Graphs with Small Spectral Radius

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Selected Topics on Spectral Graph Theory (I)
Nankai University, Tianjin, May 16, 2014
Selected Topics on Spectral Graph Theory

1. Graphs with Small Spectral Radius
   Time: Friday (May 16) 4pm.-5:30p.m.

2. Laplacian and Random Walks on Graphs
   Time: Thursday (May 22) 4pm.-5:30p.m.

3. Spectra of Random Graphs
   Time: Thursday (May 29) 4pm.-5:30p.m.

4. Hypergraphs with Small Spectral Radius
   Time: Friday (June 6) 4pm.-5:30p.m.

5. Laplacian of Random Hypergraphs
   Time: Thursday (June 12) 4pm.-5:30p.m.
Backgrounds

I: Spectral Graph Theory    II: Random Graph Theory
III: Random Matrix Theory
Given an $n \times n$ real matrix $A$, if $A\alpha = \lambda \alpha$, then $\alpha$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$. 
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If $A$ is a real symmetric matrix, (i.e., $A' = A$), then $A$ has $n$ real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. There exists an orthogonal matrix $O$ such that

$$A = O^{-1} \Lambda O.$$ 

Here $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. 
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Spectral norm (or spectral radius)

$$\rho(A) = \left(\text{maximum eigenvalue of } A'A\right)^{1/2}.$$
Basic Linear Algebra

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If $A$ is real symmetric, then $\rho(A) = \max\{|\lambda_1|, |\lambda_n|\}$. 
Perron-Frobenius theorem

- \( A = (a_{ij}) \) is **non-negative** if \( a_{ij} \geq 0 \).
- \( A \) is **irreducible** if there exists a \( m \) such that \( A^m \) is positive.
- \( A \) is **aperiodic** if the greatest common divisor of all natural numbers \( m \) such that \((A^m)_{ii} > 0\) is 1.
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**Perron-Frobenius theorem:** If $A$ is an aperiodic irreducible non-negative matrix with spectral radius $r$, then $r$ is the largest eigenvalue in absolute value of $A$, and $A$ has an eigenvector $\alpha$ with eigenvalue $r$ whose components are all positive.
Basic Graph Notation

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 0 & 1 & 1 & 1 \\
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\[
\phi_{S_4} = \lambda^4 - 3\lambda^2 \quad \rho(S_4) = \sqrt{3}
\]
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$$\delta(G) \leq d(G) \leq \rho(G) \leq \Delta(G).$$
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If $G$ is $d$-regular (i.e., all degrees equal to $d$), then $\rho(G) = d$. 
Easy facts

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Graphs with small spectral radius
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- In particular, $\rho(G) \geq \sqrt{\Delta(G)}$. 
An application

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**Wilf’s Theorem [1967]:** $\chi(G) \leq 1 + \rho(G)$. 
The chromatic number $\chi(G)$ of a graph $G$ is the smallest number of colors needed to color the vertices of $G$ so that no two adjacent vertices share the same color.

Wilf’s Theorem [1967]: $\chi(G) \leq 1 + \rho(G)$.

Proof: Let $k = \max_{H \subseteq G} \delta(H)$, where $\delta(H)$ is the minimum degree of $H$. Order the vertices $v_1, v_2, \ldots, v_n$ so that each vertex $v_i$ has at most $k$ neighbors in $v_1, \ldots, v_{i-1}$. The greedy algorithm shows that $G$ is $(k + 1)$-colorable. Hence

$$\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$$

$$\leq 1 + \max_{H \subseteq G} \rho(H)$$

$$\leq 1 + \rho(G). \quad \Box$$
Graphs with $\rho(G) < 2$

Smith [1970]: $\rho(G) < 2$ if and only if $G$ is a simply-laced Dynkin diagram.

- $A_n$
- $D_n$
- $E_6$
- $E_7$
- $E_8$
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Smith’s theorem gives an equivalent graph-theory definition for the simply-laced Dynkin diagrams.
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Then \( \alpha_1, \ldots, \alpha_n \) forms a base of a root system.

Classifying irreducible simple-laced root systems is equivalent to classifying the connected graphs with \( \rho(G) < 2 \).
Graphs with $\rho(G) = 2$

**Smith [1970]:** $\rho(G) = 2$ if and only if $G$ is a simply-laced extended Dynkin diagram.

- $\tilde{A}_n$
- $\tilde{D}_n$
- $\tilde{E}_6$
- $\tilde{E}_7$
- $\tilde{E}_8$
Graphs: \[ 2 \leq \rho(G) < \sqrt{2 + \sqrt{5}} \]

Cvetkovic-Doob-Gutman [1982], completed by Brouwer-Neumaier [1989]:

\( T(1, b, c), \ b \geq 2, \ c \geq 6: \)

\[ \bullet \ -- \bullet \ -- \bullet \ -- \bullet \]

\( T(2, 2, c), \ c \geq 3: \)

\[ \bullet \ -- \bullet \ -- \bullet \]

