Complex Graphs and Networks

Lecture 6: Spectrum of random graphs with given degrees

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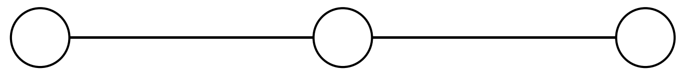
Overview of talks

- Lecture 1: Overview and outlines
- Lecture 2: Generative models - preferential attachment schemes
- Lecture 3: Duplication models for biological networks
- Lecture 4: The rise of the giant component
- Lecture 5: The small world phenomenon: average distance and diameter
- Lecture 6: Spectrum of random graphs with given degrees

Three spectra of a graph

A graph G:



(1) Adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Eigenvalues are

$$-\sqrt{2}, 0, \sqrt{2}.$$

Three spectra of a graph

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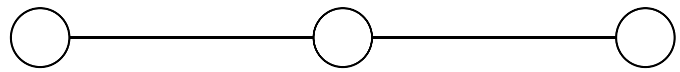

(2) Combinatorial Laplacian

$$D - A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Eigenvalues are

$$0, 1, 3.$$

Three spectra of a graph

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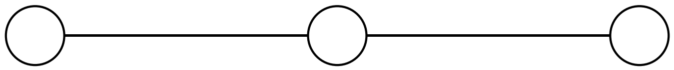

(3) Normalized Laplacian

$$I - D^{-1/2} A D^{-1/2} = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 1 & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & 1 \end{pmatrix}$$

Eigenvalues are

$$0, 1, 2.$$

Relations

If G is a d-regular graph, then three spectra are related by linear translations.

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\begin{aligned}
D - A &= dI - A \\
D - A &= d(I - D^{-1/2}AD^{-1/2}) \\
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But they are quite different for general graphs.

Laplacian Spectrum

The (normalized) Laplacian is defined to be the matrix

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2}.$$

1. All eigenvalues of $\mathcal{L}$ are between 0 and 2.

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq 2.$$

2. G is connected if and only if $\lambda_1 > 0$.

3. G is bipartite if and only if $\lambda_{n-1} = 2$.

Cheeger constant

The Cheeger constant h_G of a graph G is defined by

$$h_G = \inf_S \frac{|\partial(S)|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

where $\partial(S)$ denotes the set of edges leaving S.
Cheeger's inequality states

$$2h_G \geq \lambda_1 \geq \frac{h_G^2}{2}.$$

Diameter

Let $D(G)$ be the diameter of G, then

$$D(G) \le \left\lceil \frac{\log \frac{\mathrm{vol}(G)}{\min_x d_x}}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil .$$

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In general, let $D(X,Y)$ denote the distance between two subsets X and Y. Then

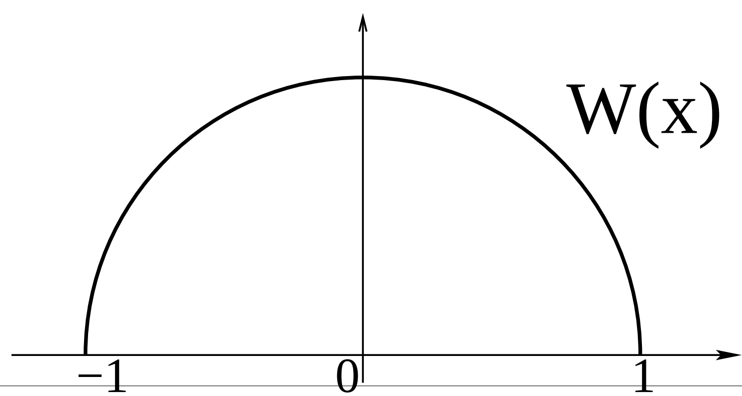
$$D(X,Y) \leq \left\lceil \frac{\log \frac{\mathrm{vol}(G)}{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)}}}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil .$$

Wigner's semicircle law

Wigner (1958)

