# Graphs with Small Spectral Radius

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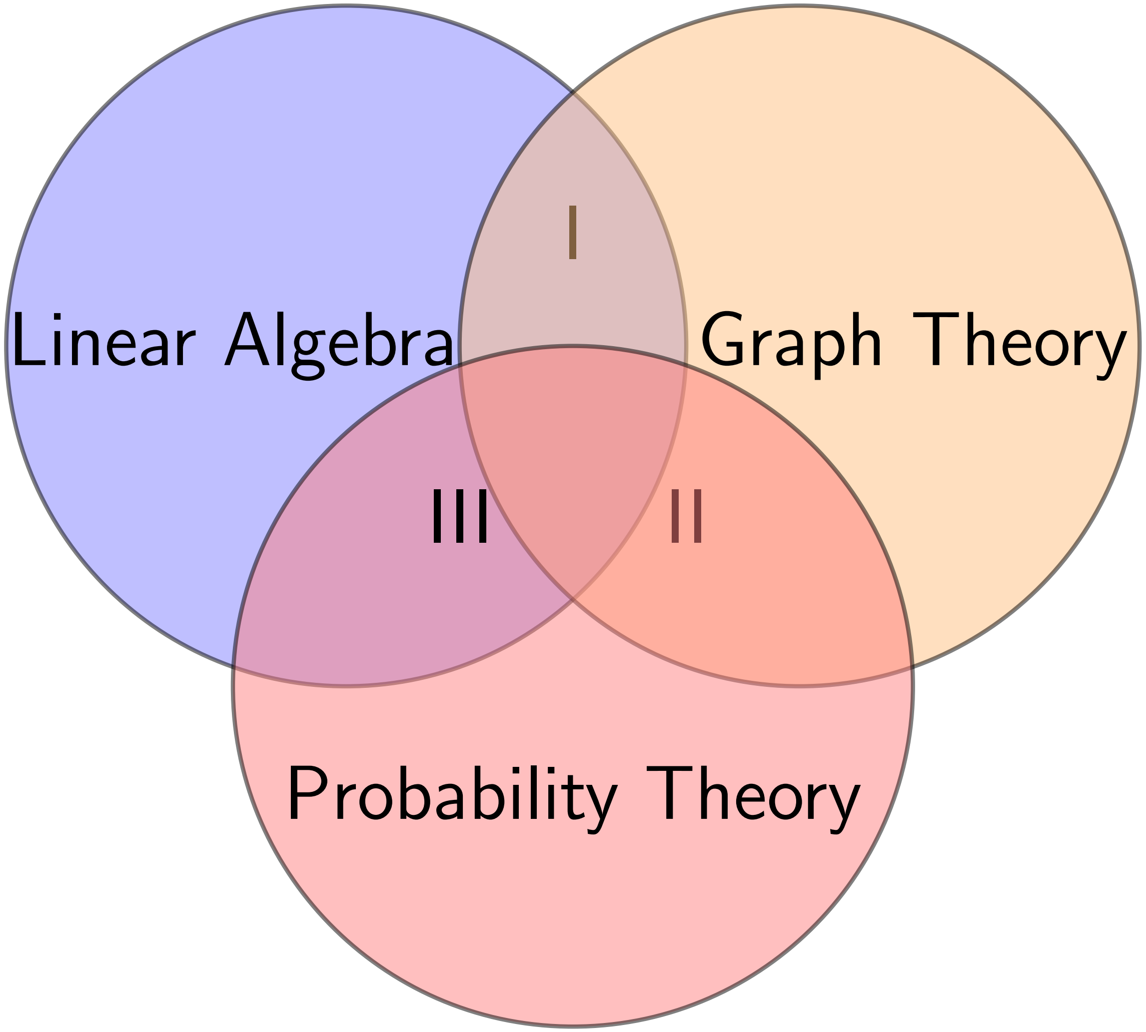
Selected Topics on Spectral Graph Theory (I)
Nankai University, Tianjin, May 16, 2014

# Five talks

**Selected Topics on Spectral Graph Theory**

1. Graphs with Small Spectral Radius
   Time: Friday (May 16) 4pm.-5:30p.m.
2. Laplacian and Random Walks on Graphs
   Time: Thursday (May 22) 4pm.-5:30p.m.
3. Spectra of Random Graphs
   Time: Thursday (May 29) 4pm.-5:30p.m.
4. Hypergraphs with Small Spectral Radius
   Time: Friday (June 6) 4pm.-5:30p.m.
5. Lapalacian of Random Hypergraphs
   Time: Thursday (June 12) 4pm.-5:30p.m.

# Backgrounds




I: Spectral Graph Theory II: Random Graph Theory
III: Random Matrix Theory

# Basic Linear Algebra

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  If $A$ is real symmetric, then $\rho(A) = \max\{|\lambda_1|, |\lambda_n|\}$.

# Perron-Frobenius theorem

- $A = (a_{ij})$ is **non-negative** if $a_{ij} \geq 0$.
- $A$ is **irreducible** if there exists a $m$ such that $A^m$ is positive.
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**Perron-Frobenius theorem:** If $A$ is an aperiodic irreducible non-negative matrix with spectral radius $r$, then $r$ is the largest eigenvalue in absolute value of $A$, and $A$ has an eigenvector $\alpha$ with eigenvalue $r$ whose components are all positive.

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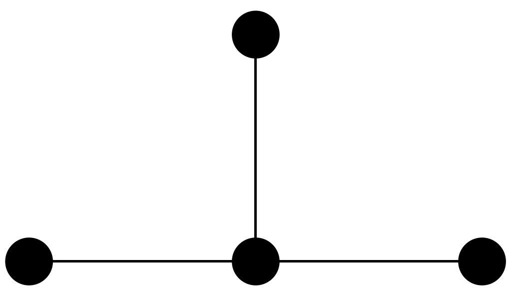

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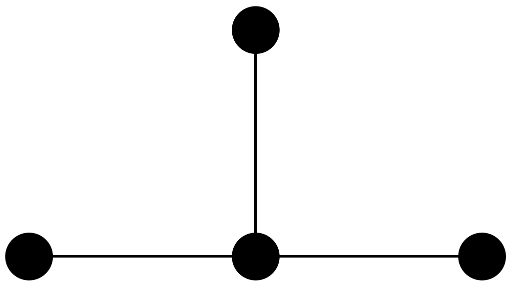

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$$\phi_{S_4} = \lambda^4 - 3\lambda^2 \qquad \rho(S_4) = \sqrt{3}$$

# Easy facts

- Let $\Delta(G)$ be the maximum degree, $d(G)$ be the average degree, and $\delta(G)$ be the minimum degree. Then

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- For the complete bipartite graph $K_{s,t}$, $\rho(K_{s,t}) = \sqrt{st}$.
- In particular, $\rho(G) \geq \sqrt{\Delta(G)}$.

# An application

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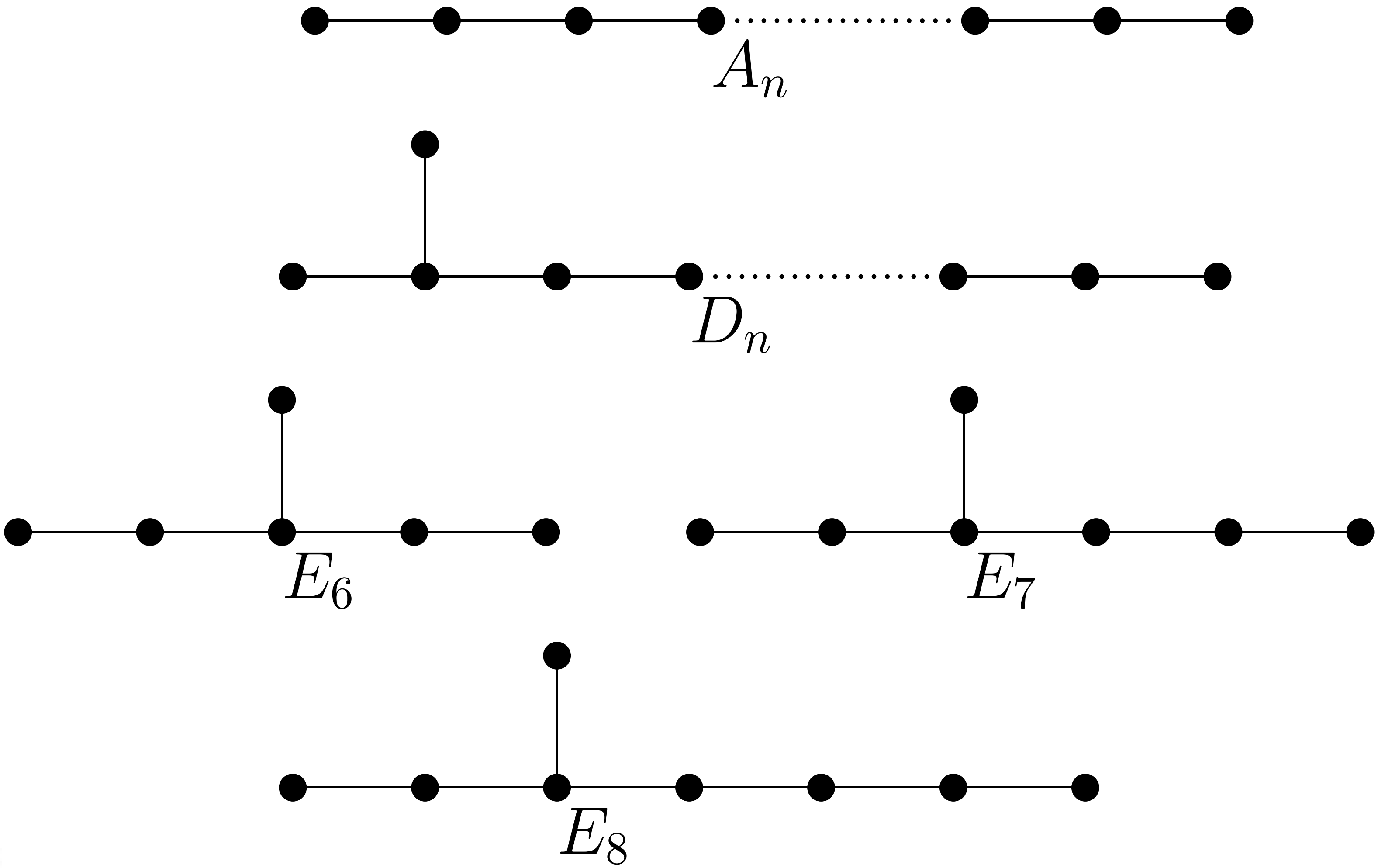
**Wilf's Theorem [1967]:** $\chi(G) \leq 1 + \rho(G)$.

**Proof:** Let $k = \max_{H \subseteq G} \delta(H)$, where $\delta(H)$ is the minimum degree of $H$. Order the vertices $v_1, v_2, \ldots, v_n$ so that each vertex $v_i$ has at most $k$ neighbors in $v_1, \ldots, v_{i-1}$. The greedy algorithm shows that $G$ is $(k+1)$-colorable. Hence

$$\begin{aligned} \chi(G) &\leq 1 + \max_{H \subseteq G} \delta(H) \\ &\leq 1 + \max_{H \subseteq G} \rho(H) \\ &\leq 1 + \rho(G). \qquad \square \end{aligned}$$

# Graphs with $\rho(G) < 2$

**Smith [1970]:** $\rho(G) < 2$ if and only if $G$ is a simply-laced Dynkin diagram.



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- If all roots have the same length, then the root system is said to be simply laced; this occurs in the cases $A$, $D$ and $E$.
- Smith's theorem gives an equivalent graph-theory definition for the simply-laced Dynkin diagrams.

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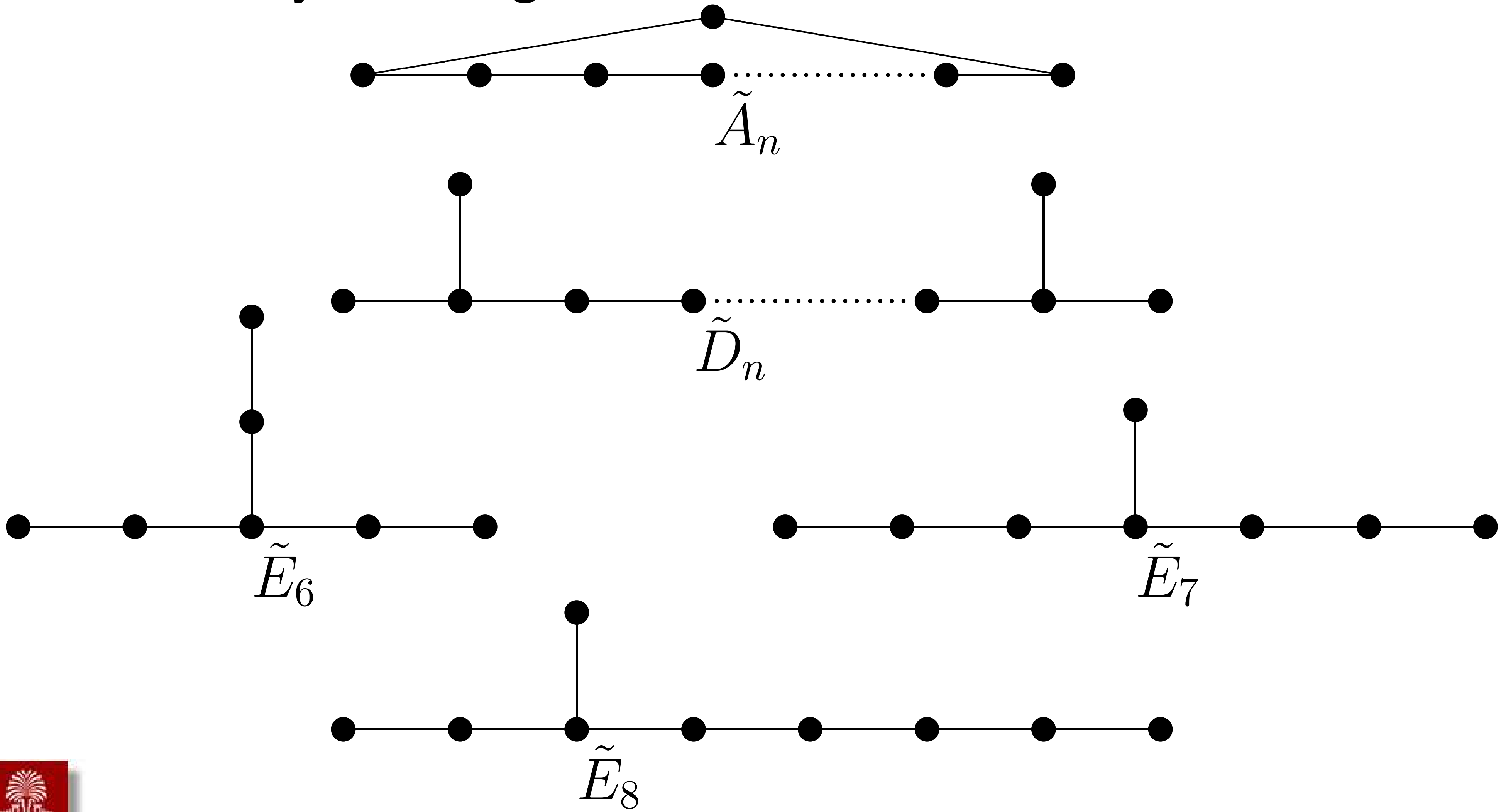
Classifying irreducible simple-laced root systems is equivalent to classifying the connected graphs with $\rho(G) < 2$.