\( Q(a, b, c), \ a \geq 3, \ c \geq 2, \ b > a + c: \)

\[ \bullet \ -- \bullet \ -- \bullet \]

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Shearer [1989]: For every number $\lambda \geq \sqrt{2 + \sqrt{5}} = 2.058171027...$, there exists a sequence of graphs $\{G_n\}$ such that $\lambda = \lim_{n \to \infty} \rho(G_n)$. 
Shearer [1989]: For every number $\lambda \geq \sqrt{2 + \sqrt{5}} = 2.058171027...$, there exists a sequence of graphs $\{G_n\}$ such that $\lambda = \lim_{n \to \infty} \rho(G_n)$.

$$\lim_{b,c \to \infty} \rho(T(1, b, c)) = \sqrt{2 + \sqrt{5}}.$$  
$$\lim_{c \to \infty} \rho(T(2, 2, c)) = \sqrt{2 + \sqrt{5}}.$$  
$$\lim_{n \to \infty} \rho(Q(n, 2n + 1, n)) = \sqrt{2 + \sqrt{5}}.$$
If $G_2$ is a proper subgraph of $G_1$, then $\rho(G_1) > \rho(G_2)$. 
Properties

- If $G_2$ is a proper subgraph of $G_1$, then $\rho(G_1) > \rho(G_2)$.
- Let $G'$ be a graph obtained from $G$ by subdividing an edge $uv$ of $G$. Then
  1. $\rho(G') > \rho(G)$ if $uv$ is not on an internal path and $G \neq C_n$.
  2. $\rho(G') < \rho(G)$ if $uv$ is on an internal path and $G \neq \tilde{D}_n$. 
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An internal path
Notation of an open quipus:

\[ P^{m_1,m_2,\ldots,m_t}_{n_1,n_2,\ldots,n_t,p}. \]

\[ P_{n_1} \quad \ldots \quad P_{n_t} \]

0 1 \quad m_1 \quad m_t \quad p - 1
In 2007, van Dam and Kooij posed the following question: Which connected graph on $n$ vertices and a given diameter $D$ has minimal spectral radius?
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They solved this problem for 
\[ D \in \{1, 2, \lfloor n/2 \rfloor, n - 3, n - 2, n - 1\} \] and for almost all graphs on at most 20 vertices by a computer search.
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They solved this problem for \( D \in \{1, 2, \lfloor n/2 \rfloor, n - 3, n - 2, n - 1\} \) and for almost all graphs on at most 20 vertices by a computer search.

Among all connected graphs on \( n \) vertices and a given diameter \( D \), let \( \mathcal{G}^{\text{min}}_{n,D} \) be a minimum graph having the smallest spectral radius.
Van Dam - Kooij [2007]:

- For $D = 2$ and $n \geq 3$, $G_{n,2}^{\min}$ is either a star $S_n$ or a Moore graph.
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- For $D = \lfloor n/2 \rfloor$ and $n \geq 7$, $G_{n,\lfloor n/2 \rfloor}^{\text{min}} = C_n$. 
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- For $D = n - 2$, $G_{n,n-2}^{\text{min}} = D_n$. 

![Graph diagram]
Van Dam - Kooij [2007]:

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- For $D = \lfloor n/2 \rfloor$ and $n \geq 7$, $G_{n,\lfloor n/2 \rfloor}^{\min} = C_n$.
- For $D = n - 2$, $G_{n,n-2}^{\min} = D_n$.
- For $D = n - 3$, $G_{n,n-3}^{\min} = \tilde{D}_n$. 
Van Dam and Kooij [2007] conjectured that for any $e \geq 2$ and $n$ large enough, $G_{\min}^{n,n-e} = P_{\left\lceil \frac{e-1}{2} \right\rceil, n-e-\left\lfloor \frac{e-1}{2} \right\rfloor \left\lceil \frac{e-1}{2} \right\rceil, \left\lfloor \frac{e-1}{2} \right\rfloor, n-e+1}$. 

What about $D = n - e$?
Yuan-Shao-Liu [2008] proved this conjecture holds for $D = n - 4$. Namely, $G^{\min}_{n,n-4} = P_{2,1,n-3}^{2,n-5}$. 
The cases $D = n - 5$

Cioabă-van Dam-Koolen-Lee [2010] proved this conjecture holds for $D = n - 5$. Namely, $G_{n,n-4}^{\text{min}} = P_{2,2,n-4}^{2,n-e-2}$. 
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They also disproved this conjecture for all $e \geq 6$ and $n$ large enough.
Theorem [Cioabă-van Dam-Koolen-Lee 2010] For fixed integer $e \geq 6$, $\rho(G_{n,n-e}^{\min}) \to \sqrt{2 + \sqrt{5}}$ as $n \to \infty$. Moreover, $G_{n,n-e}^{\min}$ must be contained in one of the three families for $n$ large enough.