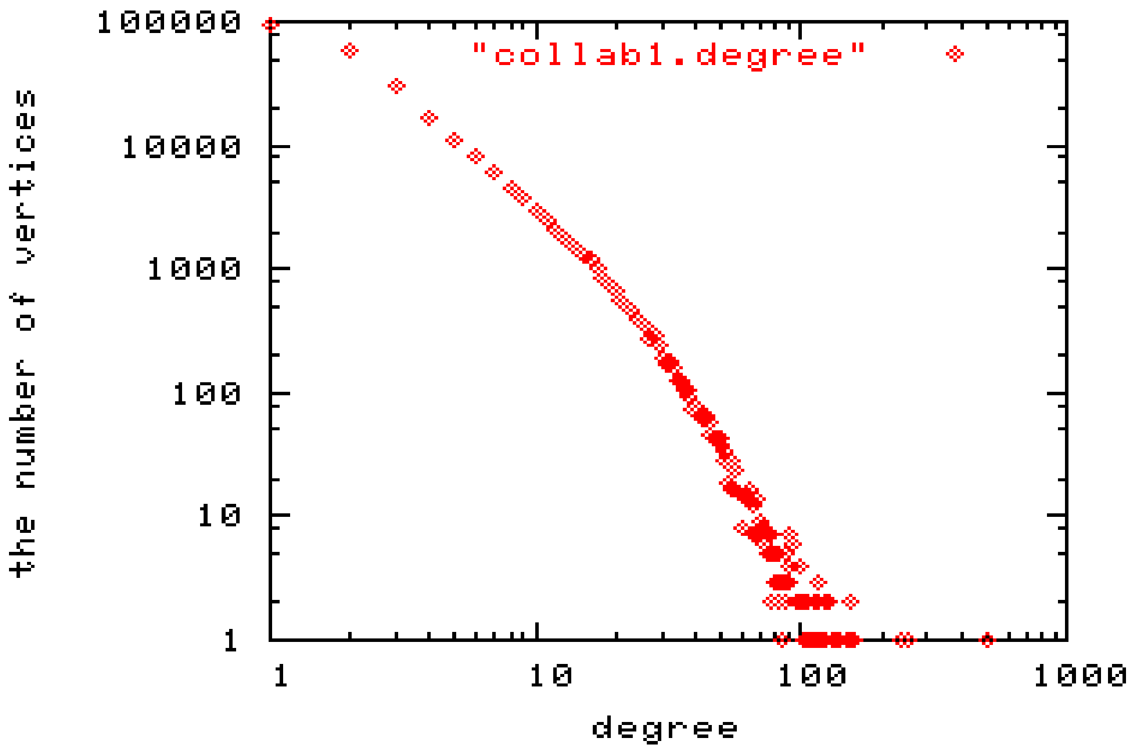
- A is a real symmetric $n \times n$ matrix.
- Entries a_{ij} are independent random variables.
- $E(a_{ij}^{2k+1}) = 0$.
- $E(a_{ij}^2) = m^2$.
- $E(a_{ij}^{2k}) < M$.

The distribution of eigenvalues of A converges into a semicircle distribution of radius $2m\sqrt{n}$.



The power law

The number of vertices of degree k is approximately proportional to $k^{-\beta}$ for some positive β.



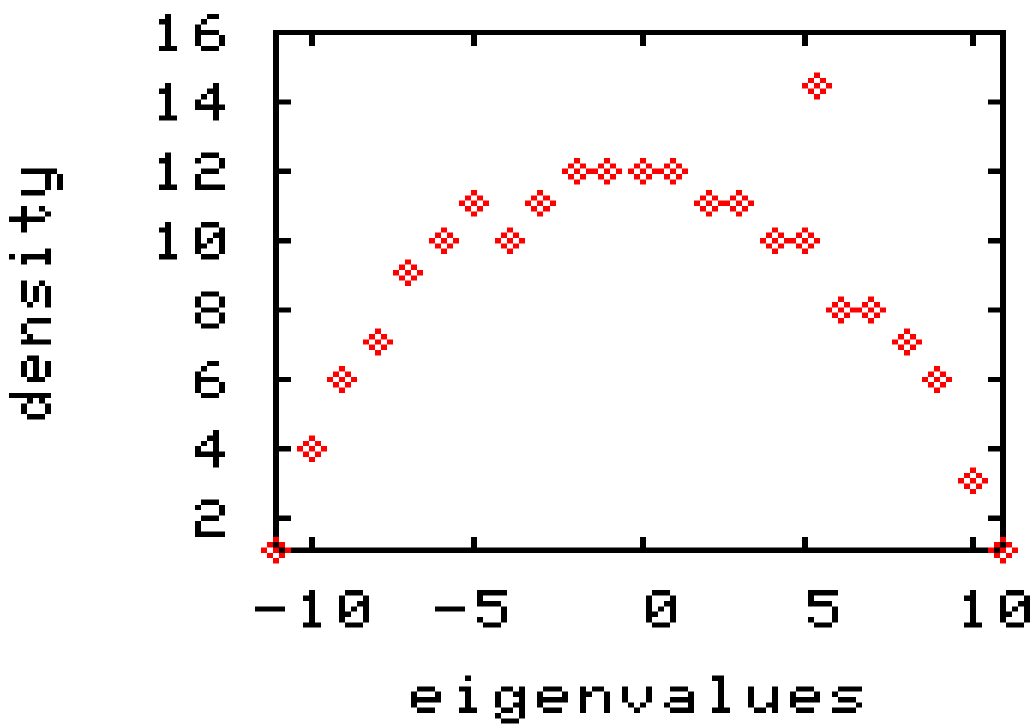
A power law graph is a graph which satisfies the power law.

