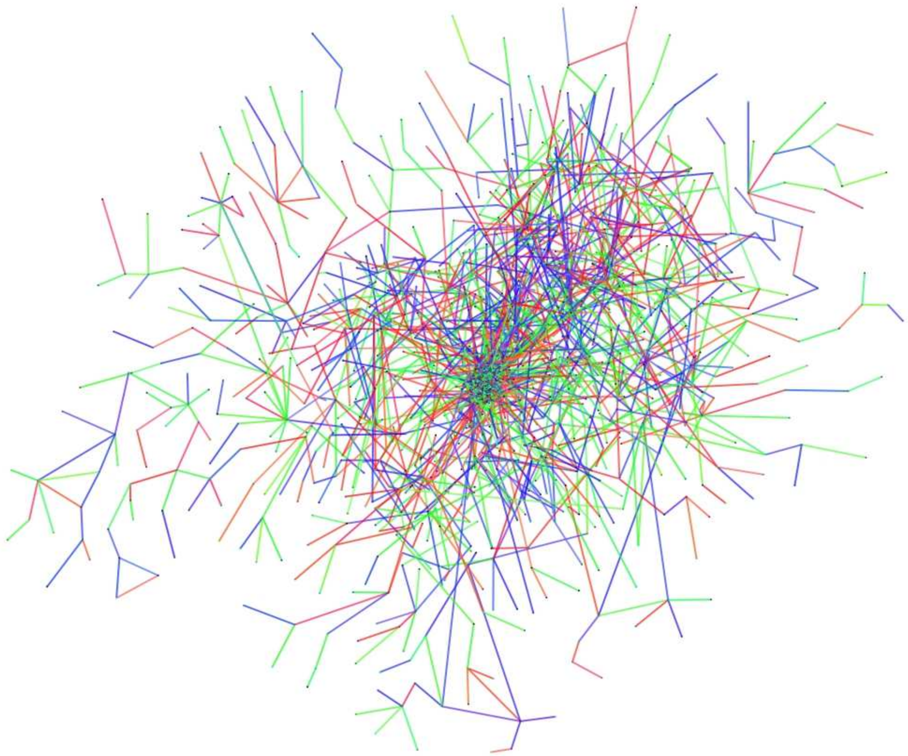

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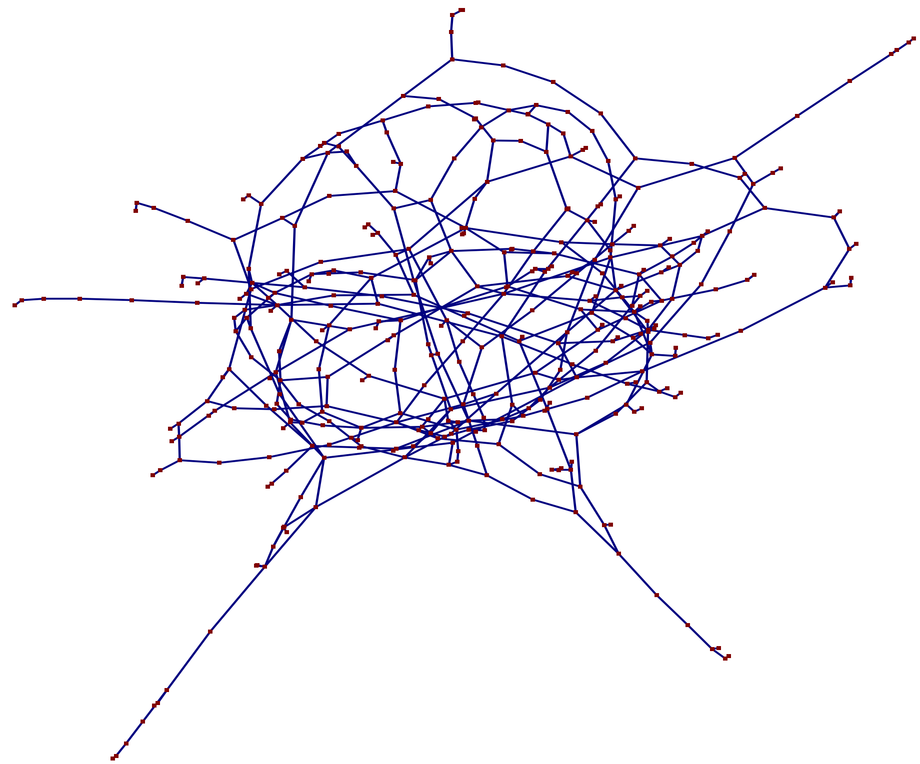

A Connected component of $G(n, p)$ with $n = 500$ and $p = 0.002$.

- Dense core for non-admissible graphs.
- No dense core for admissible graphs.

Power law graphs with $\beta \in (2,3)$

Chung, Lu (2002)

- Examples: the WWW graph, Collaboration graph, etc.

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- There are some vertices at the distance of $O(\log n)$.

The diameter is $\Theta(\log n)$, while the average distance is $O(\log\log n)$.