\[
\mathcal{P}_{n,e} = \{ P_{2,1,\ldots,1,2,n-e+1}^{2,m_2,\ldots,m_{e-4},n-e-2} \mid 2<m_2<\ldots<m_{e-4}<n-e-2 \} \\
\mathcal{P}'_{n,e} = \{ P_{2,1,\ldots,1,1,n-e+1}^{2,m_2,\ldots,m_{e-3},n-e-1} \mid 2<m_2<\ldots<m_{e-4}<n-e-1 \} \\
\mathcal{P}''_{n,e} = \{ P_{1,1,\ldots,1,1,n-e+1}^{1,m_2,\ldots,m_{e-2},n-e-1} \mid 1<m_2<\ldots<m_{e-4}<n-e-1 \}.
\]
Three families

\[ T(k_1, k_2, \ldots, k_r) \]

\[ T'(k_1, k_2, \ldots, k_r) \]

\[ T''(k_1, k_2, \ldots, k_r) \]
Three conjectures

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- **Conjecture 1**: $G_{n,n-e}^{\text{min}}$ is in $\mathcal{P}_{n,e}$. 
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- **Conjecture 1:** $G_{n,n-e}^{\min}$ is in $\mathcal{P}_{n,e}$.

- **Conjecture 2:** For $D = n - 6$ and $n$ large enough,
  
  $$G_{n,n-6}^{\min} = P_{2,\left\lceil \frac{D-1}{2} \right\rceil, D-2}^{2,1,2,n-5}.$$
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- **Conjecture 2:** For $D = n - 6$ and $n$ large enough,
  $G_{n,n-6}^{\min} = P_{2,1,2,n-5}^2,\left\lfloor \frac{D-1}{2} \right\rfloor, D-2$.
- **Conjecture 3:** For $D = n - 7$ and $n$ large enough,
  $G_{n,n-7}^{\min} = P_{2,1,1,2,n-6}^2,\left\lfloor \frac{D-2}{3} \right\rfloor, D-\left\lfloor \frac{D-2}{3} \right\rfloor, D-2$. 

Graphs with small spectral radius
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  $G_{n,n-7}^{\text{min}} = P_{2,1,1,2,n-6}^{2,\left\lfloor \frac{D+2}{3} \right\rfloor,D-\left\lceil \frac{D+2}{3} \right\rceil,D-2}$. 


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  \[ G_{n,n-6}^{\text{min}} = P_2,\left\lceil \frac{D-1}{2} \right\rceil, D-2 \]

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  \[ G_{n,n-7}^{\text{min}} = P_2,\left\lfloor \frac{D+2}{3} \right\rfloor, D-\left\lfloor \frac{D+2}{3} \right\rfloor, D-2 \]

We settled all three conjectures positively.
Theorem 1 [Lan-Lu-Shi 2012] Given $e \geq 6$, if $n \geq 4e^2 - 24e + 38$, then $G_{n,n-e}^{\min} = T(k_1,\ldots,k_r) \in \mathcal{P}_{n,e}$. 

\[
\begin{array}{cccccccc}
& \cdots & & \cdots & & \cdots & & \\
& \therefore & & \therefore & & \therefore & & \\
& k_1 & & k_2 & & \ldots & & k_r \\
\end{array}
\]
Our results

**Theorem 1 [Lan-Lu-Shi 2012]** Given $e \geq 6$, if $n \geq 4e^2 - 24e + 38$, then $G_{n,n-e}^{\min} = T(k_1,\ldots,k_r) \in \mathcal{P}_{n,e}$.

Moreover, let $r = e - 4$ and $s = \frac{\sum_{i=1}^{r} k_i}{r} + \frac{2}{r}$. We have

1. $\lfloor s \rfloor \leq k_i \leq \lceil s \rceil + 1$ for $i = 2, \ldots, r - 1$ and $\lfloor s \rfloor - 1 \leq k_i \leq \lceil s \rceil$ for $i = 1, r$. 

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then
\[ G_{n,n-e}^{\text{min}} = T(k_1,\ldots,k_r) \in \mathcal{P}_{n,e}. \]

Moreover, let $r = e - 4$ and $s = \frac{\sum_{i=1}^{r} k_i}{r} + \frac{2}{r}$. We have

1. $\lfloor s \rfloor \leq k_i \leq \lfloor s \rfloor + 1$ for $i = 2, \ldots, r - 1$ and $\lfloor s \rfloor - 1 \leq k_i \leq \lfloor s \rfloor$ for $i = 1, r$.
2. $|k_i - k_j| \leq 1$ for $2 \leq i, j \leq r - 1$.
3. $0 \leq k_i - k_j \leq 2$ for $2 \leq i \leq r - 1$ and $j = 1, r$. 
A special case

**Theorem 2 [Lan-Lu-Shi 2012]** For fixed \( e \geq 7 \),
\[ n = (e - 4)k - 2 + 2e, \text{ and } k \text{ large enough}, \]
\[ G_{n,n-e}^{\text{min}} = T(k-1,k,...,k,k-1). \]
A special case