# Graphs with $\rho(G) = 2$

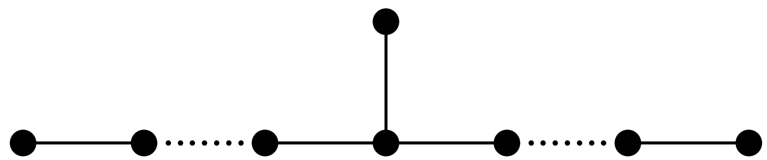
**Smith [1970]:** $\rho(G) = 2$ if and only if $G$ is a simply-laced extended Dynkin diagram.



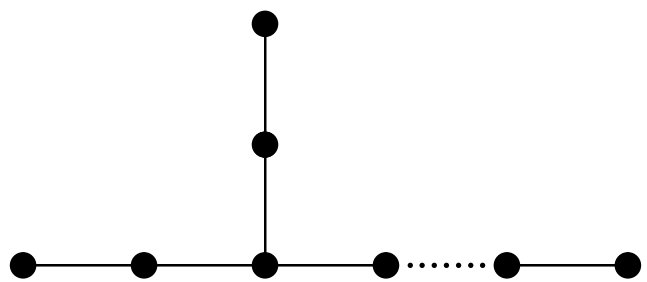
# Graphs: $2 \leq \rho(G) < \sqrt{2+\sqrt{5}}$

**Cvetkovic-Doob-Gutman [1982]**, completed by **Brouwer-Neumaier [1989]:**

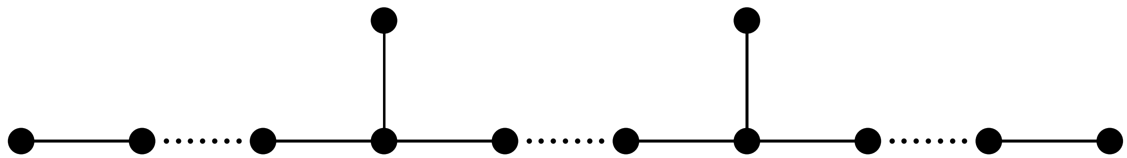
$T(1, b, c)$, $b \geq 2$, $c \geq 6$:



$T(2, 2, c)$, $c \geq 3$:



$Q(a, b, c)$, $a \geq 3$, $c \geq 2$, $b > a + c$:



# Limit points of spectral radii

**Shearer [1989]:** For every number $\lambda \geq \sqrt{2+\sqrt{5}}$ $= 2.058171027...$, there exists a sequence of graphs $\{G_n\}$ such that $\lambda = \lim_{n\to\infty} \rho(G_n)$.

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$$\lim_{b,c\to\infty} \rho(T(1,b,c)) = \sqrt{2+\sqrt{5}}.$$

$$\lim_{c\to\infty} \rho(T(2,2,c)) = \sqrt{2+\sqrt{5}}.$$

$$\lim_{n\to\infty} \rho(Q(n,2n+1,n)) = \sqrt{2+\sqrt{5}}.$$

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  1. $\rho(G') > \rho(G)$ if $uv$ is not on an internal path and $G \neq C_n$.
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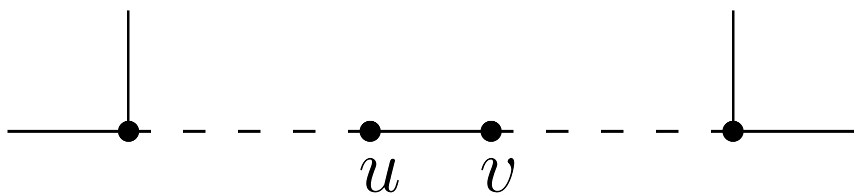
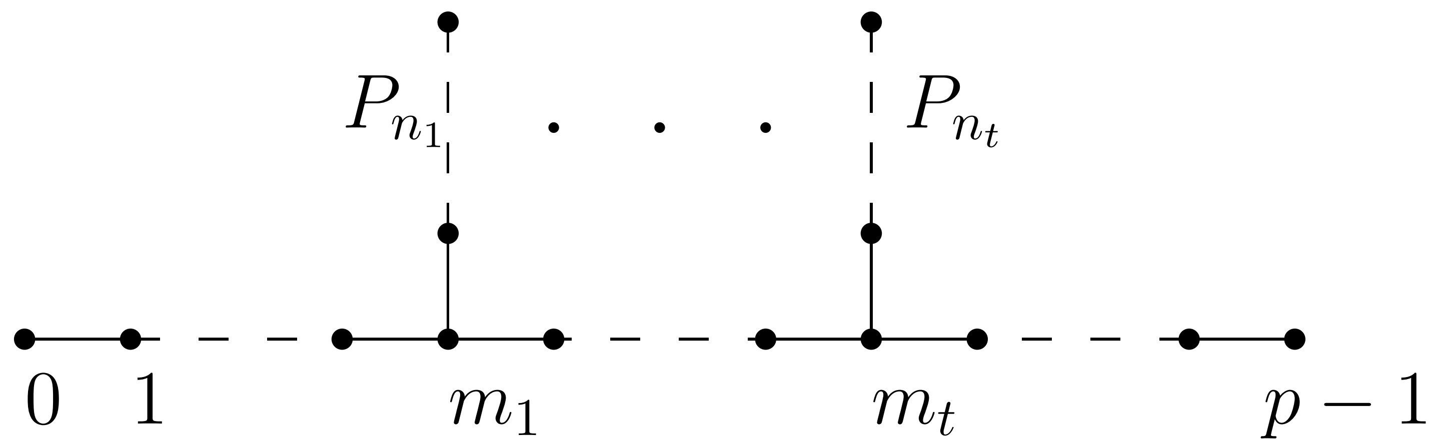



An internal path

# Open quipus

Notation of an open quipus:

$$P^{m_1,m_2,\ldots,m_t}_{n_1,n_2,\ldots,n_t,p}.$$



# Diameter and spectral radius

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Among all connected graphs on $n$ vertices and a given diameter $D$, let $G_{n,D}^{min}$ be a minimum graph having the smallest spectral radius.

# Previous results

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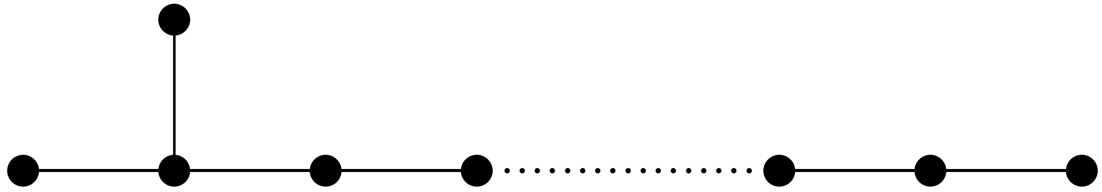
- For $D = n - 2$, $G_{n,n-2}^{min} = D_n$.



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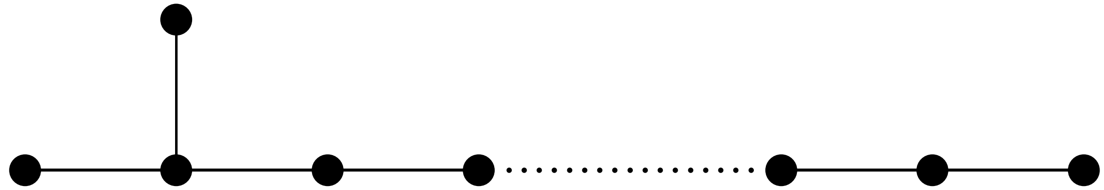
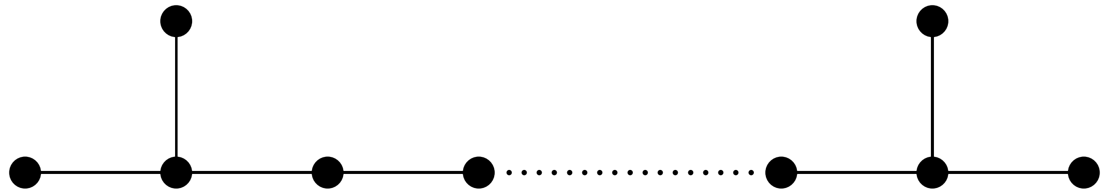
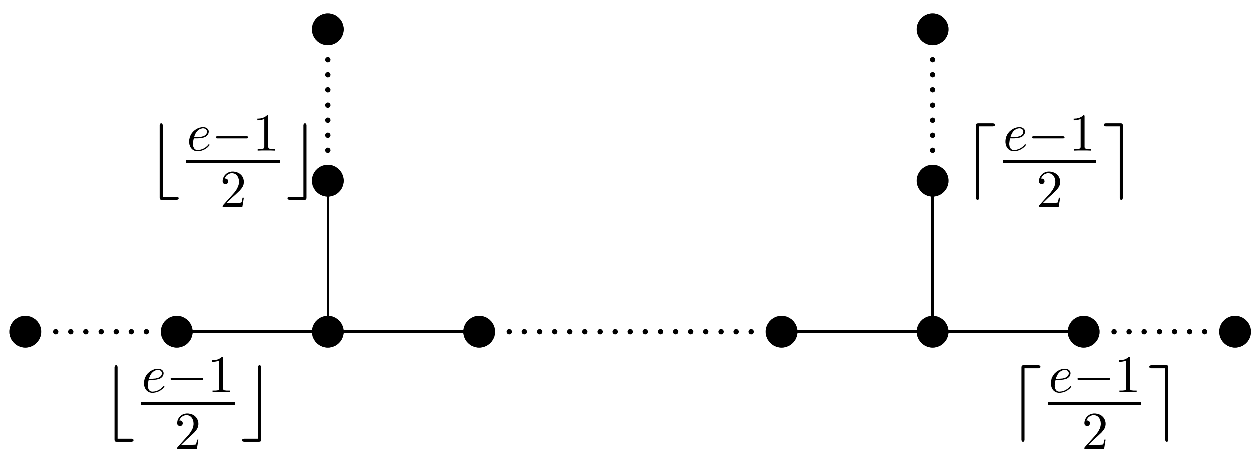


- For $D = n - 3$, $G^{min}_{n,n-3} = \tilde{D}_n$.



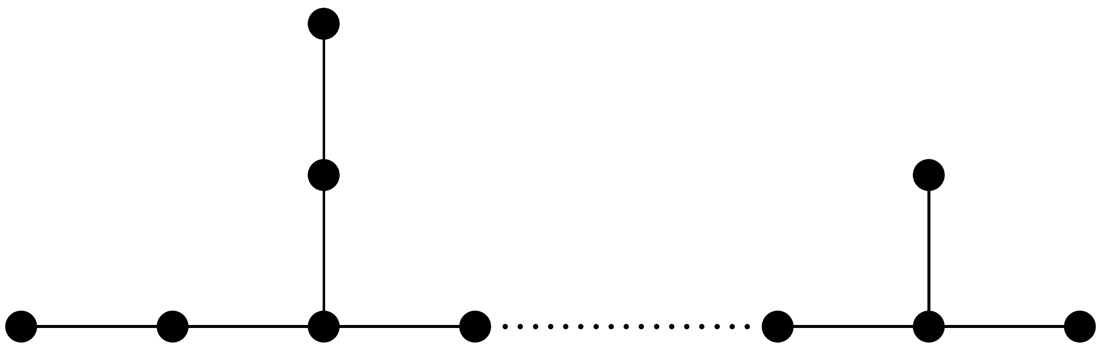
# What about $D = n - e$?

**Van Dam and Kooij [2007]** conjectured that for any $e \geq 2$ and $n$ large enough, $G^{min}_{n,n-e} = P^{\lfloor \frac{e-1}{2} \rfloor, n-e-\lceil \frac{e-1}{2} \rceil}_{\lfloor \frac{e-1}{2} \rfloor, \lceil \frac{e-1}{2} \rceil, n-e+1}$.



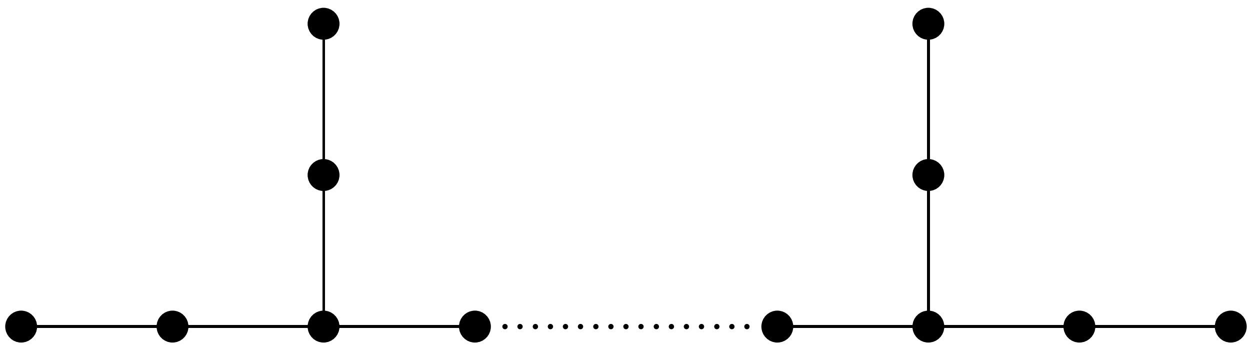
# The case $D = n - 4$

**Yuan-Shao-Liu [2008]** proved this conjecture holds for $D = n - 4$. Namely, $G^{min}_{n,n-4} = P^{2,n-5}_{2,1,n-3}$.



# The cases $D = n - 5$

**Cioabă-van Dam-Koolen-Lee [2010]** proved this conjecture holds for $D = n - 5$. Namely, $G_{n,n-4}^{min} = P_{2,2,n-4}^{2,n-e-2}$.