Experimental results

- **Faloutsos et al. (1999)** The eigenvalues of the Internet graph do not follow the semicircle law.
- **Farkas et. al. (2001), Goh et. al. (2001)** The spectrum of a power law graph follows a "triangular-like" distribution.
- **Mihail and Papadimitriou (2002)** They showed that the large eigenvalues are determined by the large degrees. Thus, the significant part of the spectrum of a power law graph follows the power law.

$$\mu_i \approx \sqrt{d_i}.$$

Eigenvalues of $G(w_1, \ldots, w_n)$

Chung, Vu, and Lu (2003)
Suppose $w_1 \geq w_2 \geq \ldots \geq w_n$. Let μ_i be i-th largest eigenvalue of $G(w_1, w_2, \ldots, w_n)$. Let $m = w_1$ and $\tilde{d} = \sum_{i=1}^{n} w_i^2 \rho$. Almost surely we have:

- $(1-o(1)) \max\{\sqrt{m}, \tilde{d}\} \leq \mu_1 \leq 7\sqrt{\log n} \cdot \max\{\sqrt{m}, \tilde{d}\}$.

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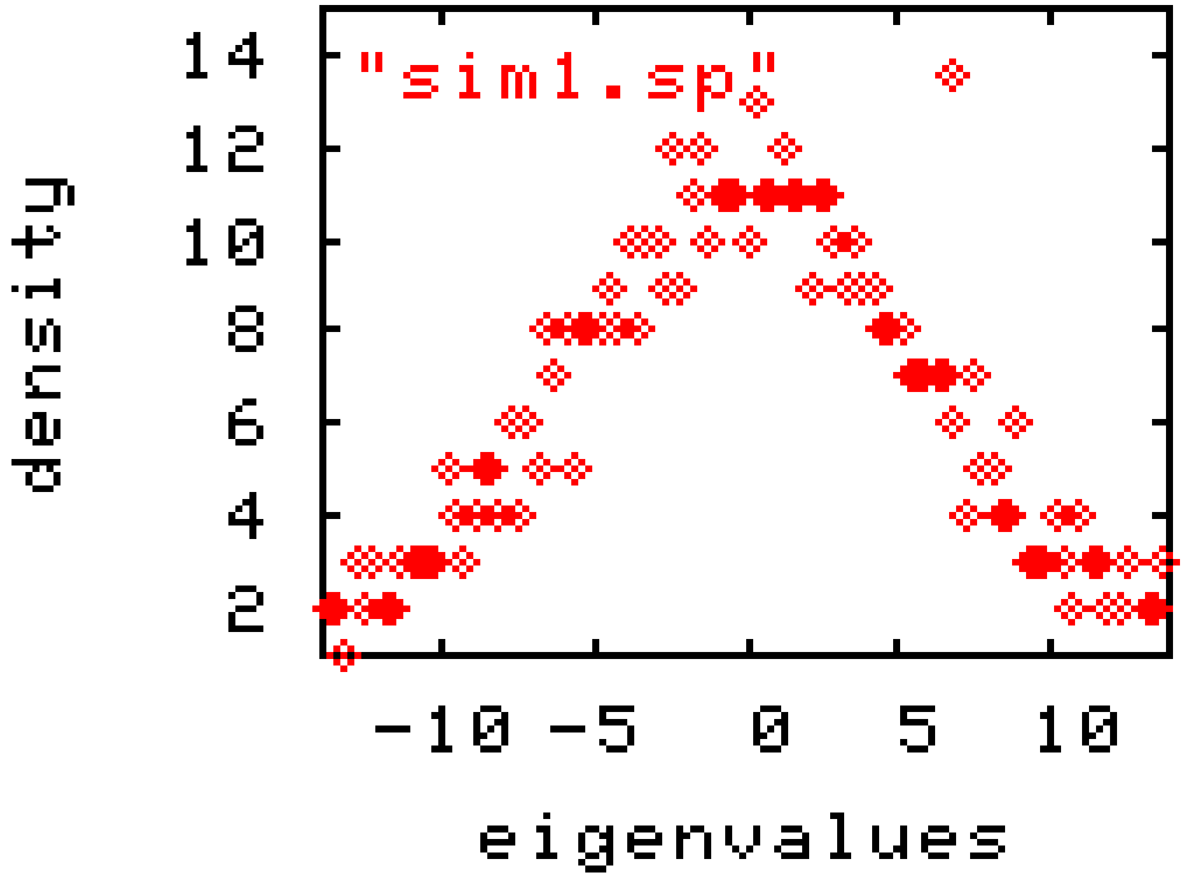
- $\mu_1 = (1 + o(1))\sqrt{m}$, if $\sqrt{m} > \tilde{d} \log^2 n$.
- $\mu_k \approx \sqrt{w_k}$ and $\mu_{n+1-k} \approx -\sqrt{w_k}$, if $\sqrt{w_k} > \tilde{d} \log^2 n$.

Random power law graphs

The first k and last k eigenvalues of the random power law graph with $\beta > 2.5$ follows the power law distribution with exponent $2\beta - 1$. It results a "triangular-like" shape.