**Theorem 2 [Lan-Lu-Shi 2012]** For fixed $e \geq 7$,

$n = (e - 4)k - 2 + 2e$, and $k$ large enough,

$$G_{n,n-e}^{\min} = T(k-1,k,...,k,k-1).$$

\[ \rho(T(k-1,k,...,k,k-1)) \text{ only depends on } k, \text{ not on } r. \]
Let $x_1, x_2 \ (x_1 \leq x_2)$ be two roots of $x^2 - \lambda x + 1 = 0$. Let $d_2 = x_2^3 - \lambda$. Then

$\lambda = \sqrt{2 + \sqrt{5}}$ is the largest root of $d_2 = 0$. 
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- $\lambda = \sqrt{2 + \sqrt{5}}$ is the largest root of $d_2 = 0$.
- $d_2(\lambda)$ is increasing on $[\sqrt{2 + \sqrt{5}}, \infty)$. 
Let $x_1, x_2 \ (x_1 \leq x_2)$ be two roots of $x^2 - \lambda x + 1 = 0$. Let $d_2 = x_2^3 - \lambda$. Then

- $\lambda = \sqrt{2 + \sqrt{5}}$ is the largest root of $d_2 = 0$.
- $d_2(\lambda)$ is increasing on $[\sqrt{2 + \sqrt{5}}, \infty)$.
- $\rho(T_{(k-1,k,...,k,k-1)})$ is the largest root of the equation

\[
d_2 = \frac{2x_1^k}{1 - x_1^k}.
\]
Our results

**Theorem 3 [Lan-Lu-Shi 2012]** For fixed $e \geq 7$ and $n$ large enough, let $s = \frac{n-2e+2}{e-4}$. We have

$$\frac{2x_1^s}{1 - x_1^s} \leq d_2(\rho(G_{n,n-e}^{\min})) \leq \frac{2x_1^{|s|}}{1 - x_1^{|s|}}.$$
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The equality holds if $s$ is an integer. In this case,

$$G_{n,n-e}^{\min} = T(k-1,k,...,k,k-1).$$
Theorem 3 [Lan-Lu-Shi 2012] For fixed $e \geq 7$ and $n$ large enough, let $s = \frac{n-2e+2}{e-4}$. We have

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The equality holds if $s$ is an integer. In this case, $G_{n,n-e}^{\min} = T(k-1,k,...,k,k-1)$.

Corollary: $\rho(G_{n,n-e}^{\min}) = \sqrt{2 + \sqrt{5}} + O(\tau^{-s/2})$. Here $\tau = \frac{\sqrt{5}+1}{2} = 1.618...$ is the golden ratio.
Our results for $D = n - 6$

**Theorem 4 [Lan-Lu-Shi 2012]** For $D = n - 6$ and $n$ large enough, $G_{n,n-6}^{\text{min}}$ is unique up to a graph isomorphism.

![Diagram of graphs $\kappa_1$ and $\kappa_2$]
Our results for $D = n - 6$

**Theorem 4 [Lan-Lu-Shi 2012]** For $D = n - 6$ and $n$ large enough, $G^\text{min}_{n,n-6}$ is unique up to a graph isomorphism.

- If $n = 2k + 12$, then $G^\text{min}_{n,n-6} = T_{k,k}$. 
Our results for $D = n - 6$

**Theorem 4 [Lan-Lu-Shi 2012]** For $D = n - 6$ and $n$ large enough, $G_{n,n-6}^{\min}$ is unique up to a graph isomorphism.

- If $n = 2k + 12$, then $G_{n,n-6}^{\min} = T_{k,k}$.
- If $n = 2k + 13$, then $G_{n,n-6}^{\min} = T_{k,k+1}$. 
Our results for $D = n - 7$ 

**Theorem 5 [Lan-Lu-Shi 2012]** For $D = n - 7$ and $n$ large enough, $G_{n,e}^{\text{min}}$ is unique up to a graph isomorphism.
Our results for $D = n - 7$

**Theorem 5 [Lan-Lu-Shi 2012]** For $D = n - 7$ and $n$ large enough, $G^\min_{n,e}$ is unique up to a graph isomorphism.

- If $n = 3k + 14$, then $G^\min_{n,e} = T(k,k,k)$. 

![Graph Diagram]
Our results for $D = n - 7$

Theorem 5 [Lan-Lu-Shi 2012] For $D = n - 7$ and $n$ large enough, $G_{n,e}^{\min}$ is unique up to a graph isomorphism.

- If $n = 3k + 14$, then $G_{n,e}^{\min} = T(k,k,k)$.
- If $n = 3k + 15$, then $G_{n,e}^{\min} = T(k,k+1,k)$. 
Our results for $D = n - 7$

**Theorem 5 [Lan-Lu-Shi 2012]** For $D = n - 7$ and $n$ large enough, $G_{n,e}^{\text{min}}$ is unique up to a graph isomorphism.