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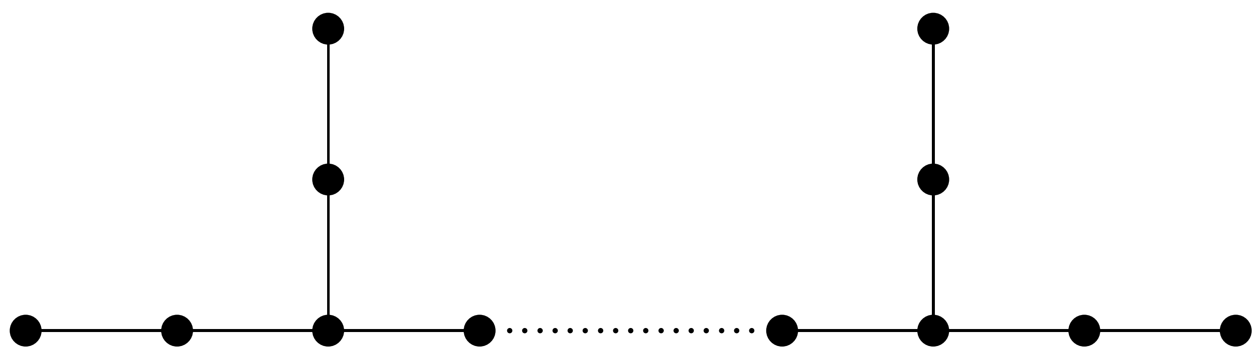


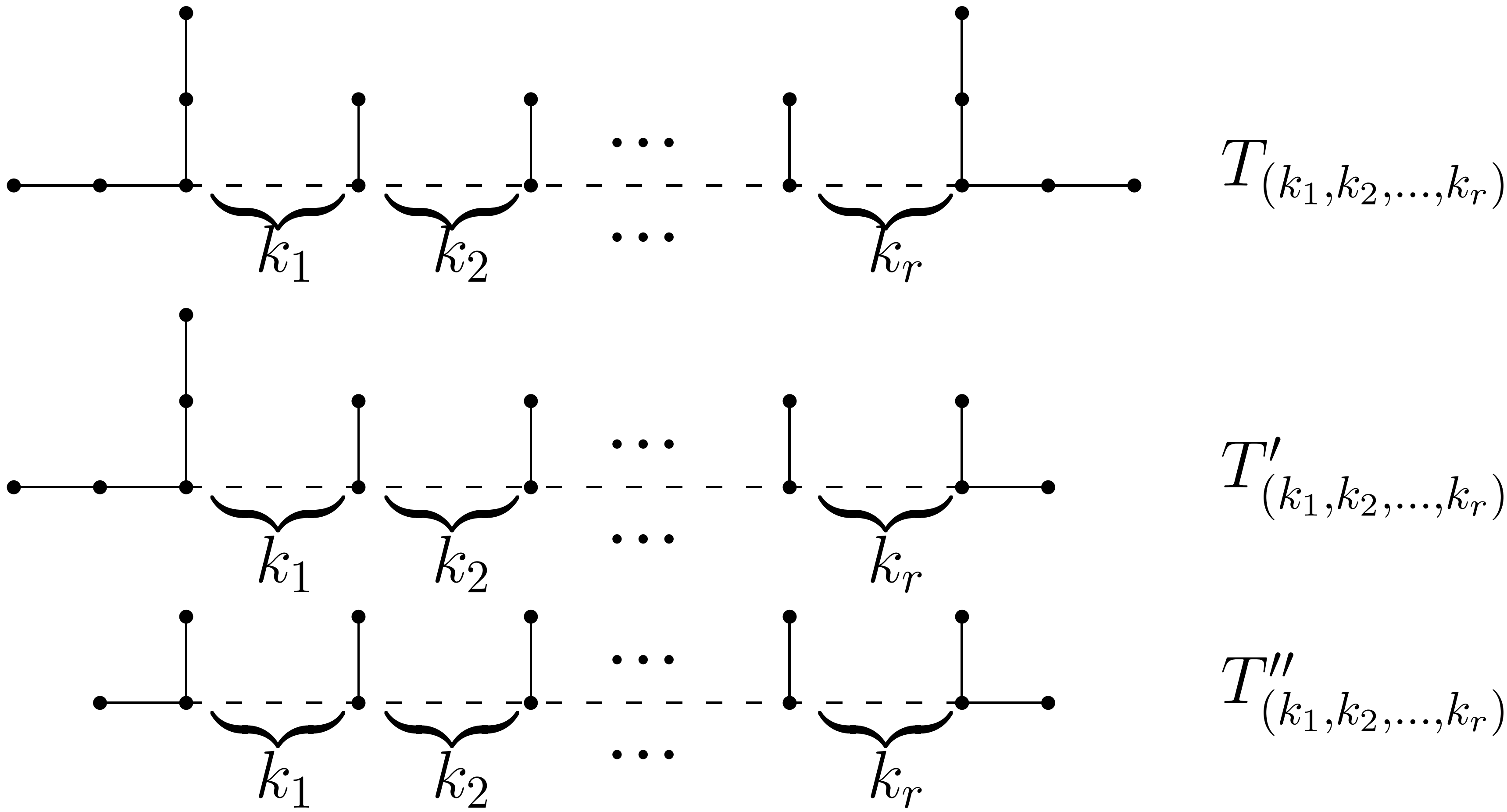
They also **disproved** this conjecture for all $e \geq 6$ and $n$ large enough.

# Previous results

**Theorem [Cioabă-van Dam-Koolen-Lee 2010]** For fixed integer $e \geq 6$, $\rho(G^{min}_{n,n-e}) \to \sqrt{2+\sqrt{5}}$ as $n \to \infty$. Moreover, $G^{min}_{n,n-e}$ must be contained in one of the three families for $n$ large enough.

$$\mathcal{P}_{n,e} = \{P^{2,m_2,\ldots,m_{e-4},n-e-2}_{2,1,\ldots 1,2,n-e+1} \mid 2<m_2<\ldots<m_{e-4}<n-e-2\}$$
$$\mathcal{P}'_{n,e} = \{P^{2,m_2,\ldots,m_{e-3},n-e-1}_{2,1,\ldots 1,1,n-e+1} \mid 2<m_2<\ldots<m_{e-4}<n-e-1\}$$
$$\mathcal{P}''_{n,e} = \{P^{1,m_2,\ldots,m_{e-2},n-e-1}_{1,1,\ldots 1,1,n-e+1} \mid 1<m_2<\ldots<m_{e-4}<n-e-1\}.$$

# Three families



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- **Conjecture 2:** For $D = n - 6$ and $n$ large enough, $G_{n,n-6}^{min} = P_{2,1,2,n-5}^{2,\lceil \frac{D-1}{2} \rceil, D-2}$.

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- **Conjecture 3:** For $D = n - 7$ and $n$ large enough, $G^{min}_{n,n-7} = P^{2,\lfloor\frac{D-2}{3}\rfloor,D-\lfloor\frac{D-2}{3}\rfloor,D-2}_{2,1,1,2,n-6}$.

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- **Conjecture 3:** For $D = n - 7$ and $n$ large enough, $G^{min}_{n,n-7} = P^{2,\lfloor \frac{D+2}{3} \rfloor, D-\lfloor \frac{D+2}{3} \rfloor, D-2}_{2,1,1,2,n-6}$.

# Three conjectures

**Cioabă-van Dam-Koolen-Lee [2010]** made the following three conjectures.

- **Conjecture 1:** $G^{min}_{n,n-e}$ is in $\mathcal{P}_{n,e}$.
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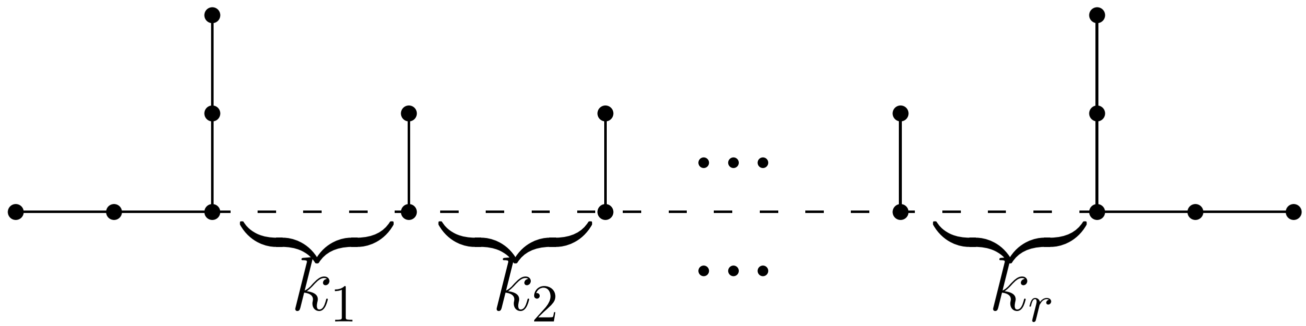
**We settled all three conjectures positively.**

# Our results

**Theorem 1 [Lan-Lu-Shi 2012]** Given $e \geq 6$, if $n \geq 4e^2 - 24e + 38$, then $G^{min}_{n,n-e} = T_{(k_1,\ldots,k_r)} \in \mathcal{P}_{n,e}$.



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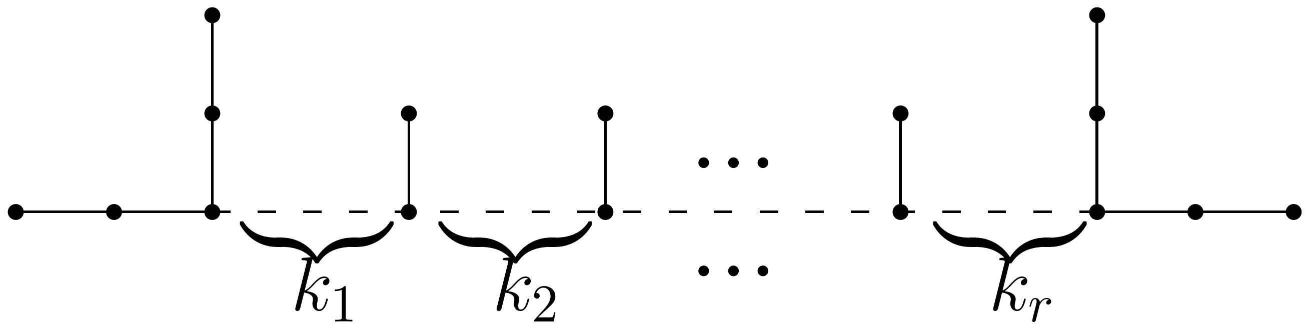


Moreover, let $r = e - 4$ and $s = \frac{\sum_{i=1}^{r} k_i}{r} + \frac{2}{r}$. We have

**1.** $\lfloor s \rfloor \leq k_i \leq \lceil s \rceil + 1$ for $i = 2, \ldots, r-1$ and $\lfloor s \rfloor - 1 \leq k_i \leq \lfloor s \rfloor$ for $i = 1, r$.

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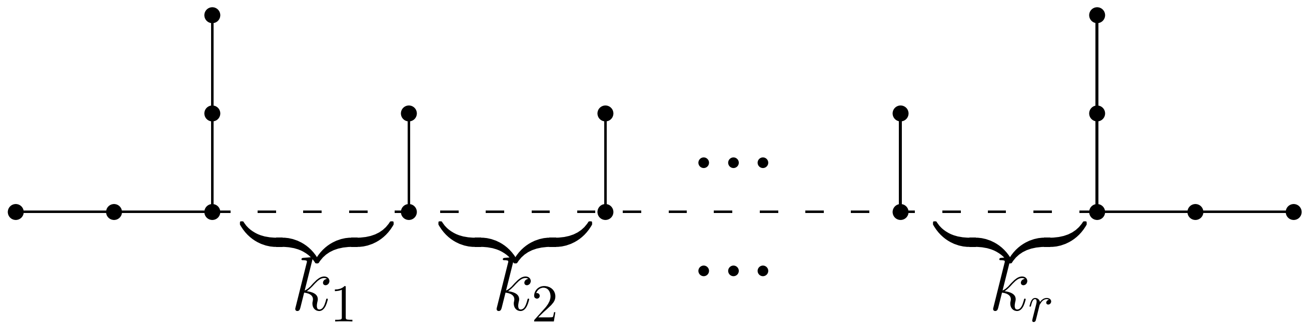


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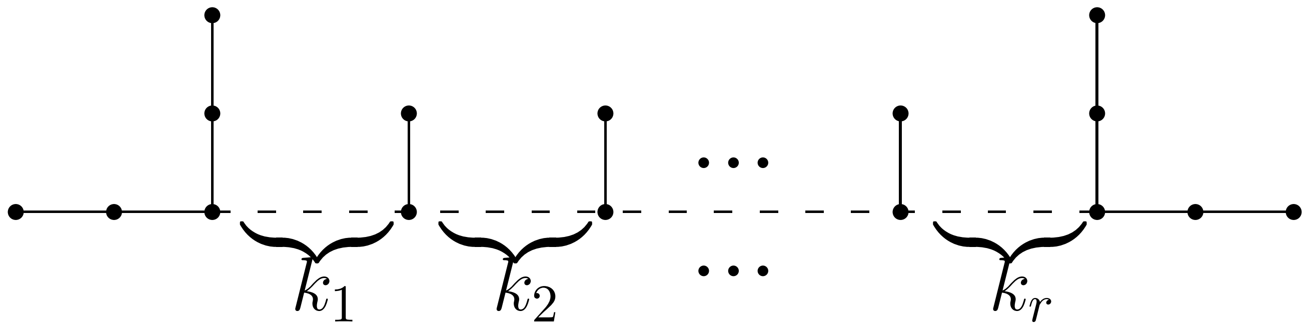


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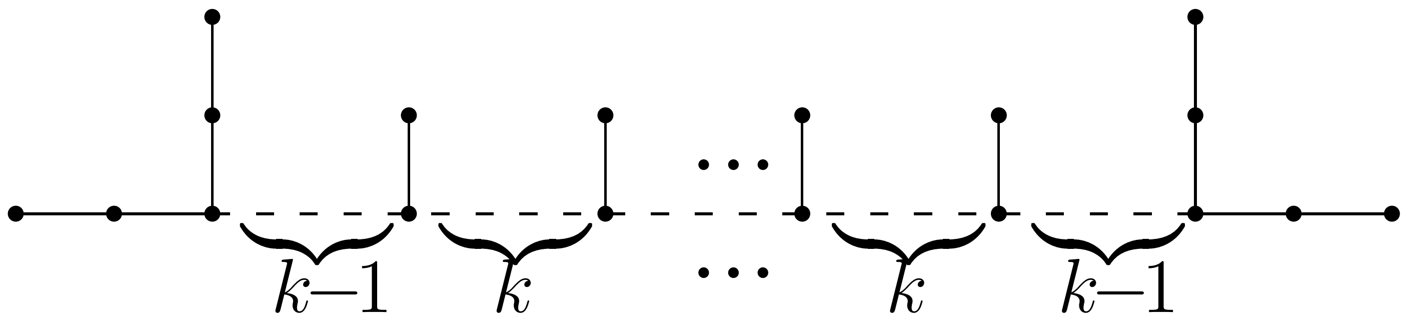
**3.** $0 \leq k_i - k_j \leq 2$ for $2 \leq i \leq r - 1$ and $j = 1, r$.

# A special case

**Theorem 2 [Lan-Lu-Shi 2012]** For fixed $e \geq 7$, $n = (e-4)k - 2 + 2e$, and $k$ large enough, $G^{min}_{n,n-e} = T_{(k-1,k,\ldots,k,k-1)}$.