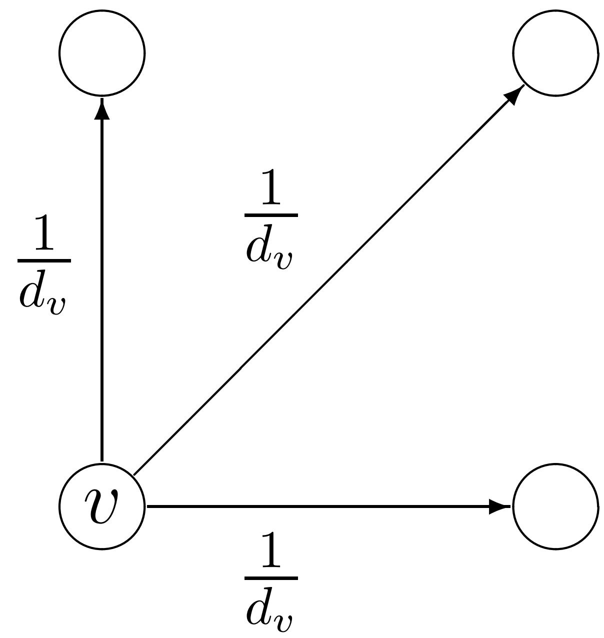


Laplacian spectrum

Random walks on a graph G:

$$\pi_{k+1} = AD^{-1}\pi_k.$$

$$AD^{-1} \sim D^{-1/2}AD^{-1/2}.$$



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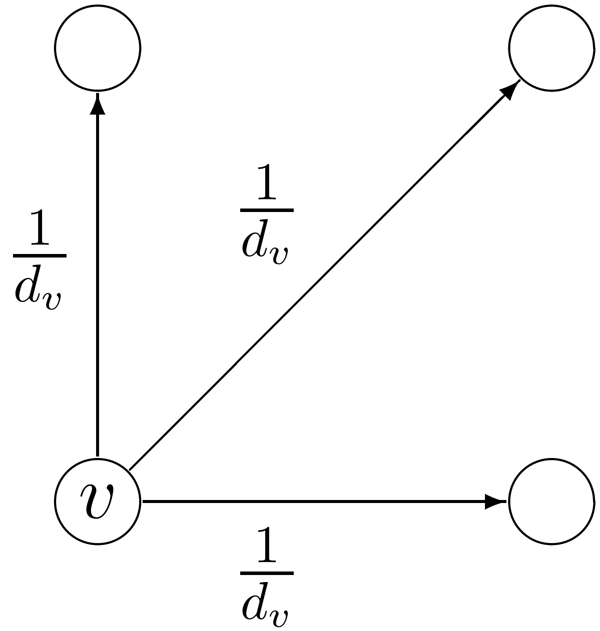

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$$0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1} \le 2$$

are the eigenvalues of $L = I - D^{-1/2}AD^{-1/2}$.

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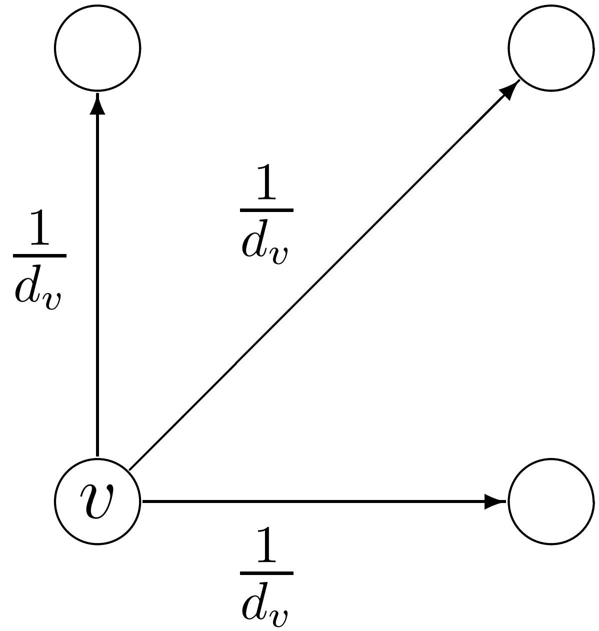

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are the eigenvalues of $L = I - D^{-1/2}AD^{-1/2}$.
The eigenvalues of AD^{-1} are $1, 1-\lambda_1, \ldots, 1-\lambda_{n-1}$.

Laplacian Spectral Radius

Let

- $w_{min} = \min\{w_1, \ldots, w_n\}$,
- $d = \frac{1}{n}\sum_{i=1}^{n} w_i$,
- $g(n)$ — a function tending to infinity arbitrarily slowly.

Chung, Vu, and Lu (2003)

- If $w_{\min} \gg \log^2 n$, then almost surely the Laplacian spectrum λ_i's of $G(w_1, \ldots, w_n)$ satisfy

$$\max_{i\neq 0} |1 - \lambda_i| \leq (1 + o(1))\frac{4}{\sqrt{d}} + \frac{g(n)\log^2 n}{w_{\min}}.$$

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- If $w_{\min} \gg \sqrt{d}$, the Laplacian spectrum follows the semi-circle distribution with radius $r \approx \frac{2}{\sqrt{d}}$.

General random graphs

General edge-independent random graphs:

- n: the number of vertices.
- p_{ij}: a probability for ij being an edge.
- Edges are mutually independent.

Question: What can we say about the spectrum of the adjacency matrix and the Laplacian matrix?

Notation

- A: adjacency matrix
- $\bar{A} := (p_{ij})$: the expectation of A
- Δ: the maximum expected degree
- δ: the minimum expected degree
- D: the diagonal matrix of degrees
- $\bar{D}$: the expectation of D
- $L := I - D^{-1/2}AD^{-1/2}$: the normalized Laplacian
- $\bar{L} := I - \bar{D}^{-1/2}\bar{A}\bar{D}^{-1/2}$: the Laplacian of $\bar{A}$

Known results

Oliveira [2010]: For $\Delta \geq C \ln n$, with high probability we have

$$|\lambda_i(A) - \lambda_i(\bar{A})| \leq 4\sqrt{\Delta \ln n}.$$

For $\delta \geq C \ln n$, with high probability we have

$$\lambda_i(L) - \lambda_i(\bar{L}) \leq 14\sqrt{\ln(4n)/\delta}.$$

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For $\delta \geq C \ln n$, with high probability we have

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Chung-Radcliffe [2011] reduces the constant coefficient using a new matrix Chernoff inequality.