- If $n = 3k + 14$, then $G_{n,e}^{\text{min}} = T(k,k,k)$.
- If $n = 3k + 15$, then $G_{n,e}^{\text{min}} = T(k,k+1,k)$.
- If $n = 3k + 16$, then $G_{n,e}^{\text{min}} = T(k,k+2,k)$. 
Our results for $D = n - 8$

**Theorem 6 [Lan-Lu-Shi 2012]** For $D = n - 8$ and $n$ large enough, $G_{n,e}^{\text{min}}$ is determined up to a graph isomorphism as follows.

```
   V
  / \  / \  / \  / \\
n  |   |   |   |   |
  \  \  \  \  \  \
O---O---O---O---O
     \   \   \   \\
      k_1 k_2 k_3 k_4
```
Our results for $D = n - 8$

**Theorem 6 [Lan-Lu-Shi 2012]** For $D = n - 8$ and $n$ large enough, $G_{n,e}^{\min}$ is determined up to a graph isomorphism as follows.

If $n = 4k + 16$, then $G_{n,e}^{\min}$ is one of three graphs $T(k,k,k,k)$, $T(k,k,k+1,k-1)$, and $T(k-1,k+1,k+1,k-1)$. 
Our results for $D = n - 8$

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- If $n = 4k + 16$, then $G_{n,e}^{\text{min}}$ is one of three graphs $T(k,k,k,k)$, $T(k,k,k+1,k-1)$, and $T(k-1,k+1,k+1,k-1)$.
- If $n = 4k + 17$, then $G_{n,e}^{\text{min}} = T(k,k+1,k,k)$. 

[Diagram of the graphs $T(k,k,k,k)$, $T(k,k,k+1,k-1)$, and $T(k-1,k+1,k+1,k-1)$ with labels $k_1$, $k_2$, $k_3$, and $k_4$.]
Our results for $D = n - 8$

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If $n = 4k + 16$, then $G_{n,e}^{\text{min}}$ is one of three graphs $T(k,k,k,k)$, $T(k,k,k+1,k-1)$, and $T(k-1,k+1,k+1,k-1)$.

If $n = 4k + 17$, then $G_{n,e}^{\text{min}} = T(k,k+1,k)$.

If $n = 4k + 18$, then $G_{n,e}^{\text{min}} = T(k,k+1,k+1,k)$. 
Our results for $D = n - 8$

**Theorem 6 [Lan-Lu-Shi 2012]** For $D = n - 8$ and $n$ large enough, $G_{n,e}^{\min}$ is determined up to a graph isomorphism as follows.

If $n = 4k + 16$, then $G_{n,e}^{\min}$ is one of three graphs $T(k,k,k,k)$, $T(k,k,k+1,k-1)$, and $T(k-1,k+1,k+1,k-1)$.

If $n = 4k + 17$, then $G_{n,e}^{\min} = T(k,k+1,k,k)$.

If $n = 4k + 18$, then $G_{n,e}^{\min} = T(k,k+1,k+1,k)$.

If $n = 4k + 19$, then $G_{n,e}^{\min} = T(k,k+1,k+2,k)$. 

Graphs with small spectral radius
Consider three basic operations to extend a rooted graph

\[ \psi_i : (H, v') \rightarrow (G, v) \]

for \( i = 1, 2, 3 \).
Any tree in three families $\mathcal{P}_{n,e}$, $\mathcal{P}_{n,e}$, and $\mathcal{P}_{n,e}$ can be built from a single vertex graph using above operations recursively.
Any tree in three families $\mathcal{P}_{n,e}$, $\mathcal{P}_{n,e}$, and $\mathcal{P}_{n,e}$ can be built from a single vertex graph using above operations recursively.

$\phi_G, \phi_{G-v}$ can be computed from $\phi_H, \phi_{H-v'}$.

\[
\begin{pmatrix}
\phi_G \\
\phi_{G-v}
\end{pmatrix} = M_i \begin{pmatrix}
\phi_H \\
\phi_{H-v'}
\end{pmatrix}
\]

$M_i$ are $2\times2$-matrices with entries in $\mathbb{Z}[\lambda]$. 
Let \( x_1 \leq x_2 \) be two root of \( x^2 - \lambda x + 1 = 0 \). Let

\[
\begin{pmatrix}
P(G,v) \\ Q(G,v)
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ x_2 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} \phi_G \\ \phi_{G-v} \end{pmatrix}.
\]
Let \( x_1 \leq x_2 \) be two root of \( x^2 - \lambda x + 1 = 0 \). Let

\[
\begin{pmatrix}
\mathcal{P}(G,v) \\
\mathcal{Q}(G,v)
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ x_2 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} \phi_G \\ \phi_{G-v} \end{pmatrix}.
\]