A spectrum question

Do the eigenvalues of a power law graph follow the semicircle law or do the eigenvalues have a power law distribution?

Evidence for the semicircle law for power law graphs

The eigenvalues of an Erdős-Rényi random graph follow the semicircle law. (Füredi and Komlós, 1981)



Experimental results

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Experimental results

- **Faloutsos et al. (1999)** The eigenvalues of the Internet graph do not follow the semicircle law.
- **Farkas et. al. (2001), Goh et. al. (2001)** The spectrum of a power law graph follows a "triangular-like" distribution.
- **Mihail and Papadimitriou (2002)** They showed that the large eigenvalues are determined by the large degrees. Thus, the significant part of the spectrum of a power law graph follows the power law.

$$\mu_i \approx \sqrt{d_i}.$$

Model $G(w_1, w_2, \ldots, w_n)$

Random graph model with given expected degree sequence

- n nodes with weights $w_1, w_2, \ldots, w_n$.

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- The graph H has probability

$$\prod_{ij \in E(H)} p_{ij} \prod_{ij \notin E(H)} (1 - p_{ij}).$$

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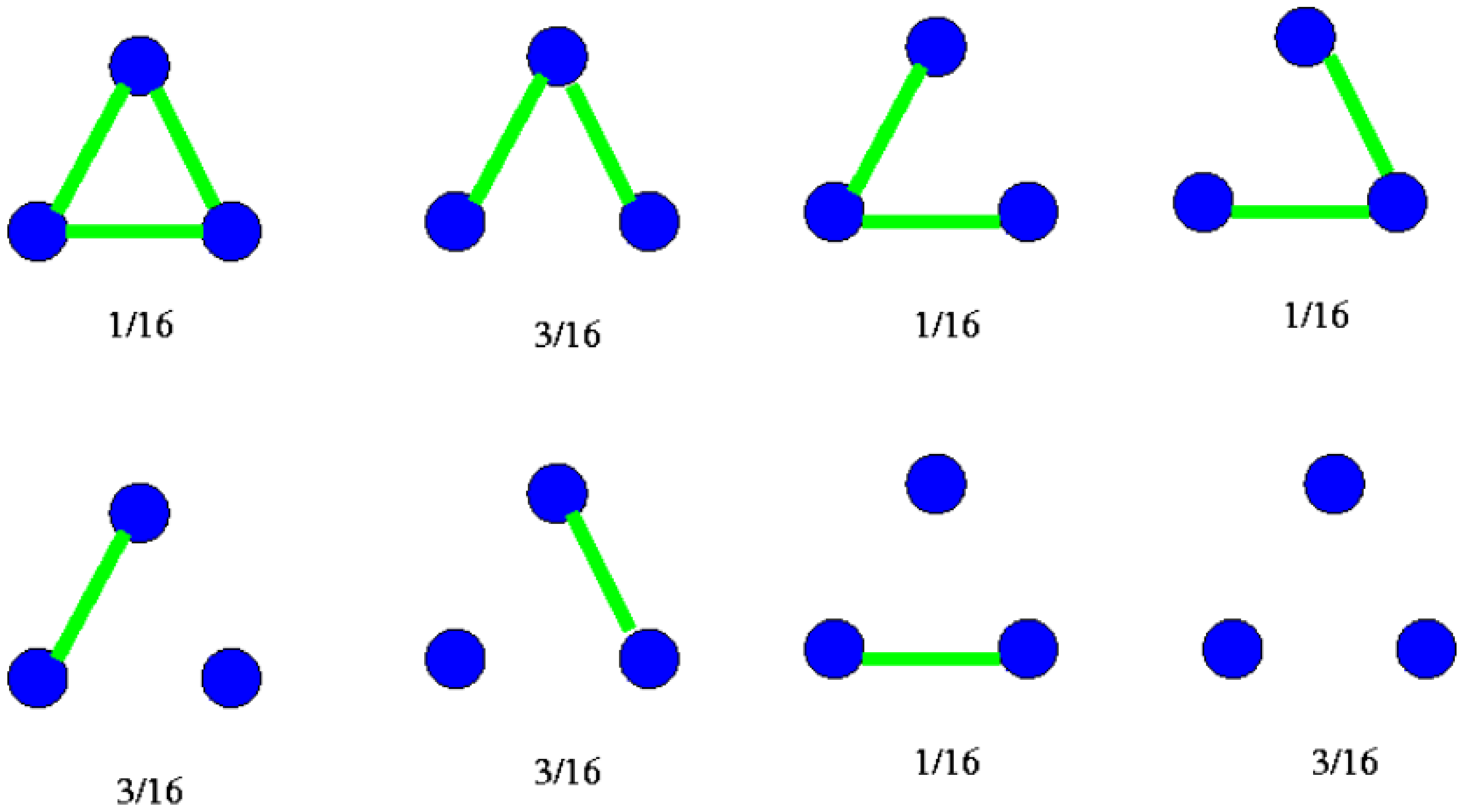
- The expected degree of vertex i is w_i.