Our results

Lu-Peng [2012+]: If $\Delta \gg \ln^4 n$, then almost surely

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Let $\Lambda := \{\lambda_i(\bar{L}) \colon |1 - \lambda_i(\bar{L})| = \omega(1/\sqrt{\ln n})\}$.
If $\delta \gg \max\{|\Lambda|, \ln^4 n\}$, then almost surely

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In both case, we remove the multiplicative factor $\sqrt{\ln n}$.

Random symmetric matrices

$B = (b_{ij})$ is a random symmetric matrix satisfying:

- b_{ij}: independent, but not necessary identical,
- $|b_{ij}| \leq K$,
- $\mathrm{E}(b_{ij}) = 0$,
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Vu [2007]:

$$\|B\| \leq 2\sigma\sqrt{n} + c(K\sigma)^{1/2}n^{1/4}\ln n.$$

Our result

Lu-Peng [2012+]: We further assume $\mathrm{Var}(b_{ij}) \leq \sigma_{ij}^2$. Let $\Delta := \max_{1\leq i\leq n}\sum_{j=1}^{n}\sigma_{ij}^2$. If $\Delta \geq C'K^2\ln^4 n$, then asymptotically almost surely

$$\|B\| \leq 2\sqrt{\Delta} + C\sqrt{K}\Delta^{1/4}\ln n.$$

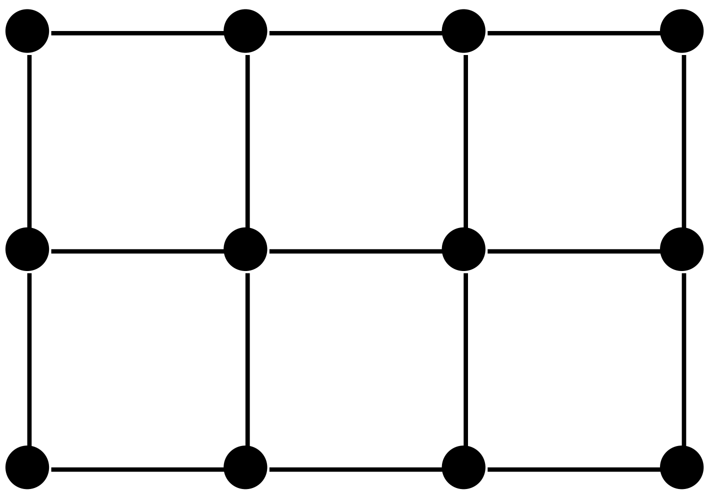
- It generalizes Vu's theorem.
- This result is asymptotically tight.

Graph percolation

- G: a connected graph on n vertices
- p: a probability ($0 \leq p \leq 1$)

G_p: a random spanning subgraph of G, obtained as follows: for each edge f of G, independently,

$$\Pr(f \text{ is an edge of } G_p) = p.$$



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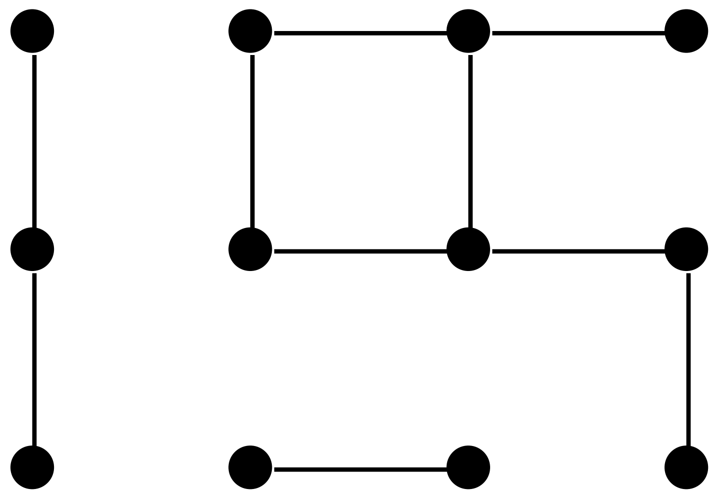

Spectrum of G_p

Lu-Peng [2012+]:

- If $p \gg \frac{\ln^4 n}{\Delta}$, then almost surely we have

$$|\lambda_i(A(G_p)) - p\lambda_i(A(G))| \leq (2 + o(1))\sqrt{p\Delta}.$$

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- Suppose that all but k Laplacian eigenvalues λ of G satisfies $|1-\lambda| = o(\frac{1}{\sqrt{\ln n}})$. If $\delta \gg \max\{k, \ln^4 n\}$, then for $p \gg \max\{\frac{k}{\delta}, \frac{\ln^4 n}{\delta}\}$, almost surely we have $|\lambda_i(L(G_p)) - \lambda_i(L(G))| \leq (2 + \sqrt{\sum_{i=1}^k (1-\lambda_i)^2} + o(1))) \frac{1}{\sqrt{p\delta}}$.

□

Method

We will illustrate Wigner's trace method through the sketch proof of the following result.

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- $\mathrm{E}(b_{ij}) = 0$,
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then almost surely

$$\|B\| \leq 2\sqrt{\Delta} + C\sqrt{K}\Delta^{1/4}\ln n,$$

where $\Delta := \max_{1\leq i\leq n} \sum_{j=1}^{n} \sigma_{ij}^2$.