For any \( G \) in the three families \( \mathcal{P}_{n,e}, \mathcal{P}'_{n,e}, \mathcal{P}''_{n,e} \), we can write \( \phi_G \) as the product of some matrices.
The first operation

\begin{align*}
\begin{pmatrix}
  p(G,v) \\
  q(G,v)
\end{pmatrix}
&= \begin{pmatrix}
x_1 & 0 \\
0 & x_2
\end{pmatrix}
\begin{pmatrix}
p(H,v') \\
q(H,v')
\end{pmatrix}
\end{align*}
The second operation

\[
\begin{pmatrix}
    p(G,v) \\
    q(G,v)
\end{pmatrix}
= \frac{1}{x_2 - x_1} \begin{pmatrix}
    \lambda - x_1^3 & x_1 \\
    -x_2 & x_2^3 - \lambda
\end{pmatrix}
\begin{pmatrix}
    p(H,v') \\
    q(H,v')
\end{pmatrix}
\]
The second operation

Let \( d_1 = \lambda - x_1^3 \) and \( d_2 = x_2^3 - \lambda \).
The third operation

\[
\begin{pmatrix}
p(G,v) \\ q(G,v)
\end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix}
x_1^4 + \lambda^2 - 1 & \lambda x_1 \\ -\lambda x_2 & x_2^4 - \lambda^2 + 1
\end{pmatrix} \begin{pmatrix}
p(H,v') \\ q(H,v')
\end{pmatrix}
\]
Lemma 1: Let $\rho''_{k_0} = \lim_{i,j \to \infty} \rho(T''_{(i,k_0,j)})$. Then $\rho''_{k_0}$ is the largest root of

$$d_2 = x_1^{k_0}.$$
Lemma 2 Let $\rho'_{k_0} = \lim_{j \to \infty} \rho(T'_{(k_0,j)})$. Then $\rho'_{k_0}$ is the largest root of

$$d_2 = d_1^{\frac{1}{2}} x_1^{k_0 + \frac{1}{2}}.$$
Otherwise, $G_{n,e}^{\min}$ has at least one internal length $k_i \ll k = \left\lfloor \frac{n-2e+2}{e-4} \right\rfloor$. 
Otherwise, \( G_{n,e}^{\min} \) has at least one internal length \( k_i \ll k = \left\lceil \frac{n-2e+2}{e-4} \right\rceil \).

**Case 1:** \( k_i \) is not at the end.

\[ \rho(G_{n,e}^{\min}) \geq \rho(T''_{(\infty,k_i,\infty)}) \geq \rho(T_{k-1,k,\ldots,k,k-1}). \]

Contradiction.
Otherwise, $G_{n,e}^{\text{min}}$ has at least one internal length
\[ k_i \ll k = \left\lfloor \frac{n-2e+2}{e-4} \right\rfloor. \]

**Case 1:** $k_i$ is not at the end.

\[ \rho(G_{n,e}^{\text{min}}) \geq \rho(T''_{(\infty,k_i,\infty)}) \geq \rho(T_{k-1,k,...,k,k-1}). \]

Contradiction.

**Case 2:** $k_i$ is at the end.

\[ \rho(G_{n,e}^{\text{min}}) \geq \rho(T'_{(\infty,k_i)}) \geq \rho(T_{k-1,k,...,k,k-1}). \]

Contradiction.
The number $\frac{3}{2}\sqrt{2}$ is the limit of the spectral radius of the following graphs:
\( \frac{3}{2} \sqrt{2} \) as a spectral limit

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Woo-Neumaier [2007]: If $\rho(G) \leq \frac{3}{2}\sqrt{2}$, then $G$ is one of the following graphs:

- A Dagger:
Woo-Neumaier [2007]: If $\rho(G) \leq \frac{3}{2}\sqrt{2}$, then $G$ is one of the following graphs:

- A **dagger**:

```
      n-4
```

- An **open quipu**:

```
\begin{array}{ccccccc}
  k_0 & m_0 & k_1 & m_1 & \ldots & m_{i-1} & k_i & m_i & \ldots & m_{r-1} & k_r & m_r & k_{r+1}
\end{array}
```

Graphs: $\rho(G) \leq \frac{3}{2}\sqrt{2}$
Graphs: \( \rho(G) \leq \frac{3}{2} \sqrt{2} \)

Woo-Neumaier [2007]: If \( \rho(G) \leq \frac{3}{2} \sqrt{2} \), then \( G \) is one of the following graphs:

- **A dagger**: 

- **An open quipu**:

- **A closed quipu**:
If $G$ has a vertex of degree 4 and $\rho(G) \leq \frac{3}{2}\sqrt{2}$, then $G$ is a dagger.
If $G$ has a vertex of degree 4 and $\rho(G) \leq \frac{3}{2} \sqrt{2}$, then $G$ is a dagger.

All daggers have spectral radius less than $\frac{3}{2} \sqrt{2}$. 
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All daggers have spectral radius less than $\frac{3}{2}\sqrt{2}$.

The dagger on $n$ vertices has diameter $n - 3$. 
If $G$ is a tree with degrees at most 3 and $\rho(G) \leq \frac{3}{2} \sqrt{2}$, then $G$ is an open quipu.
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Not all open quipus satisfy $\rho(G) \leq \frac{3}{2}\sqrt{2}$. 
If $G$ contains a cycle and $\rho(G) \leq \frac{3}{2} \sqrt{2}$, then $G$ is a closed quipu.
Closed quipus

If $G$ contains a cycle and $\rho(G) \leq \frac{3}{2} \sqrt{2}$, then $G$ is a closed quipu.