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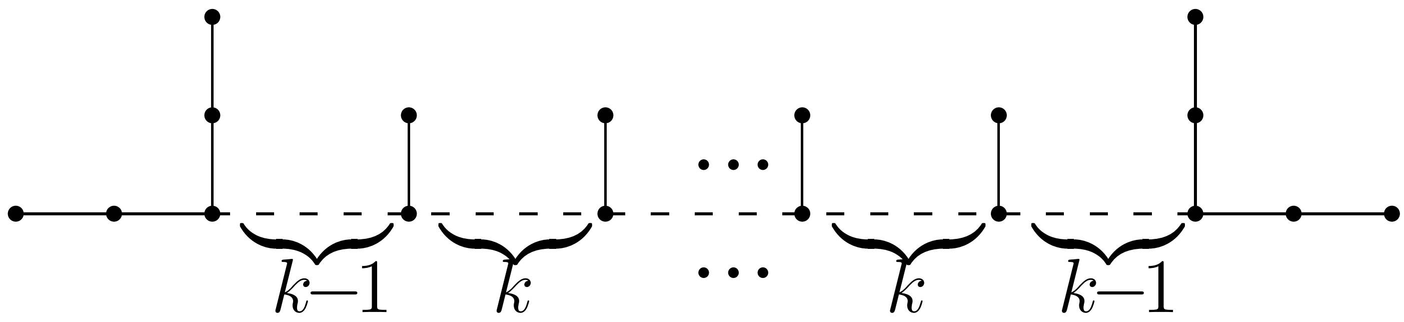


$\rho(T_{(k-1,k,\ldots,k,k-1)})$ only depends on $k$, not on $r$.

# Useful parameters

Let $x_1, x_2$ $(x_1 \le x_2)$ be two roots of $x^2 - \lambda x + 1 = 0$. Let $d_2 = x_2^3 - \lambda$. Then

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- $d_2(\lambda)$ is increasing on $[\sqrt{2+\sqrt{5}}, \infty)$.
- $\rho(T_{(k-1,k,\ldots,k,k-1)})$ is the largest root of the equation

$$d_2 = \frac{2x_1^k}{1-x_1^k}.$$

# Our results

**Theorem 3 [Lan-Lu-Shi 2012]** For fixed $e \geq 7$ and $n$ large enough, let $s = \frac{n-2e+2}{e-4}$. We have

$$\frac{2x_1^s}{1-x_1^s} \leq d_2(\rho(G_{n,n-e}^{min})) \leq \frac{2x_1^{\lfloor s \rfloor}}{1-x_1^{\lfloor s \rfloor}}.$$

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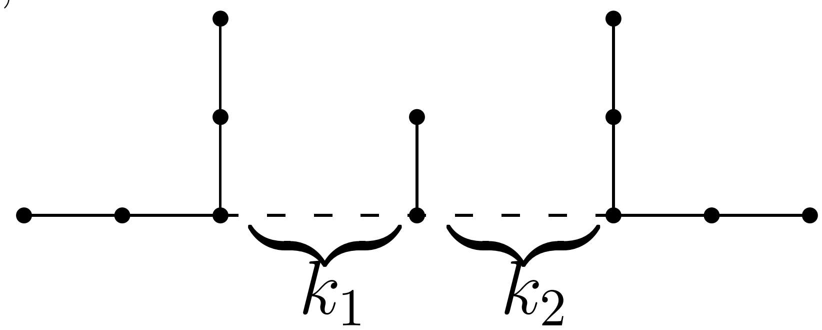
The equality holds if $s$ is an integer. In this case, $G_{n,n-e}^{min} = T_{(k-1,k,\ldots,k,k-1)}$.

**Corollary:** $\rho(G_{n,n-e}^{min}) = \sqrt{2+\sqrt{5}} + O(\tau^{-s/2})$.

Here $\tau = \frac{\sqrt{5}+1}{2} = 1.618\ldots$ is the golden ratio.

# Our results for $D = n - 6$

**Theorem 4 [Lan-Lu-Shi 2012]** For $D = n - 6$ and $n$ large enough, $G^{min}_{n,n-6}$ is unique up to a graph isomorphism.



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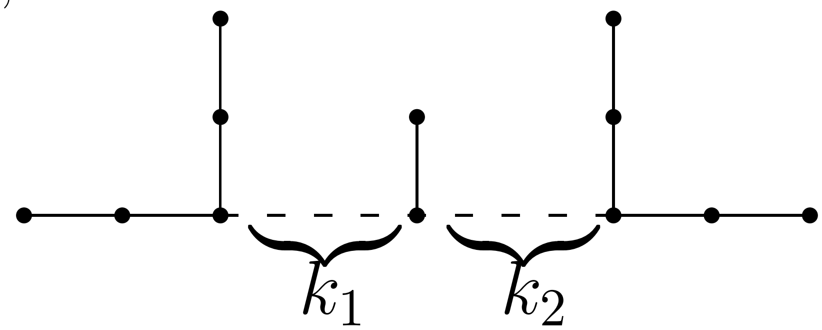


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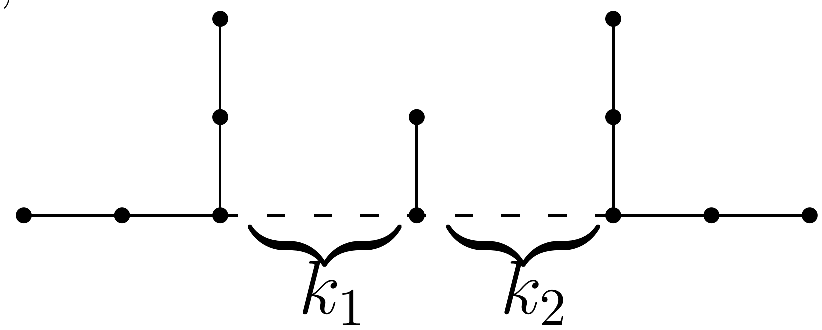


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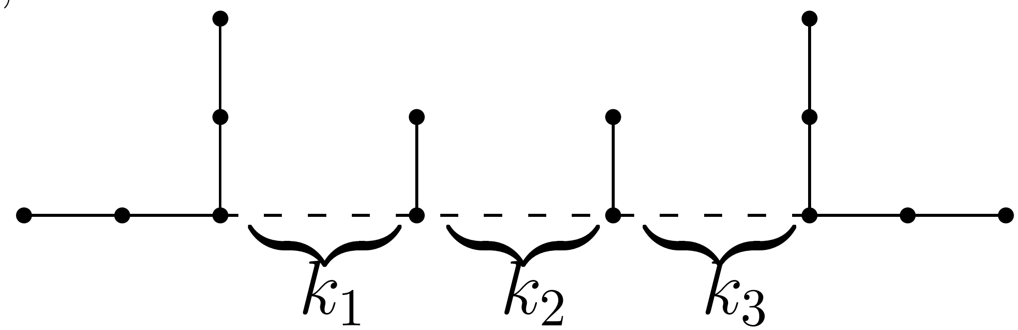
- If $n = 2k + 13$, then $G^{min}_{n,n-6} = T_{k,k+1}$.

# Our results for $D = n - 7$

**Theorem 5 [Lan-Lu-Shi 2012]** For $D = n - 7$ and $n$ large enough, $G_{n,e}^{min}$ is unique up to a graph isomorphism.



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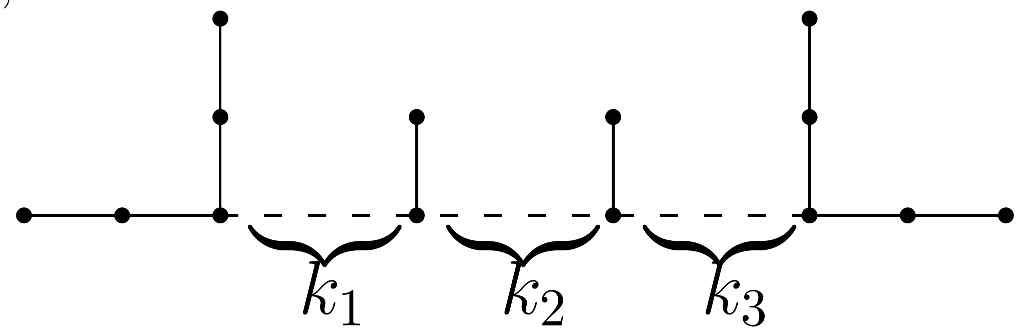


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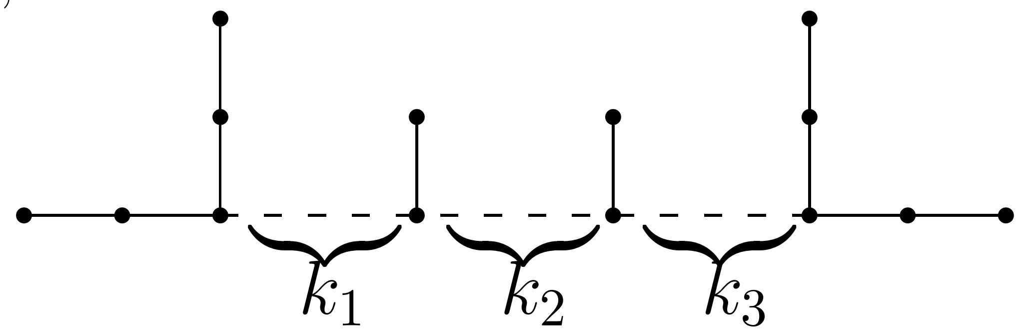


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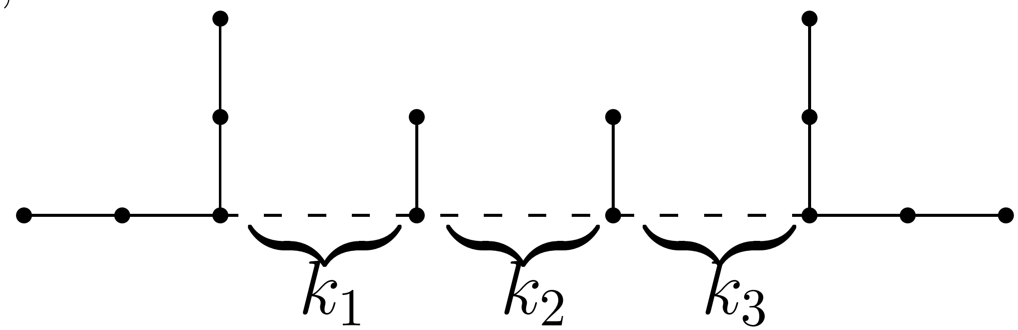


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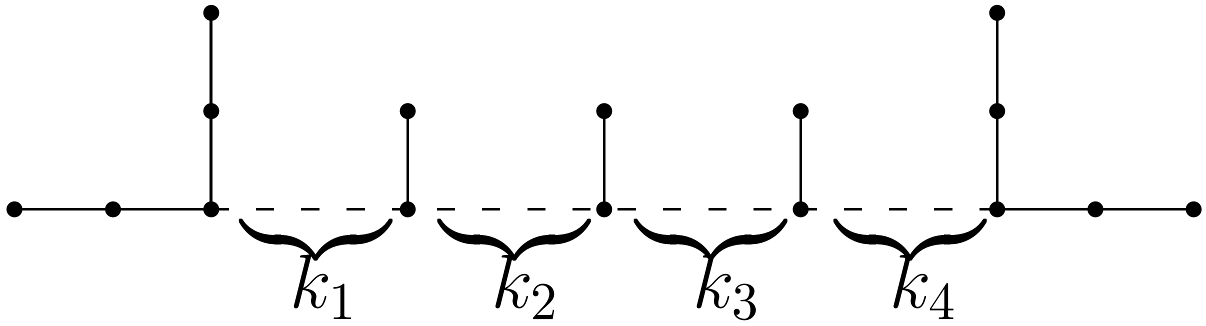
# Our results for $D = n - 8$

**Theorem 6 [Lan-Lu-Shi 2012]** For $D = n - 8$ and $n$ large enough, $G_{n,e}^{min}$ is determined up to a graph isomorphism as follows.



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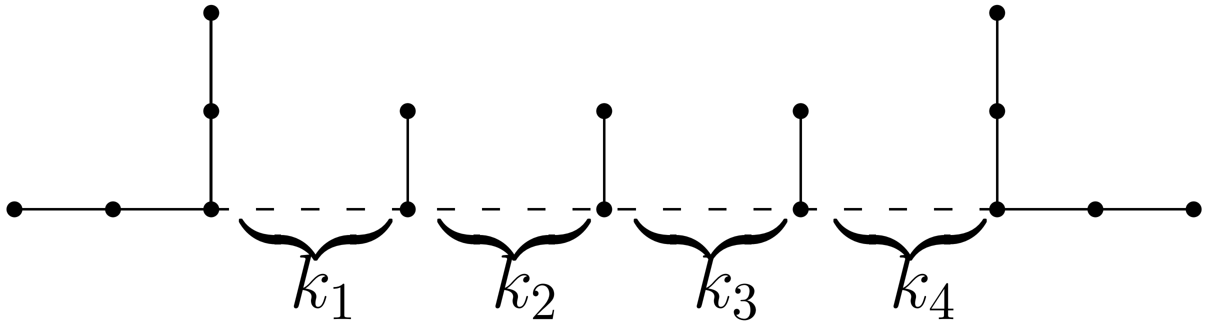


- If $n = 4k + 16$, then $G_{n,e}^{min}$ is one of three graphs $T_{(k,k,k,k)}$, $T_{(k,k,k+1,k-1)}$, and $T_{(k-1,k+1,k+1,k-1)}$.