Sketch proof

WLOG, we can assume $K = 1$ and $b_{ii} = 0$. Using Wigner's trace method, we have

$$\mathrm{E}\left(\mathrm{Trace}(B^k)\right) = \sum_{i_1, i_2, \ldots, i_k} \mathrm{E}(b_{i_1 i_2} b_{i_2 i_3} \ldots b_{i_{k-1} i_k} b_{i_k i_1})$$

$$= \sum_{p=2}^{\lfloor k/2 \rfloor + 1} \sum_{w \in \mathcal{G}(n,k,p)} \prod_{e \in E(w)} \mathrm{E}(b_e^{q_e}).$$

Here $\mathcal{G}(n, k, p)$ is the set of "good" closed walks w in K_n of length k on p vertices, where each edge in w appears more than once ($q_e \geq 2$).

Continue

Let $\tilde{\mathcal{G}}(k,p)$ be the set of good closed walks w of length k on the complete graph K_p where vertices first appear in w in the order $1, 2, \ldots, p$.

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All walks in $\mathcal{G}(n,k,p)$ can be coded by a walk in $\tilde{\mathcal{G}}(k,p)$ plus the ordered p distinct vertices. Let $[n]^{\underline{p}} := \{(v_1, v_2, \ldots, v_p) \in [n]^p \colon v_1, v_2, \ldots, v_p$ are distinct$\}$.

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$[n]^{\underline{p}} := \{(v_1, v_2, \ldots, v_p) \in [n]^p \colon v_1, v_2, \ldots, v_p$ are distinct$\}$.

Define a rooted tree $T(w)$ so that the edge $i_j i_{j+1} \in E(T(w))$ if it brings in a new vertex i_{j+1} when it occurs first time.

continue

$$\sum_{w\in\mathcal{G}(n,k,p)}\prod_{e\in E(w)}\sigma_e^2 = \sum_{\tilde{w}\in\tilde{\mathcal{G}}(k,p)}\sum_{(v_1,\ldots,v_p)\in[n]^{\underline{p}}}\prod_{xy\in E(\tilde{w})}\sigma^2_{v_xv_y}$$

$$\leq \sum_{\tilde{w}\in\tilde{\mathcal{G}}(k,p)}\sum_{v_1=1}^{n}\sum_{v_2=1}^{n}\cdots\sum_{v_p=1}^{n}\prod_{xy\in E(T)}\sigma^2_{v_xv_y}$$

$$= \sum_{\tilde{w}\in\tilde{\mathcal{G}}(k,p)}\sum_{v_1=1}^{n}\sum_{v_2=1}^{n}\cdots\sum_{v_{p-1}=1}^{n}\prod_{y=2}^{p-1}\sigma^2_{v_{\eta(y)}v_y}\sum_{v_p=1}^{n}\sigma^2_{v_{\eta(p)}v_p}$$

$$\leq \Delta\sum_{\tilde{w}\in\tilde{\mathcal{G}}(k,p)}\sum_{v_1=1}^{n}\sum_{v_2=1}^{n}\cdots\sum_{v_{p-1}=1}^{n}\prod_{y=2}^{p-1}\sigma^2_{v_{\eta(y)}v_y}$$

$$\leq \cdots$$

$$\leq n\Delta^{p-1}\left|\tilde{\mathcal{G}}(k,p)\right|.$$

Continue

Vu [2007] proved

$$|\tilde{\mathcal{G}}(k,p)| \leq \binom{k}{2p-2} 2^{2k-2p+3} p^{k-2p+2} (k-2p+4)^{k-2p+2}.$$

Continue

Vu [2007] proved

$$|\tilde{\mathcal{G}}(k,p)| \leq \binom{k}{2p-2} 2^{2k-2p+3} p^{k-2p+2} (k-2p+4)^{k-2p+2}.$$

We get

$$\left|\mathrm{E}\left(\mathrm{Trace}(B^k)\right)\right| \leq \sum_{w \in \mathcal{G}(n,k)} \prod_{e \in E(w)} \sigma_e^2 \leq \sum_{p=2}^{k/2+1} n\Delta^{p-1} \left|\tilde{\mathcal{G}}(k,p)\right|$$

$$\leq n \sum_{p=2}^{k/2+1} \Delta^{p-1} \binom{k}{2p-2} 2^{2k-2p+3} p^{k-2p+2} (k-2p+4)^{k-2p+2}$$

$$:= n \sum_{p=2}^{k/2+1} S(n,k,p).$$

Continue

One can show

$$S(n,k,p-1) \leq \frac{16k^4}{\Delta} S(n,k,p).$$

Continue

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For any even integer k such that $k^4 \le \frac{\Delta}{32}$, we get

$$\begin{aligned}
\left|\mathrm{E}\left(\mathrm{Trace}(B^k)\right)\right| &\le \sum_{p=2}^{k/2+1} S(n,k,p) \\
&\le S(n,k,k/2+1) \sum_{p=2}^{k/2+1} \left(\frac{1}{2}\right)^{k/2+1-p} \\
&< 2S(n,k,k/2+1) \\
&= n2^{k+2}\Delta^{k/2}.
\end{aligned}$$

continue

For even k, we have

$$
\begin{aligned}
&\Pr(\|B\| \geq 2\sqrt{\Delta} + C\Delta^{1/4}\ln n) \\
&= \Pr(\|B\|^k \geq (2\sqrt{\Delta} + C\Delta^{1/4}\ln n)^k) \\
&\leq \Pr(\mathrm{Trace}(B^k) \geq (2\sqrt{\Delta} + C\Delta^{1/4}\ln n)^k) \\
&\leq \frac{\mathrm{E}(\mathrm{Trace}(B^k))}{(2\sqrt{\Delta} + C\Delta^{1/4}\ln n))^k} \quad \text{(Markov's inequality)} \\
&\leq \frac{n2^{k+2}\Delta^{k/2}}{(2\sqrt{\Delta} + C\Delta^{1/4}\ln n))^k} \\
&= 4ne^{-(1+o(1))\frac{C}{2}k\Delta^{-1/4}\ln n}.
\end{aligned}
$$