A example: $G(1, 2, 1)$



Loops are omitted here.

Notations

For $G = G(w_1, \ldots, w_n)$, let

- $d = \frac{1}{n}\sum_{i=1}^{n} w_i$
- $\tilde{d} = \frac{\sum_{i=1}^{n} w_i^2}{\sum_{i=1}^{n} w_i}$.
- The volume of S: $\mathrm{Vol}(S) = \sum_{i \in S} w_i$.
- The k-th volume of S: $\mathrm{Vol}_k(S) = \sum_{i \in S} w_i^k$.

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We have

$$\tilde{d} \geq d$$

"$=$" holds if and only if $w_1 = \cdots = w_n$.

Eigenvalues of $G(w_1, \ldots, w_n)$

Chung, Vu, and Lu (2003)

Suppose $w_1 \geq w_2 \geq \ldots \geq w_n$. Let μ_i be i-th largest eigenvalue of $G(w_1, w_2, \ldots, w_n)$. Let $m = w_1$ and $\tilde{d} = \sum_{i=1}^n w_i^2 \rho$. Almost surely we have:

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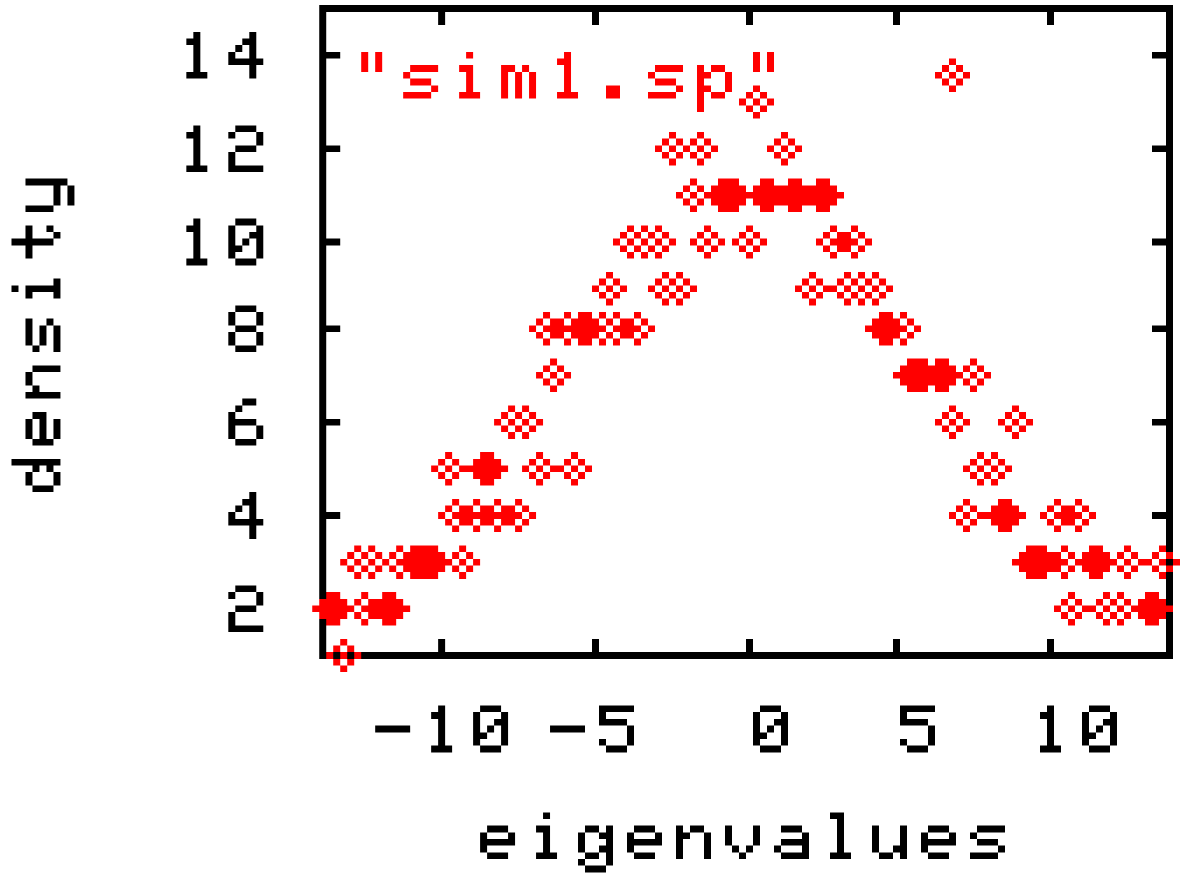
- $\mu_1 = (1 + o(1))\sqrt{m}$, if $\sqrt{m} > \tilde{d} \log^2 n$.
- $\mu_k \approx \sqrt{w_k}$ and $\mu_{n+1-k} \approx -\sqrt{w_k}$, if $\sqrt{w_k} > \tilde{d} \log^2 n$.