Not all closed quipus satisfy $\rho(G) \leq \frac{3}{2} \sqrt{2}$. 
A question

Can one describe those open (or closed) quipus with $\rho(G) \leq \frac{3}{2}\sqrt{2}$?
A question

Can one describe those open (or closed) quipus with $\rho(G) \leq \frac{3}{2}\sqrt{2}$?

We could not answer this question exactly, but we can derive information of the diameters.
Theorem 1 [Lan-Lu 2013] Suppose that $T$ is an open quipu on $n$ vertices ($n \geq 6$) with $\rho(T) < \frac{3}{2} \sqrt{2}$. Then the diameter of $T$ satisfies $D(T) \geq \frac{2n-4}{3}$. 
Our result

**Theorem 1 [Lan-Lu 2013]** Suppose that $T$ is an open quipu on $n$ vertices ($n \geq 6$) with $\rho(T) < \frac{3}{2} \sqrt{2}$. Then the diameter of $T$ satisfies $D(T) \geq \frac{2n-4}{3}$.

The equality holds if and only if $T = P_{(1,m-2,m)}^{(1,m)}$ (for $m \geq 2$).
Theorem 1 [Lan-Lu 2013] Suppose that $L$ is a closed quipu on $n$ vertices ($n \geq 13$) with $\rho(L) < \frac{3}{2} \sqrt{2}$. Then the diameter of $L$ satisfies $\frac{n}{3} < D(L) \leq \frac{2n-2}{3}$.
Theorem 1 [Lan-Lu 2013] Suppose that $L$ is a closed quipu on $n$ vertices ($n \geq 13$) with $\rho(L) < \frac{3}{2}\sqrt{2}$. Then the diameter of $L$ satisfies $\frac{n}{3} < D(L) \leq \frac{2n-2}{3}$.

Moreover, if $L$ is neither $C^{(m)}_{(2m+3)}$ nor $C^{(m)}_{(2m+5)}$, then $D(L) \leq \frac{2n-4}{3}$. 
Diameter v.s. spectral radius

\[ \Theta \]

Closed quipus

Open quipus

Graphs with small spectral radius

2

\[ \sqrt{2 + \sqrt{5}} \]

\[ \frac{3}{2} \sqrt{2} \]

\[ 2 \]

\[ \rho \]

C_n

\[ \frac{n}{3} \]

\[ \frac{n}{2} \]

\[ \frac{2n-4}{3} \]

n-1

D

\[ Q(a, b, c) \]

\[ T(2, 2, c) \]

\[ T(1, b, c) \]

\[ \tilde{D}_n \]

\[ D_n \]

\[ P_n \]
Case $D \approx \frac{n}{2}$

**Theorem [Cioabă-van Dam-Koolen-Lee, 2010]:** For $e = 1, 2, 3, 4$ and sufficiently large $n$ with $n + e$ even, $C^{(\lfloor \frac{e}{2} \rfloor, \lceil \frac{e}{2} \rceil)}(\frac{n-e-2}{2}, \frac{n-e-2}{2})$ is the unique minimizer graph $G_{min}^{n, \frac{n+e}{2}}$. 
Case $D \approx \frac{n}{2}$

Theorem [Cioabă-van Dam-Koolen-Lee, 2010]: For $e = 1, 2, 3, 4$ and sufficiently large $n$ with $n + e$ even, $C(\lfloor \frac{e}{2} \rfloor, \lceil \frac{e}{2} \rceil)$ is the unique minimizer graph $G_{n, \frac{n+e}{2}}^{\text{min}}$. They **Conjectured** that the statement above holds for any constant $e \geq 1$. 
Our result

Theorem I [Lu-Lan 2013]: For $n \geq 13$ and
\[
\frac{n}{2} \leq D \leq \frac{2n-7}{3}, \quad C^{(D-\lfloor \frac{n}{2} \rfloor, D-\lceil \frac{n}{2} \rceil)}_{(n-D-1, n-D-1)}
\] is the unique minimizer graph $G_{n,D}^{\text{min}}$.

Cioabă-van Dam-Koolen-Lee’s conjecture is settled in a stronger way.
The upper bound $\frac{2n-7}{3}$ can not replaced by $\frac{2n-3}{3}$. 

The minimizer graph $G_{n,D}^{\min}$ is determined for the following range of $D$.