Setting $k = \left(\frac{\Delta}{32}\right)^{1/4}$, this probability is $o(1)$ for sufficiently large C. □

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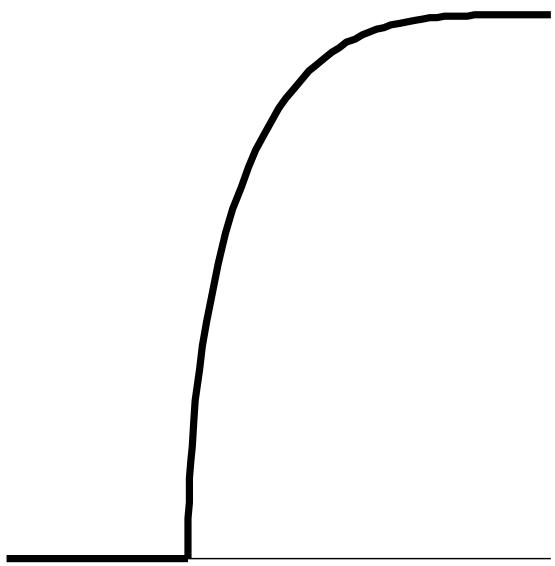
Percolation threshold p_c

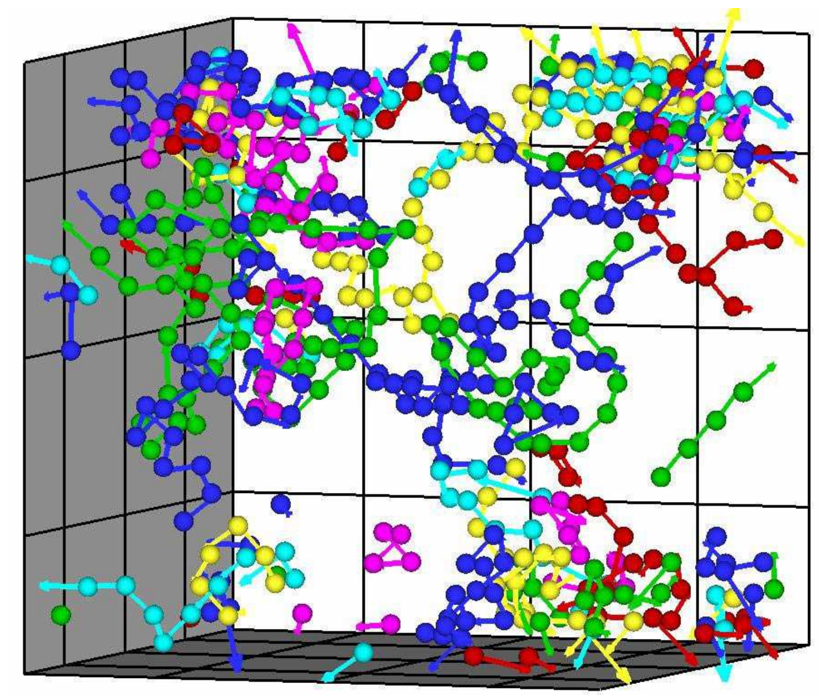
- For $p < p_c$, almost surely there is no giant component
- For $p > p_c$, almost surely there is a giant component.