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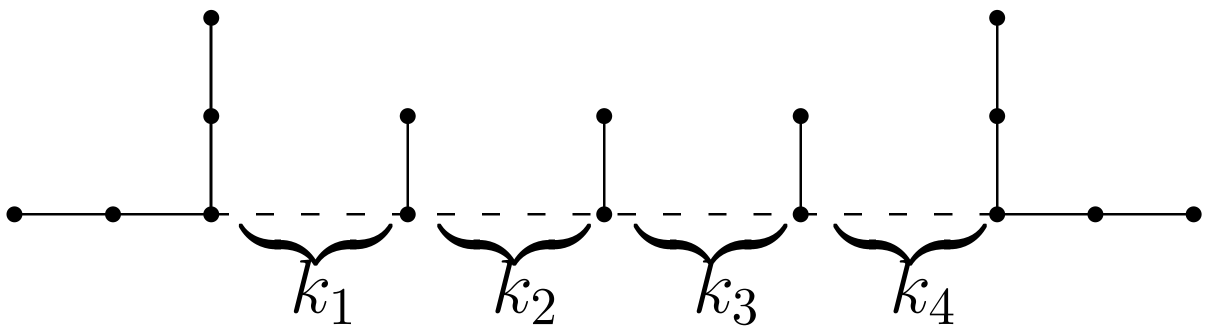


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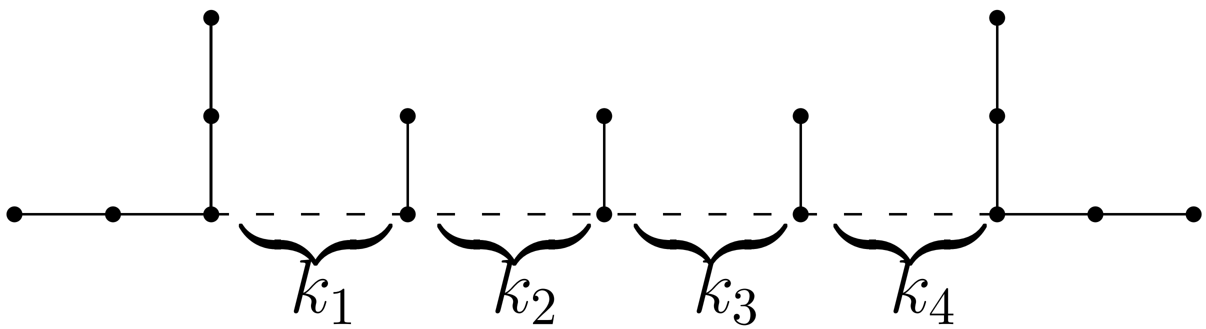


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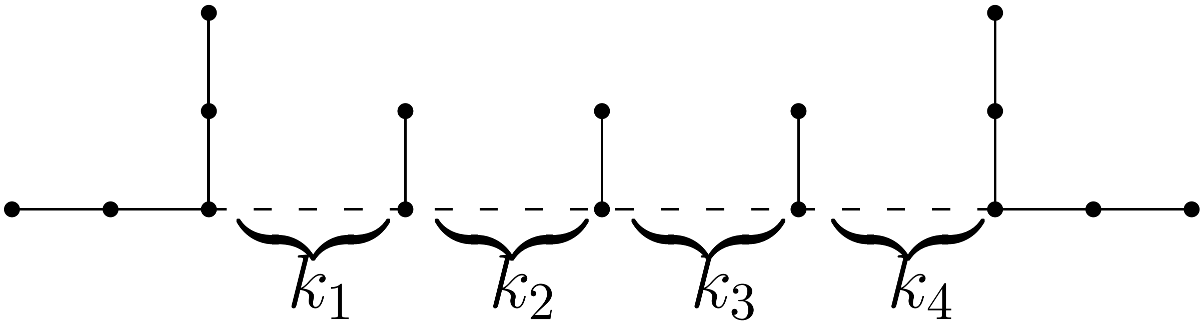


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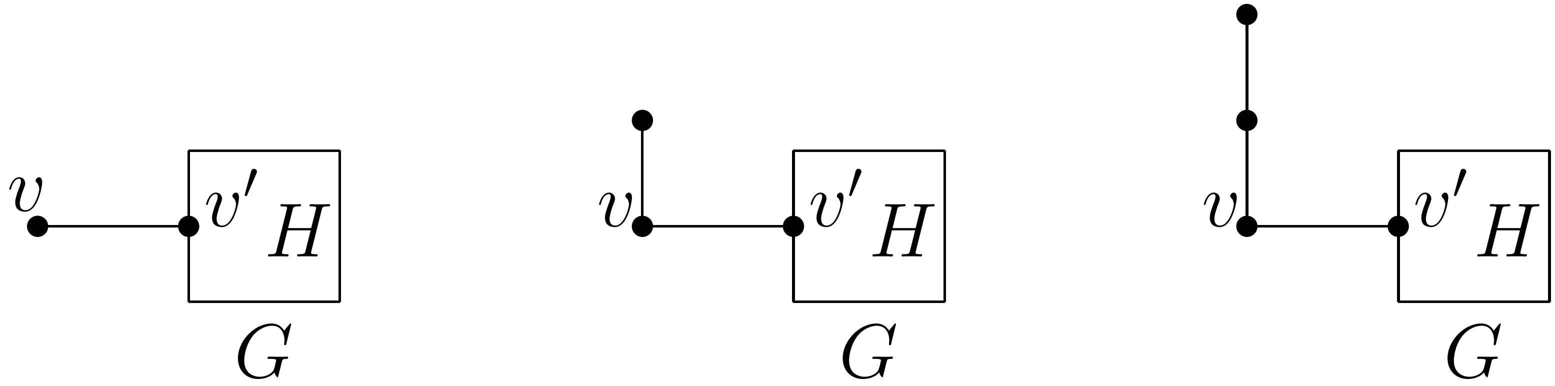
- If $n = 4k + 19$, then $G_{n,e}^{min} = T_{(k,k+1,k+2,k)}$.

# Three basic operations

Consider three basic operations to extend a rooted graph

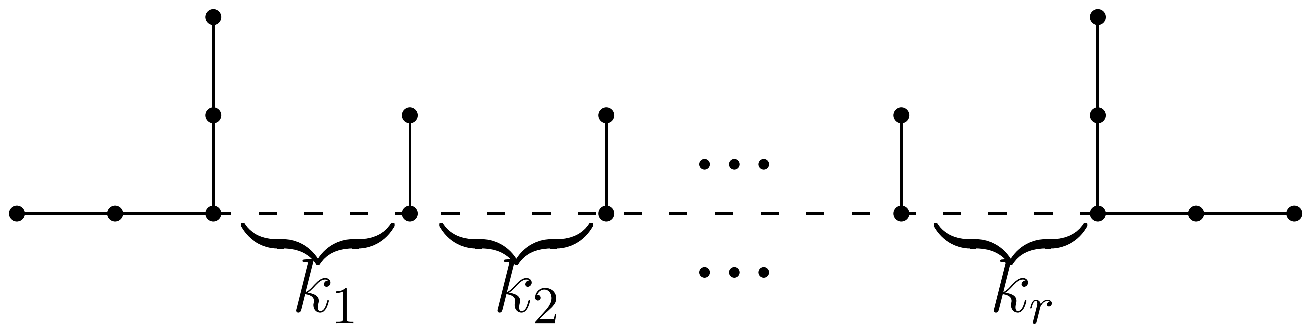
$$\psi_i \colon (H, v') \to (G, v)$$

for $i = 1, 2, 3$.



# Observations

- Any tree in three families $\mathcal{P}_{n,e}$, $\mathcal{P}_{n,e}$, and $\mathcal{P}_{n,e}$ can be built from a single vertex graph using above operations recursively.



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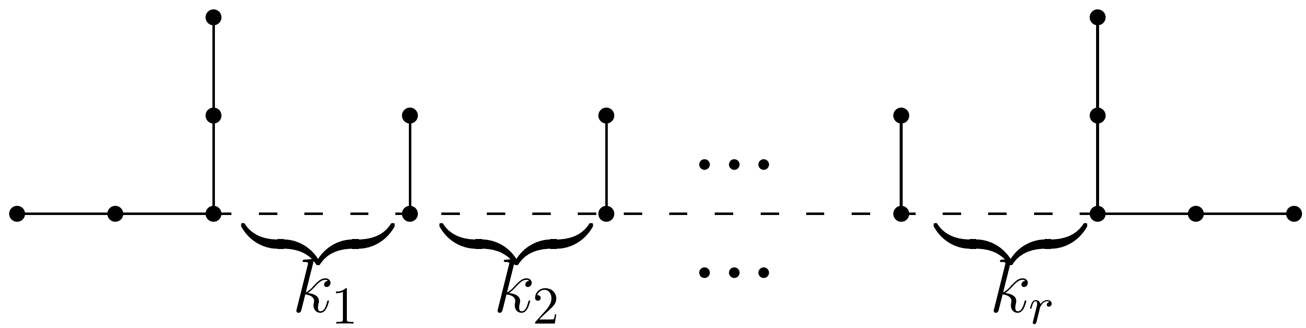


- $(\phi_G, \phi_{G-v})$ can be computed from $(\phi_H, \phi_{H-v'})$ .

$$\begin{pmatrix} \phi_G \\ \phi_{G-v} \end{pmatrix} = M_i \begin{pmatrix} \phi_H \\ \phi_{H-v'} \end{pmatrix}$$

$M_i$ are $2{\times}2$-matrices with entries in $\mathbb{Z}[\lambda]$.

# Choosing right base

Let $x_1 \leq x_2$ be two root of $x^2 - \lambda x + 1 = 0$. Let

$$\begin{pmatrix} p_{(G,v)} \\ q_{(G,v)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ x_2 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} \phi_G \\ \phi_{G-v} \end{pmatrix}.$$

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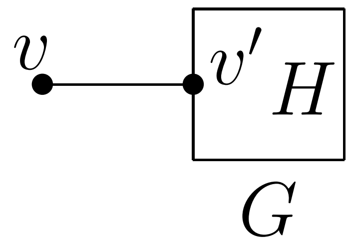
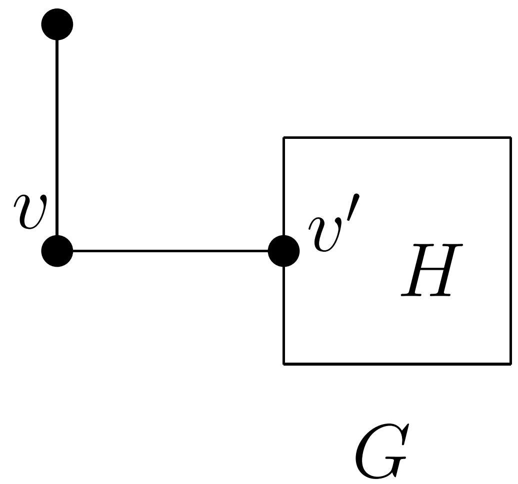
For any $G$ in the three families $\mathcal{P}_{n,e}$, $\mathcal{P}'_{n,e}$, $\mathcal{P}''_{n,e}$, we can write $\phi_G$ as the product of some matrices.

# The first operation




$$\begin{pmatrix} p_{(G,v)} \\ q_{(G,v)} \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} p_{(H,v')} \\ q_{(H,v')} \end{pmatrix}$$

# The second operation




$$\begin{pmatrix} p_{(G,v)} \\ q_{(G,v)} \end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix} \lambda - x_1^3 & x_1 \\ -x_2 & x_2^3 - \lambda \end{pmatrix} \begin{pmatrix} p_{(H,v')} \\ q_{(H,v')} \end{pmatrix}$$

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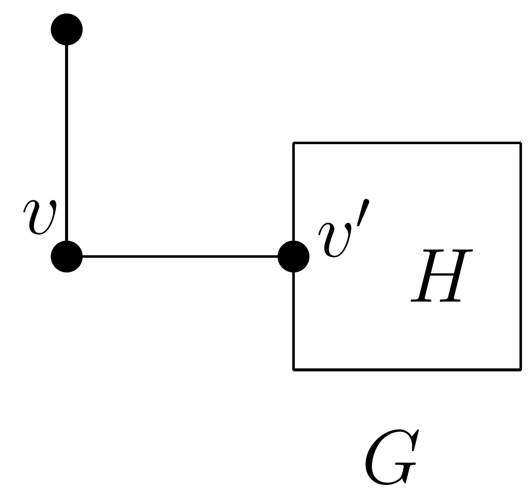


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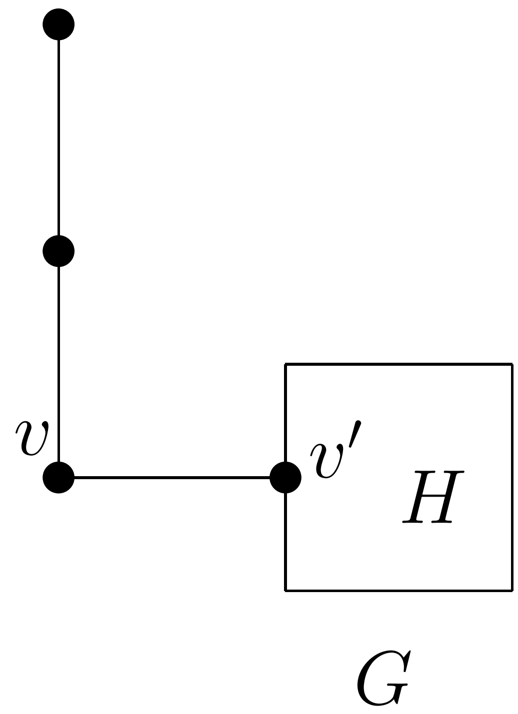
Let $d_1 = \lambda - x_1^3$ and $d_2 = x_2^3 - \lambda$.

# The third operation

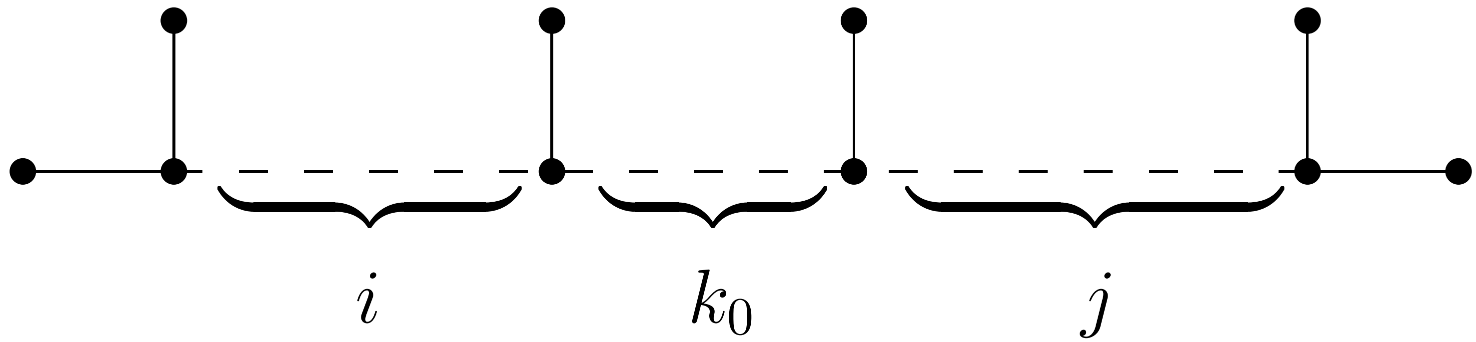



$$\begin{pmatrix} p_{(G,v)} \\ q_{(G,v)} \end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix} x_1^4 + \lambda^2 - 1 & \lambda x_1 \\ -\lambda x_2 & x_2^4 - \lambda^2 + 1 \end{pmatrix} \begin{pmatrix} p_{(H,v')} \\ q_{(H,v')} \end{pmatrix}$$