$1 \leq \frac{n}{2} \leq \frac{2n}{3} \leq n - 1$

- **Van Dam-Kooij** [2007]
- **Yuan-Shao-Liu** [2008]
- **Cioabă-van Dam-Koolen-Lee** [2010]
- **Lan-Lu-Shi** [2012]
- **Lan-Lu** [2013]
For $m \geq 0$, consider the basic operations to extend a rooted graph

$$\psi_m : (H, v') \rightarrow (G, v).$$

- Any tree open quipu can be built from a single vertex graph using above operations recursively.
- The characteristic polynomials $(\phi_G, \phi_{G-v})$ can be computed from $(\phi_H, \phi_{H-v'})$. 
Let $x_1 \leq x_2$ be two roots of $x^2 - \lambda x + 1 = 0$. Let

$$
\begin{pmatrix}
P(G,v) \\ Q(G,v)
\end{pmatrix}
= \begin{pmatrix} 1 & 1 \\ x_2 & x_1 \end{pmatrix}^{-1}
\begin{pmatrix}
\phi_G \\ \phi_{G-v}
\end{pmatrix}.
$$
Let $x_1 \leq x_2$ be two roots of $x^2 - \lambda x + 1 = 0$. Let

$$
\begin{pmatrix}
    p(G,v) \\
    q(G,v)
\end{pmatrix}
= 
\begin{pmatrix}
    1 & 1  \\
    x_2 & x_1
\end{pmatrix}^{-1}
\begin{pmatrix}
    \phi_G \\
    \phi_{G-v}
\end{pmatrix}.
$$

Then

$$
\begin{pmatrix}
    p(G_m,v) \\
    q(G_m,v)
\end{pmatrix}
= \frac{1}{x_2 - x_1}
\begin{pmatrix}
    d^{(1)}_m & x_1 \phi_{P_{m-1}} \\
    -x_2 \phi_{P_{m-1}} & d^{(2)}_m
\end{pmatrix}
\begin{pmatrix}
    p(H,v') \\
    q(H,v')
\end{pmatrix},
$$

where $\phi_{P_m} = \frac{x_2^{m+1} - x_1^{m+1}}{x_2 - x_1}$, $d^{(1)}_m = \phi_{P_m} - x_1^{m+2}$, and

$$
d^{(2)}_m = x_2^{m+2} - \phi_{P_m}.
$$
Special value $\rho_{m,k}$

Let $\rho_{m,k}$ be the largest root of the equation

$$d_m^{(2)} = \frac{2\phi_{P_{m-1}} x_1^k}{1 - x_1^{k+1}}.$$  Then, $\rho_{m,k}$ is the spectral radius of the following graphs.

- $P^{(m+1,m+1)}_{(m+1,k-2,m+1)}$
- $P^{(m+1,m,m+1)}_{(m+1,k-1,k-1,m+1)}$
- $P^{(m+1,m,...,m,m+1)}_{(m+1,k-1,k,...,k,k-1,m+1)}$
- $C^{(m)}_{(k)}$
- $C^{(m,m)}_{(k,k)}$
- $C^{(m,...,m)}_{(k,...,k)}$
Quipus with $\rho(G) = \rho_{m,k}$

$\rho_{m,k} < \frac{3}{2}\sqrt{2}$ if and only if
- “$m \geq 2$ and $k \geq 2m + 3$”,
- or “$m = 1$ and $k \geq 4$”.
A necessary condition of $\rho < \frac{3}{2} \sqrt{2}$

**Theorem [Lan-Lu 2013]** Suppose an open quipu $P^{(m_0,\ldots,m_r)}_{(m_0,k_1,\ldots,k_r,m_r)}$ has spectral radius less than $\frac{3}{2} \sqrt{2}$. Then the following statements hold.

1. For $2 \leq i \leq r - 1$, we have $k_i \geq m_{i-1} + m_i$. Moreover if $m_{i-1}, m_i \geq 2$, then $k_i \geq m_{i-1} + m_i + 1$.

2. We have $k_1 \geq m_0 + m_1$ if $m_0 \geq 2$; and $k_1 \geq m_1 - 1$ if $m_0 = 1$.

3. We have $k_r \geq m_r + m_{r-1}$ if $m_r \geq 2$; and $k_r \geq m_{r-1} - 1$ if $m_r = 1$. 
Theorem [Lan-Lu 2013] Suppose that an open quipu $P^{(m_0,\ldots,m_r)}_{(m_0,k_1,\ldots,k_r,m_r)}$ satisfies

1. $m_0, m_r \geq 2$;
2. $k_i \geq m_{i-1} + m_i + 3$ for $2 \leq i \leq r - 1$;
3. $k_j \geq m_{j-1} + m_j + 1$ for $j = 1, r$.

Then we have $\rho(P^{(m_0,\ldots,m_r)}_{(m_0,k_1,\ldots,k_r,m_r)}) < \frac{3}{2}\sqrt{2}$. 
Determine $G_{n,D}^{\min}$ for $D$ in the empty region.
Determine $G_{n,D}^{\min}$ for $D$ in the empy region.

In particular, determine $G_{n,n-e}^{\min}$ for $e = 9, 10, 11, 12, \ldots$. 

2. Linyuan Lu and Jingfen Lan, Diameter of Graphs with Spectral Radius at most $\frac{3}{2}\sqrt{2}$, *Linear Algebra and its Application*, 438, No. 11, (2013), 4382-4407.

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**Thank You**