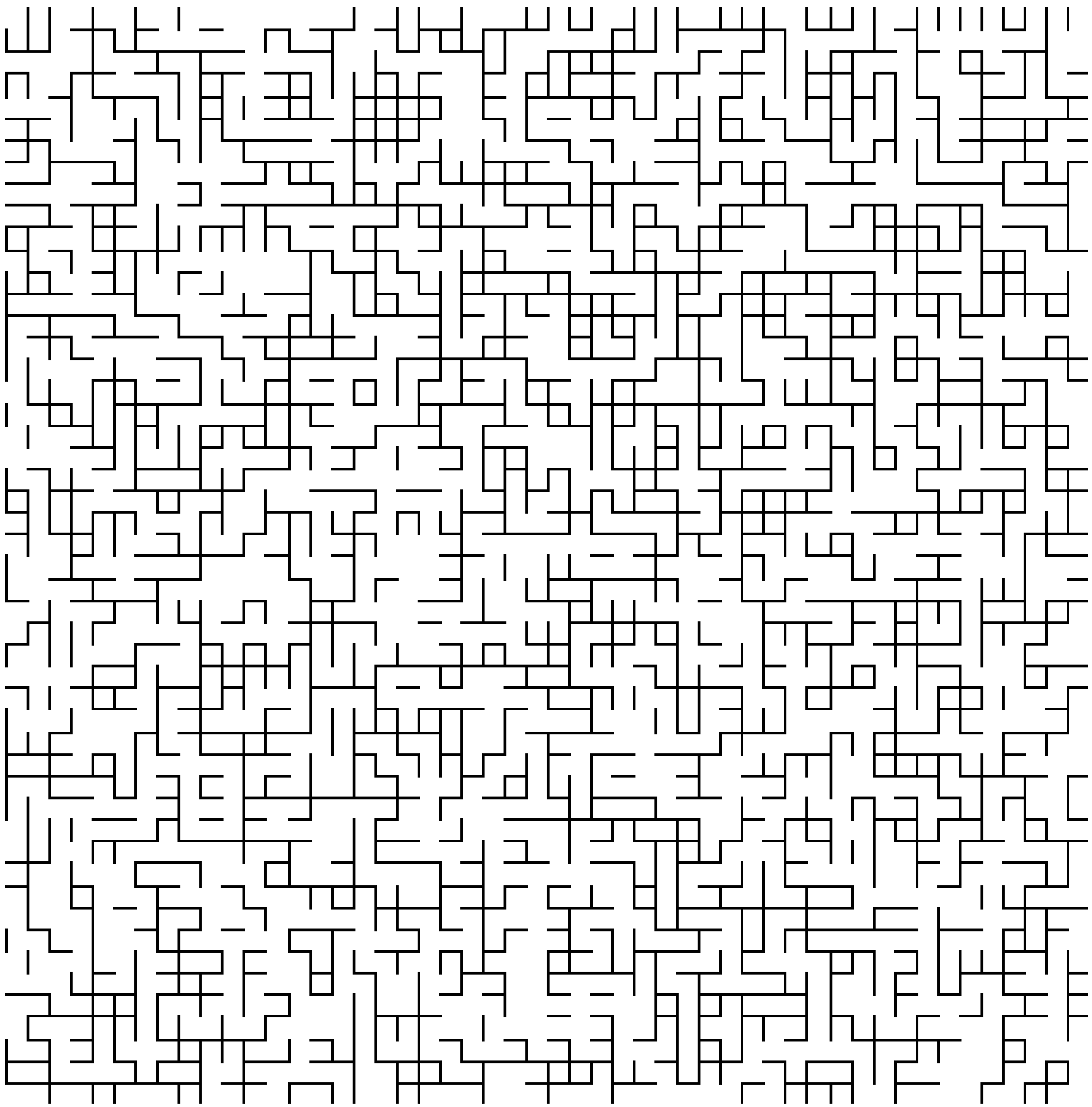
p_c

Motivations

- Graph theory: random graphs
- Theoretical physics: crystals melting
- Sociology: the spread of disease on contact networks



Percolation of Z^d



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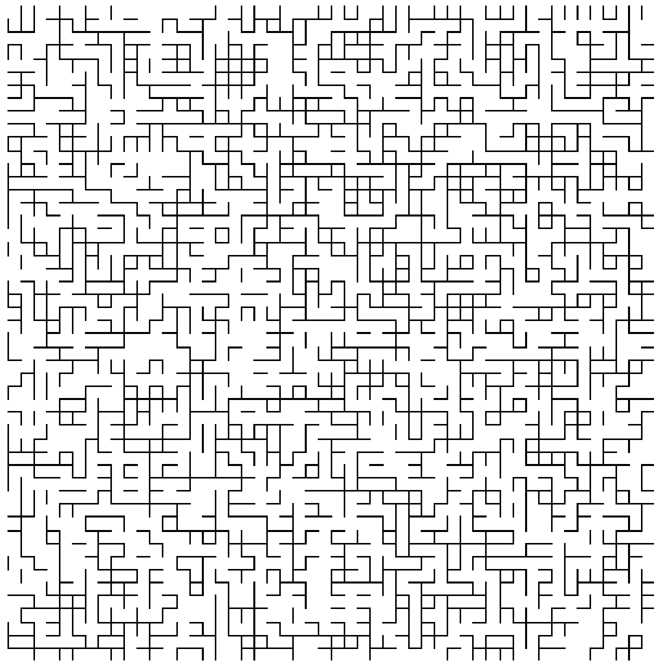

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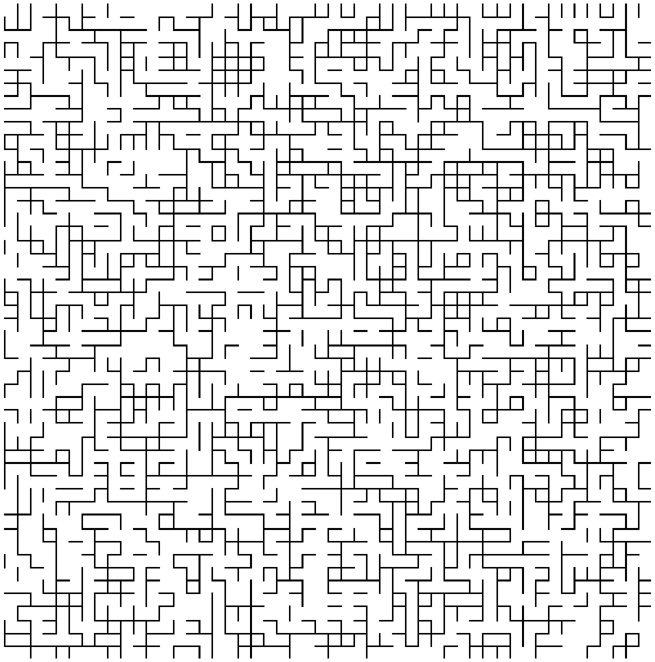