# Lemma 1

**Lemma 1:** Let $\rho''_{k_0} = \lim_{i,j\to\infty} \rho(T''_{(i,k_0,j)})$. Then $\rho''_{k_0}$ is the largest root of

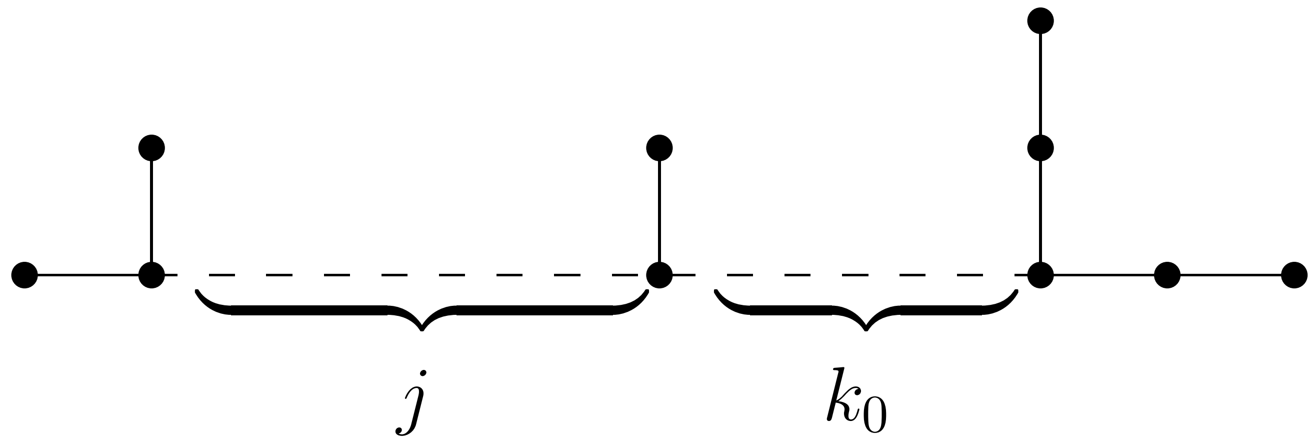
$$d_2 = x_1^{k_0}.$$



# Lemma 2

**Lemma 2** Let $\rho'_{k_0} = \lim_{j\to\infty} \rho(T'_{(k_0,j)})$. Then $\rho'_{k_0}$ is the largest root of

$$d_2 = d_1^{\frac{1}{2}} x_1^{k_0+\frac{1}{2}}.$$



# Sketched proof of $G_{n,e}^{min} \in \mathcal{P}_{n,e}$

Otherwise, $G_{n,e}^{min}$ has at least one internal length $k_i \ll k = \lceil \frac{n-2e+2}{e-4} \rceil$.

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**Case 1:** $k_i$ is not at the end.

$$\rho(G_{n,e}^{min}) \geq \rho(T''_{(\infty,k_i,\infty)}) \geq \rho(T_{k-1,k,\ldots,k,k-1}).$$

Contradiction.

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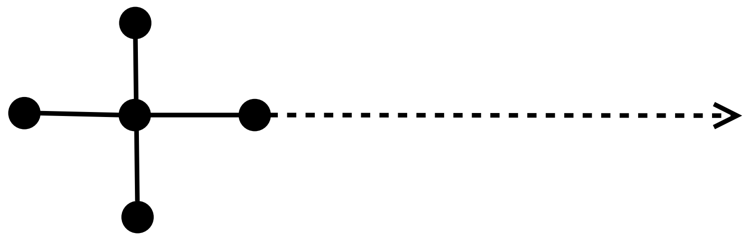
$$\rho(G_{n,e}^{min}) \geq \rho(T'_{(\infty,k_i)}) \geq \rho(T_{k-1,k,\ldots,k,k-1}).$$

Contradiction.

# $\frac{3}{2}\sqrt{2}$ as a spectral limit

The number $\frac{3}{2}\sqrt{2}$ is the limit of the spectral radius of the following graphs:



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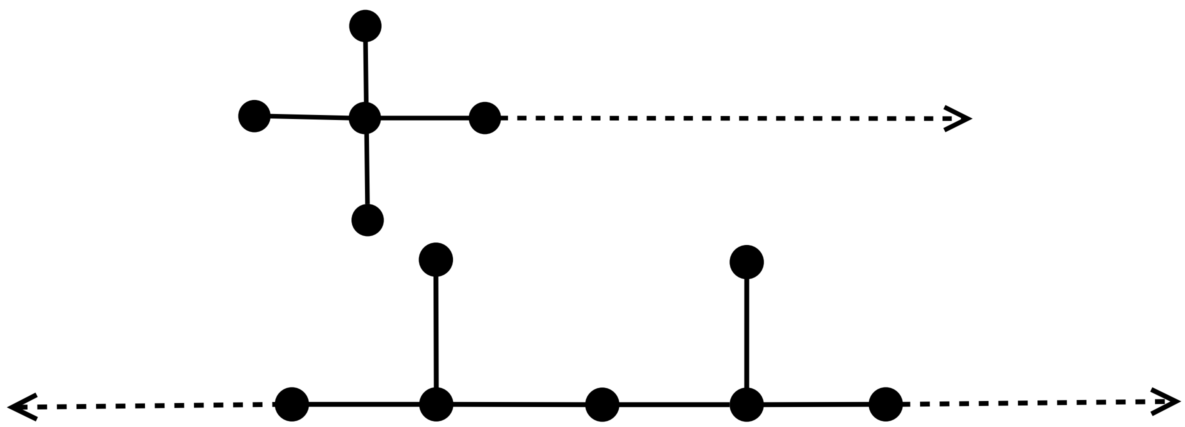

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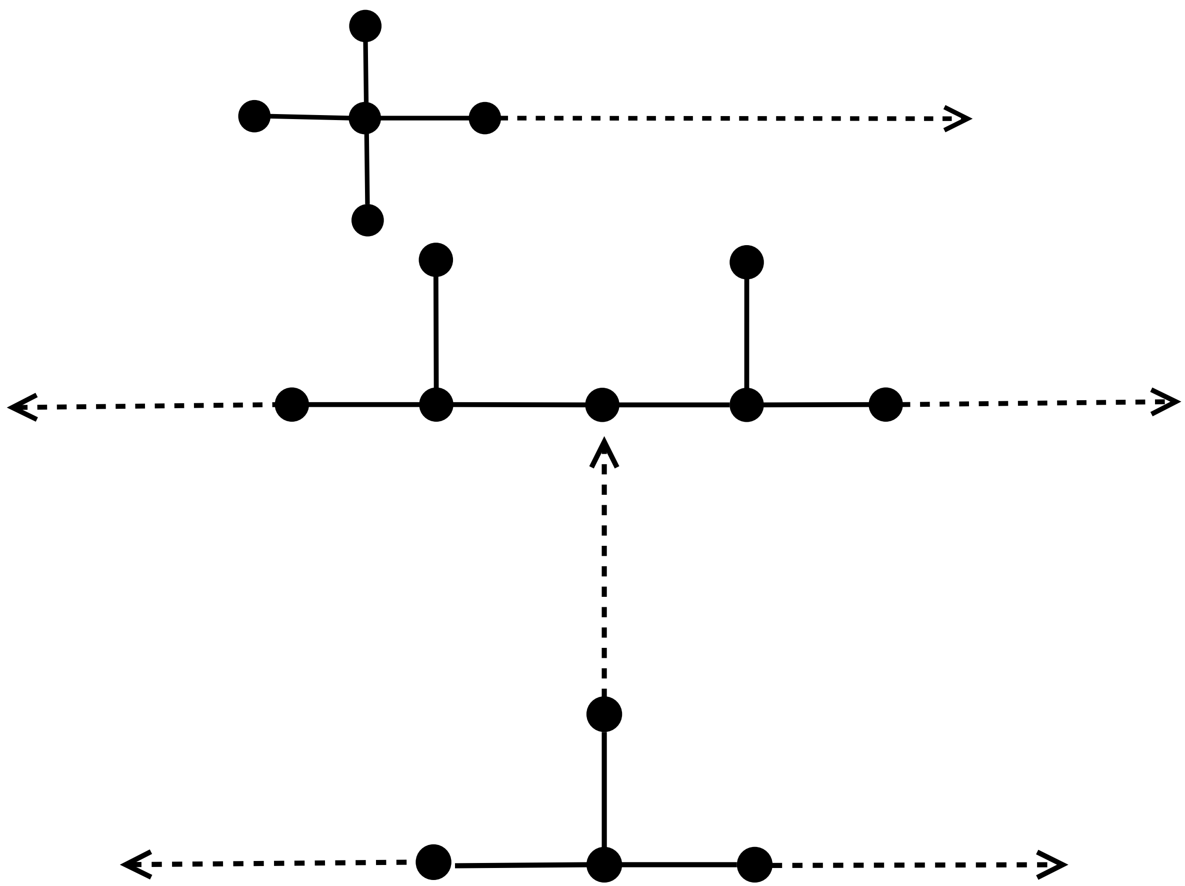
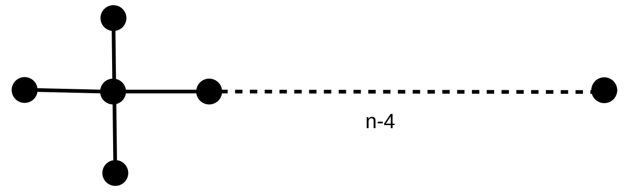


# Graphs: $\rho(G) \leq \frac{3}{2}\sqrt{2}$

**Woo-Neumaier [2007]:** If $\rho(G) \leq \frac{3}{2}\sqrt{2}$, then $G$ is one of the following graphs:

- A dagger:



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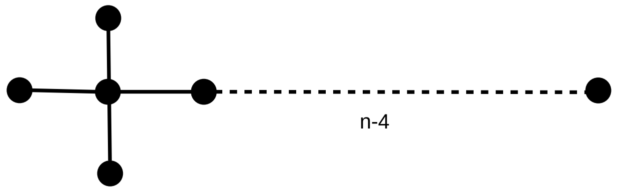
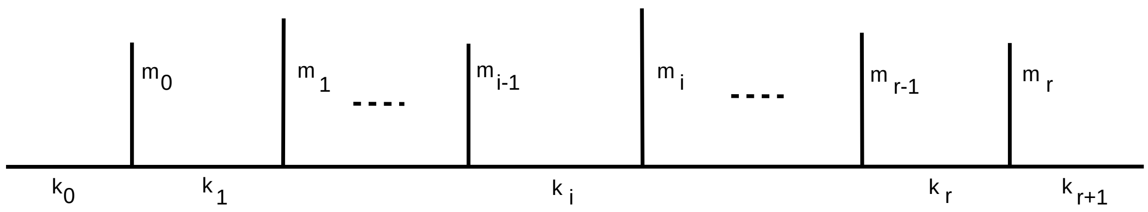



- An open quipu:



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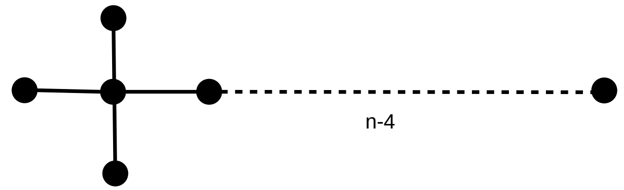


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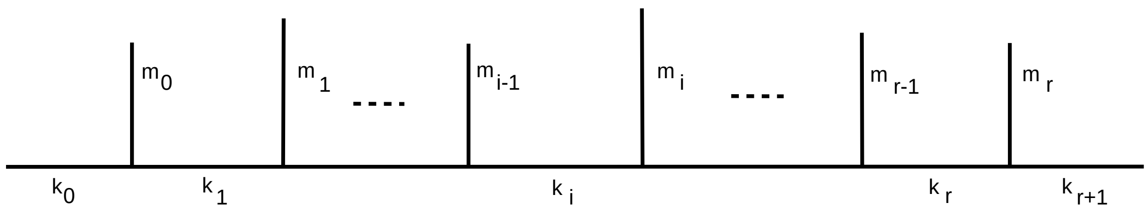
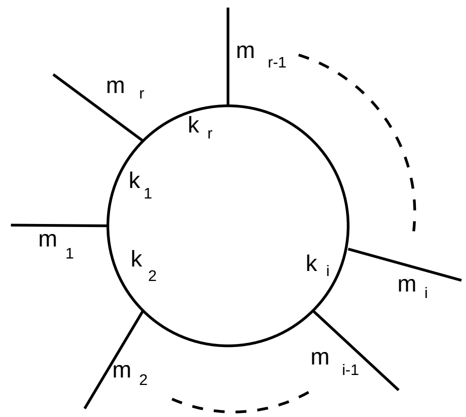



- A closed quipu:



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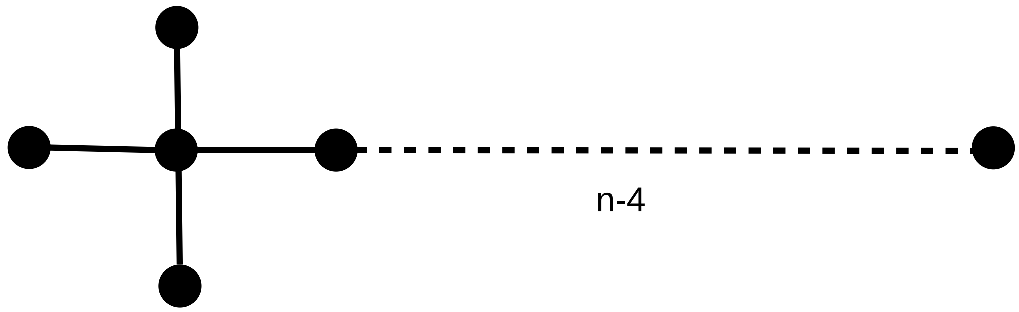
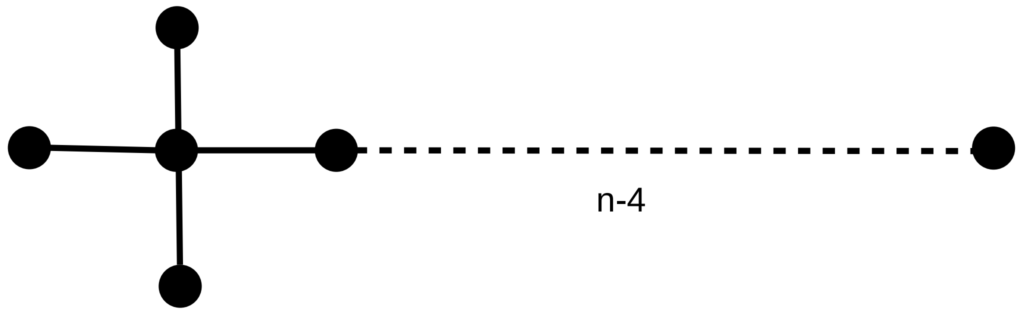



- If $G$ has a vertex of degree $4$ and $\rho(G) \leq \frac{3}{2}\sqrt{2}$, then $G$ is a dagger.