Kesten (1980): $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

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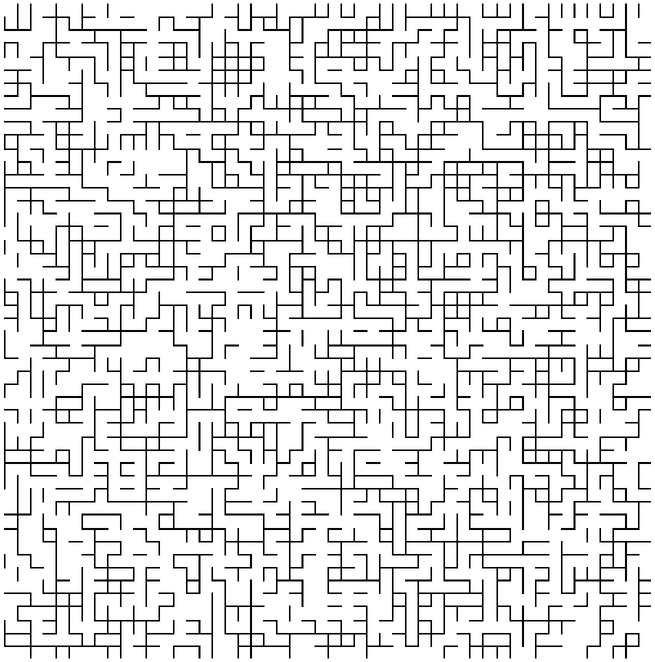

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Lorenz and Ziff (1997, simulation):
$p_c(\mathbb{Z}^3) \approx 0.2488126 \pm 0.0000005$ if it exists.

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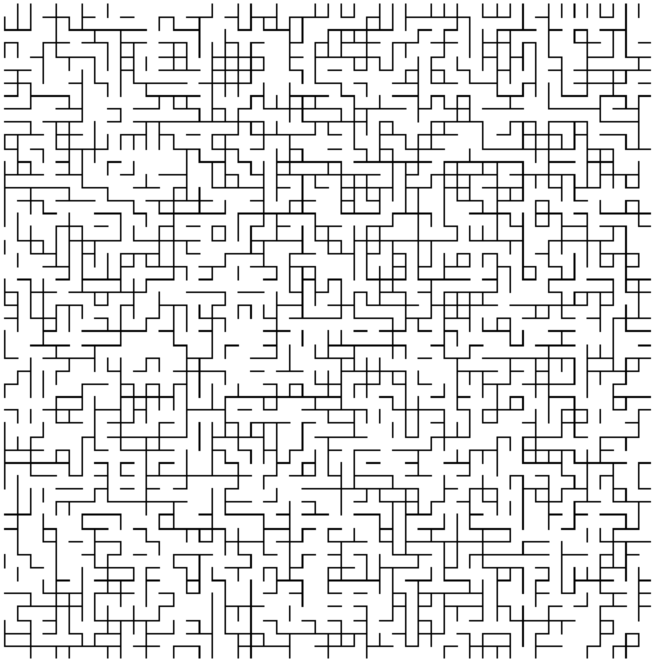

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Kesten (1990): $p_c(\mathbb{Z}^d) \sim \frac{1}{2d}$ as $d \to \infty$.

d-regular graphs

Alon, Benjamini, Stacey (2004): Suppose $d \geq 2$ and let (G_n) be a sequence of d-regular expanders with $girth(G_n) \to \infty$, then

$$p_c = \frac{1}{d-1} + o(1).$$

Percolation of dense graphs

Bollobás, Borgs, Chayes, and Riordan (2008): Suppose that G is a dense graph (i.e., average degree $d = \Theta(n)$). Let μ be the largest eigenvalue of the adjacency matrix of G. Then

$$p_c \approx \frac{1}{\mu}.$$

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Remark: The requirement of "dense graph" is essential. Their methods can not be extended to sparse graphs.

Percolation of sparse graphs

Chung, Lu, Horn [2008]:

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Chung, Lu, Horn [2008]:

- If $p < \frac{1}{\mu}$, then G_p has no giant component.
- The condition $p > \frac{1}{\mu}$ in general does not imply that G_p has a giant component.
- If $p > \frac{1}{\mu}$, $\Delta = O(d)$, and $\sigma = o(\frac{1}{\log n})$, then G_p has a giant component.

Percolation of $G(\mathbf{w})$

Bhamidi-van der Hofstad-van Leeuwaarden [2012]: Consider $G(\mathbf{w})$, where $\mathbf{w} = (w_1, \ldots, w_n)$ follows the power law of exponent β. If $\mathrm{E}(\sum_{i=1}^n w_i^2)$ converges and is bounded, then the percolation threshed is $(1 + o(1))\frac{1}{\tilde{d}}$.

- For $\beta > 4$, $\mathrm{E}(\sum_{i=1}^n w_i^3)$ converges. The largest component has the size $\Theta(n^{2/3})$ at the critical window.

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- For $\beta > 4$, $\mathrm{E}(\sum_{i=1}^n w_i^3)$ converges. The largest component has the size $\Theta(n^{2/3})$ at the critical window.
- For $2 < \beta < 3$, $\mathrm{E}(\sum_{i=1}^n w_i^3)$ diverges. The largest component has the size $\Theta(n^{\frac{\beta-2}{\beta-1}})$ at the critical window.

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Thank You