# Daggers




- If $G$ has a vertex of degree $4$ and $\rho(G) \leq \frac{3}{2}\sqrt{2}$, then $G$ is a dagger.
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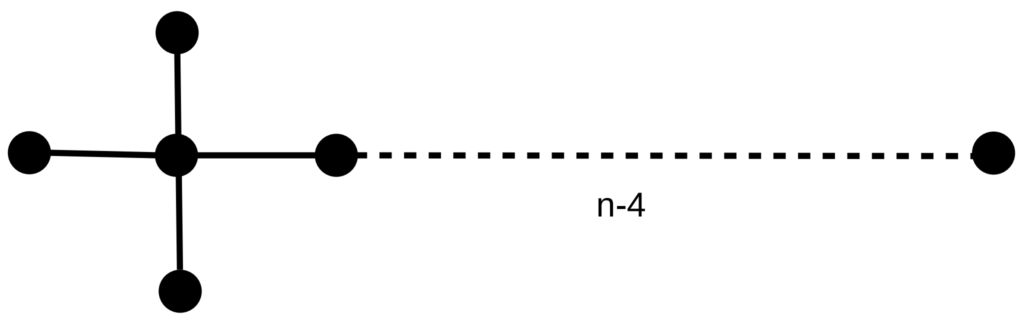
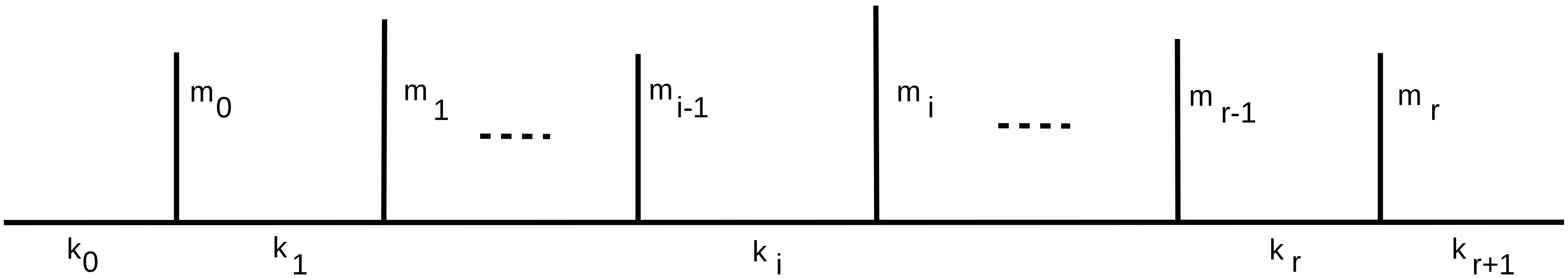



- If $G$ has a vertex of degree $4$ and $\rho(G) \le \frac{3}{2}\sqrt{2}$, then $G$ is a dagger.
- All daggers have spectral radius less than $\frac{3}{2}\sqrt{2}$.
- The dagger on $n$ vertices has diameter $n-3$.

# Open quipus




$$P_{(k_0,k_1,\ldots,k_{t+1})}^{(m_0,m_1,\ldots,m_t)}$$

- If $G$ is a tree with degrees at most $3$ and $\rho(G) \leq \frac{3}{2}\sqrt{2}$, then $G$ is an open quipu.

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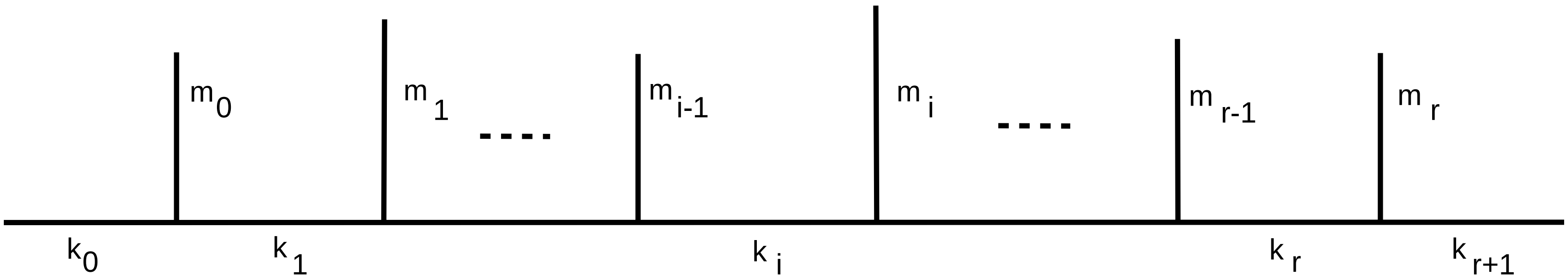
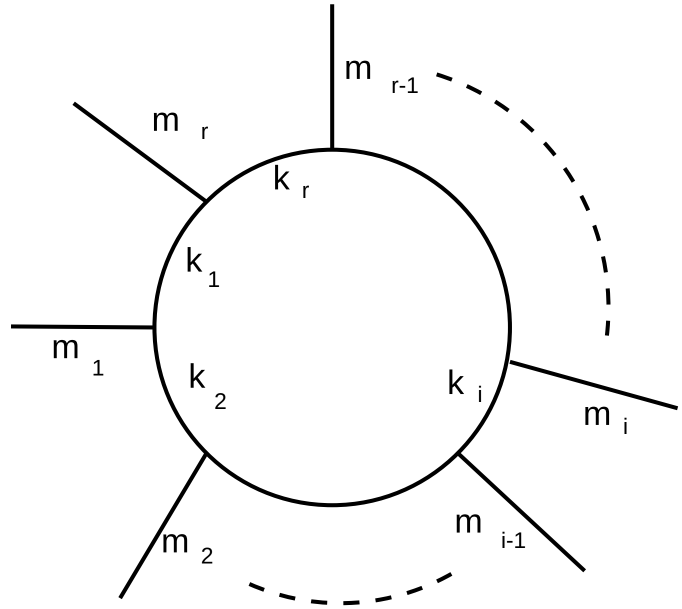



$$P^{(m_0, m_1, \ldots, m_t)}_{(k_0, k_1, \ldots, k_{t+1})}$$

- If $G$ is a tree with degrees at most $3$ and $\rho(G) \leq \frac{3}{2}\sqrt{2}$, then $G$ is an open quipu.
- Not all open quipus statisfy $\rho(G) \leq \frac{3}{2}\sqrt{2}$.

# Closed quipus




$$C_{(k_1,k_2,\ldots,k_t)}^{(m_1,m_2,\ldots,m_t)}$$

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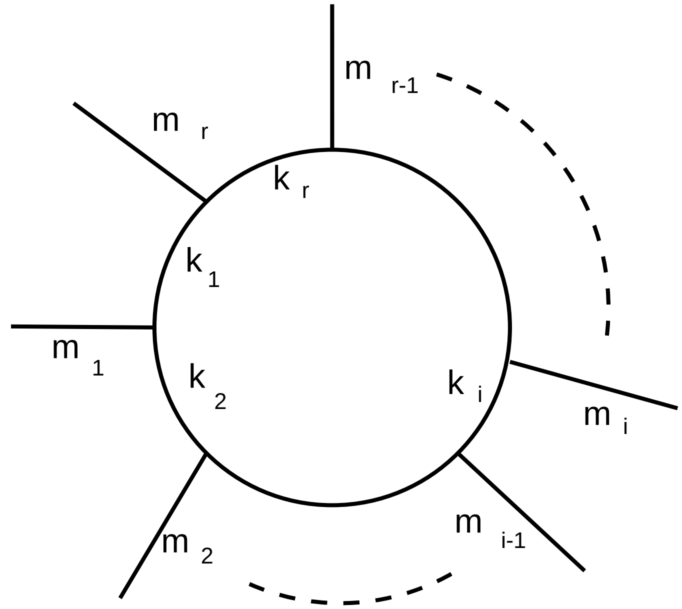


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# A question

Can one describe those open (or closed) quipus with $\rho(G) \leq \frac{3}{2}\sqrt{2}$?

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We could not answer this question exactly, but we can derive information of the diameters.

# Our result

**Theorem 1 [Lan-Lu 2013]** Suppose that $T$ is an open quipu on $n$ vertices $(n \geq 6)$ with $\rho(T) < \frac{3}{2}\sqrt{2}$. Then the diameter of $T$ satisfies $D(T) \geq \frac{2n-4}{3}$.

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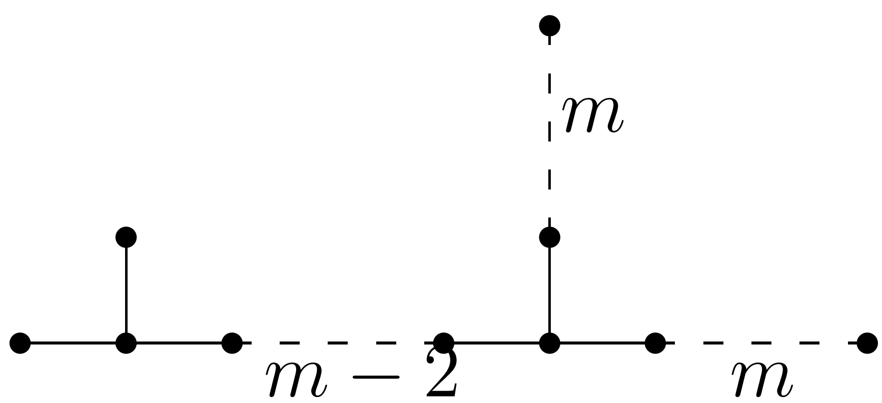
The equality holds if and only if $T = P^{(1,m)}_{(1,m-2,m)}$ (for $m \geq 2$).



# Our result

**Theorem 1 [Lan-Lu 2013]** Suppose that $L$ is a closed quipu on $n$ vertices $(n \geq 13)$ with $\rho(L) < \frac{3}{2}\sqrt{2}$. Then the diameter of $L$ satisfies $\frac{n}{3} < D(L) \leq \frac{2n-2}{3}$.

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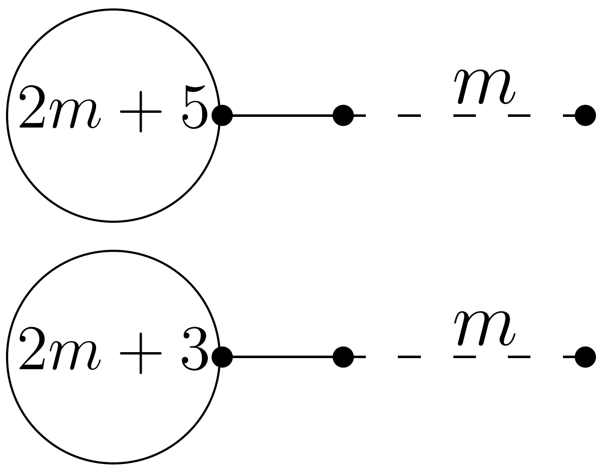
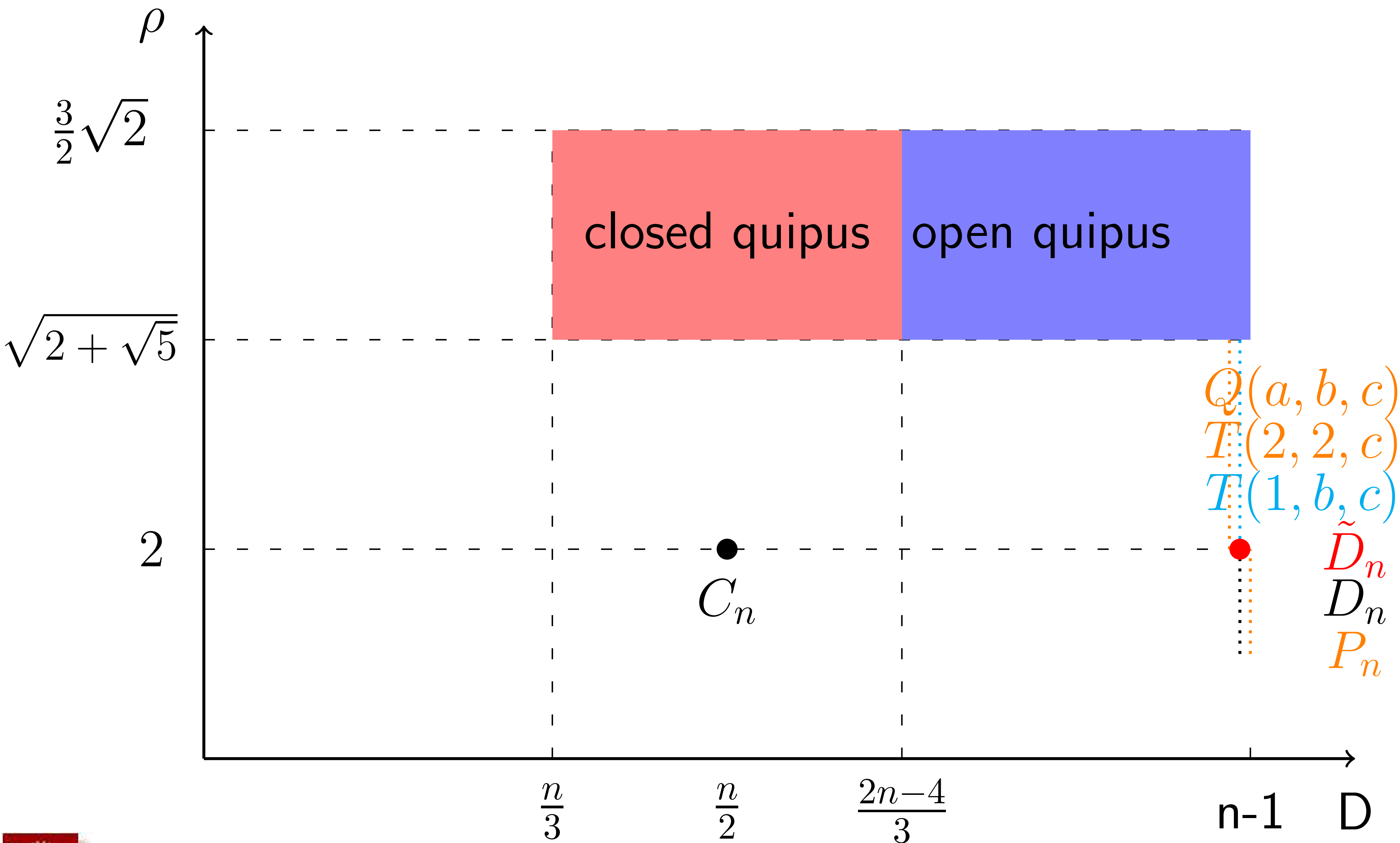
Moreover, if $L$ is neither $C^{(m)}_{(2m+3)}$ nor $C^{(m)}_{(2m+5)}$, then $D(L) \leq \frac{2n-4}{3}$.



# Diameter v.s. spectral radius



# Case $D \approx \frac{n}{2}$

**Theorem [Cioabă-van Dam-Koolen-Lee, 2010]:** For $e = 1, 2, 3, 4$ and sufficiently large $n$ with $n + e$ even, $C^{(\lfloor \frac{e}{2} \rfloor, \lceil \frac{e}{2} \rceil)}_{(\frac{n-e-2}{2}, \frac{n-e-2}{2})}$ is the unique minimizer graph $G^{min}_{n, \frac{n+e}{2}}$.

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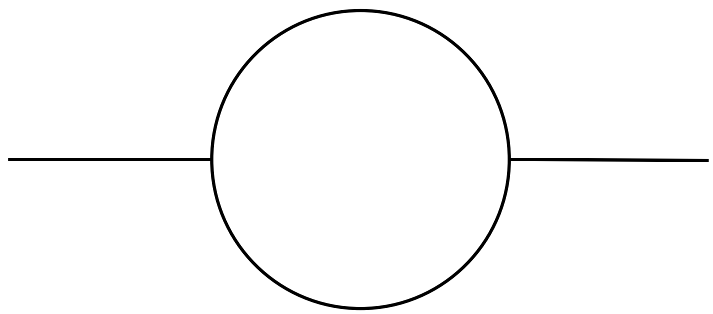
They **Conjectured** that the statement above holds for any constant $e \geq 1$.

# Our result

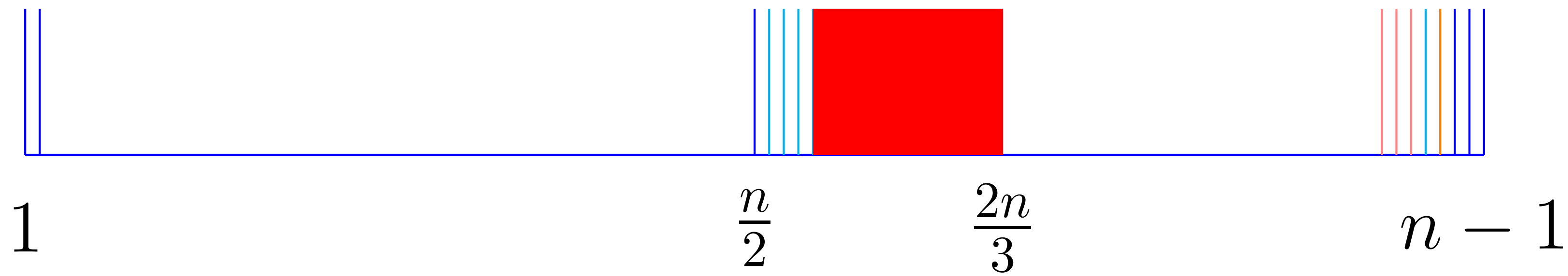
**Theorem I [Lu-Lan 2013]:** For $n \geq 13$ and $\frac{n}{2} \leq D \leq \frac{2n-7}{3}$, $C_{(n-D-1, n-D-1)}^{(D-\lfloor \frac{n}{2} \rfloor, D-\lceil \frac{n}{2} \rceil)}$ is the unique minimizer graph $G_{n,D}^{min}$.



Cioabă-van Dam-Koolen-Lee's conjecture is settled in a stronger way.
The upper bound $\frac{2n-7}{3}$ can not replaced by $\frac{2n-3}{3}$.

# Summary

The minimizer graph $G_{n,D}^{min}$ is determined for the following range of $D$.

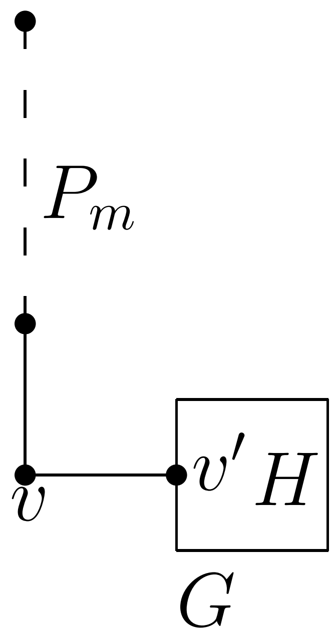



- Van Dam-Kooij [2007]
- Yuan-Shao-Liu [2008]
- Cioabă-van Dam-Koolen-Lee[2010]
- Lan-Lu-Shi[2012]
- Lan-Lu[2013]

# Recursive construction

For $m \geq 0$, consider the basic operations to extend a rooted graph

$$\psi_m \colon (H, v') \to (G, v).$$




- Any tree open quipu can be built from a single vertex graph using above operations recursively.
- The characteristic polynomials $(\phi_G, \phi_{G-v})$ can be computed from $(\phi_H, \phi_{H-v'})$.

# Choosing right base

Let $x_1 \leq x_2$ be two root of $x^2 - \lambda x + 1 = 0$. Let

$$\begin{pmatrix} p_{(G,v)} \\ q_{(G,v)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ x_2 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} \phi_G \\ \phi_{G-v} \end{pmatrix}.$$

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Then

$$\begin{pmatrix} p_{(G_m,v)} \\ q_{(G_m,v)} \end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix} d_m^{(1)} & x_1\phi_{P_{m-1}} \\ -x_2\phi_{P_{m-1}} & d_m^{(2)} \end{pmatrix} \begin{pmatrix} p_{(H,v')} \\ q_{(H,v')} \end{pmatrix},$$

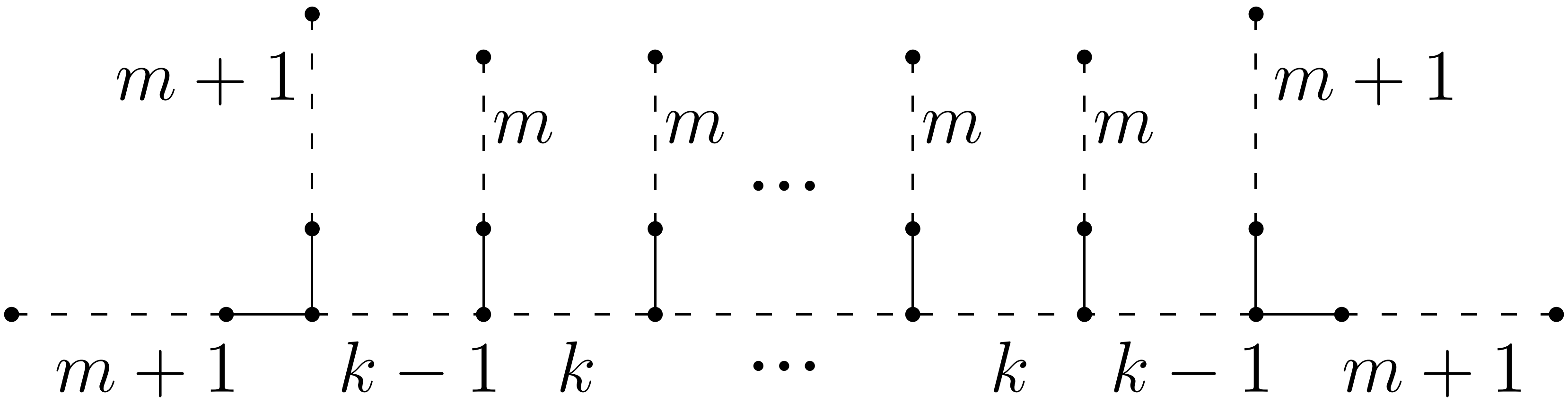
where $\phi_{P_m} = \frac{x_2^{m+1} - x_1^{m+1}}{x_2 - x_1}$, $d_m^{(1)} = \phi_{P_m} - x_1^{m+2}$, and $d_m^{(2)} = x_2^{m+2} - \phi_{P_m}$.

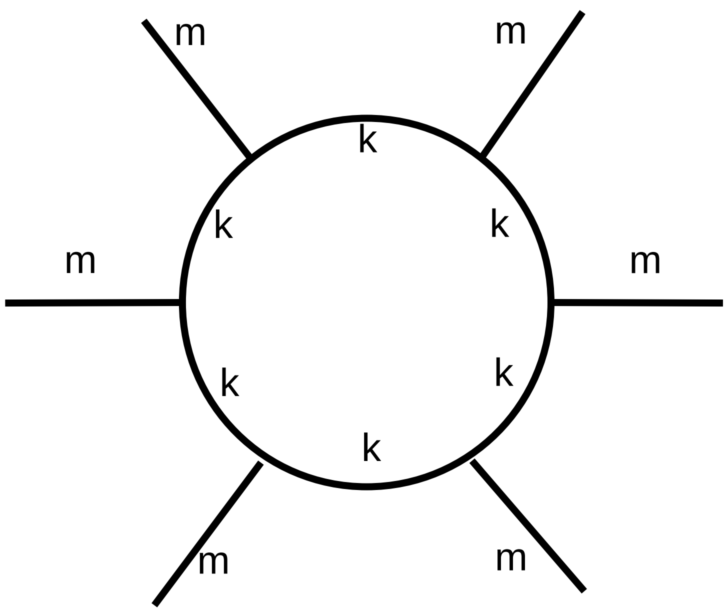
# Special value $\rho_{m,k}$

Let $\rho_{m,k}$ be the the largest root of the equation $d_m^{(2)} = \frac{2\phi_{P_{m-1}} x_1^k}{1-x_1^{k+1}}$. Then, $\rho_{m,k}$ is the spectral radius of the following graphs.

- $P^{(m+1,m+1)}_{(m+1,k-2,m+1)}$,
- $P^{(m+1,m,m+1)}_{(m+1,k-1,k-1,m+1)}$,
- $P^{(m+1,m,\ldots,m,m+1)}_{(m+1,k-1,k,\ldots,k,k-1,m+1)}$,
- $C^{(m)}_{(k)}$,
- $C^{(m,m)}_{(k,k)}$,
- $C^{(m,\ldots,m)}_{(k,\ldots,k)}$,

# Quipus with $\rho(G) = \rho_{m,k}$




$\rho_{m,k} < \frac{3}{2}\sqrt{2}$ if and only if

- "$m \geq 2$ and $k \geq 2m+3$",
- or "$m = 1$ and $k \geq 4$".

# A necessary condition of $\rho < \frac{3}{2}\sqrt{2}$

**Theorem [Lan-Lu 2013]** Suppose an open quipu $P^{(m_0,\ldots,m_r)}_{(m_0,k_1,\ldots,k_r,m_r)}$ has spectral radius less than $\frac{3}{2}\sqrt{2}$. Then the following statements hold.

1. For $2 \leq i \leq r-1$, we have $k_i \geq m_{i-1} + m_i$. Moreover if $m_{i-1}, m_i \geq 2$, then $k_i \geq m_{i-1} + m_i + 1$.
2. We have $k_1 \geq m_0 + m_1$ if $m_0 \geq 2$; and $k_1 \geq m_1 - 1$ if $m_0 = 1$.
3. We have $k_r \geq m_r + m_{r-1}$ if $m_r \geq 2$; and $k_r \geq m_{r-1} - 1$ if $m_r = 1$.

# A sufficient condition of $\rho < \frac{3}{2}\sqrt{2}$

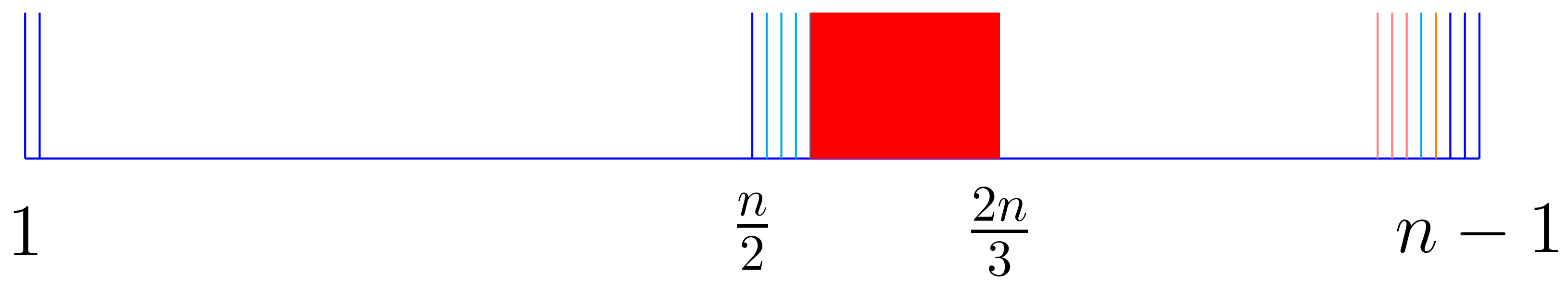
**Theorem [Lan-Lu 2013]** Suppose that an open quipu $P_{(m_0,k_1,\ldots,k_r,m_r)}^{(m_0,\ldots,m_r)}$ satisfies

1. $m_0, m_r \geq 2$;
2. $k_i \geq m_{i-1} + m_i + 3$ for $2 \leq i \leq r-1$;
3. $k_j \geq m_{j-1} + m_j + 1$ for $j = 1, r$.

Then we have $\rho(P_{(m_0,k_1,\ldots,k_r,m_r)}^{(m_0,\ldots,m_r)}) < \frac{3}{2}\sqrt{2}$.

# Open problems

Determine $G_{n,D}^{min}$ for $D$ in the empy region.



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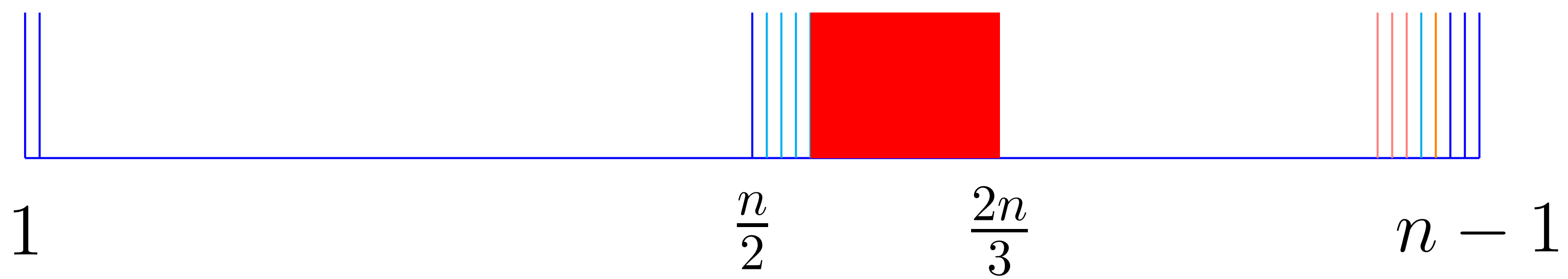


In particular, determine $G_{n,n-e}^{min}$ for $e = 9, 10, 11, 12, \ldots$.

# References

1. Jingfen Lan, Linyuan Lu, and Lingsheng Shi, Graphs with Diameter $n-e$ Minimizing the Spectral Radius, *Linear Algebra and its Application*, **437**, No. 11, (2012), 2823-2850.
2. Linyuan Lu and Jingfen Lan, Diameter of Graphs with Spectral Radius at most $\frac{3}{2}\sqrt{2}$, *Linear Algebra and its Application*, **438**, No. 11, (2013), 4382-4407.

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Thank You