Random power law graphs

The first k and last k eigenvalues of the random power law graph with $\beta > 2.5$ follows the power law distribution with exponent $2\beta - 1$. It results a "triangular-like" shape.



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Hence $\mu_1 \geq (1 + o(1))\sqrt{m}$.

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- $\mu_1 \geq X$.
- X can be written as a sum of independent random variables. $X = \frac{1}{\sum_{i=1}^n w_i^2} \sum_{i,j} w_i w_j X_{i,j}$, where X_{ij} is the 0-1 random variable with $Pr(X_{i,j} = 1) = w_i w_j \rho$.

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- $E(X) = \tilde{d}$.
- X concentrates on $E(X)$.

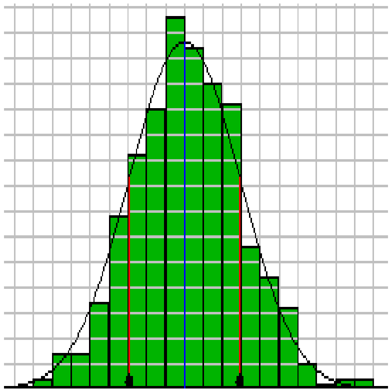
Lemma A:

Let $X_1, \ldots, X_n$ be independent random variables with

$$Pr(X_i = 1) = p_i, \qquad Pr(X_i = 0) = 1 - p_i$$

For $X = \sum_{i=1}^n a_i X_i$, we have $E(X) = \sum_{i=1}^n a_i p_i$ and we define $\nu = \sum_{i=1}^n a_i^2 p_i$. Then we have

$$\begin{aligned} Pr(X < E(X) - t) &\leq e^{-\frac{t^2}{2\nu}}; \\ Pr(X > E(X) + t) &\leq e^{-\frac{t^2}{2(Var(X) + at/3)}}; \end{aligned}$$



where a the maximum coefficient among a_i's.

Lemma B:

$$\mu_1 \leq \tilde{d} + \sqrt{6\sqrt{m \log n}(\tilde{d} + \log n)} + 3\sqrt{m \log n}.$$

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$$c_i = \begin{cases} w_i & \text{if } w_i > x \\ x & \text{otherwise.} \end{cases}$$

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The i-th row sum X_i of $C^{-1}AC$ is $X_i = \frac{1}{c_i}\sum_{j=1}^{n} c_j a_{ij}$. We have

$$E(X_i) \le \tilde{d} + x;$$

$$Var(X_i) \le \frac{m}{x}\tilde{d} + x.$$

Proof continues

By Lemma A, we have

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$$\begin{aligned}
\mu_1 &\leq \max_i \{X_i\} \\
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&\leq \tilde{d} + \sqrt{6\sqrt{m \log n}(\tilde{d} + \log n)} + 3\sqrt{m \log n}.
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The outline for proving $\mu_k = (1 + o(1))\sqrt{w_k}$.

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- S and T are disjoint.
- $G = G(\bar{S}) \cup G(\bar{T}) \cup G(S, T)$.
- Apply Lemma B to $G(\bar{S})$ and $G(\bar{T})$, we have $\mu_1(G(\bar{S})) = o(\sqrt{w_k})$ and $\mu_1(G(\bar{T})) = o(\sqrt{w_k})$.

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Putting together, for $1 \leq i \leq k$, we have

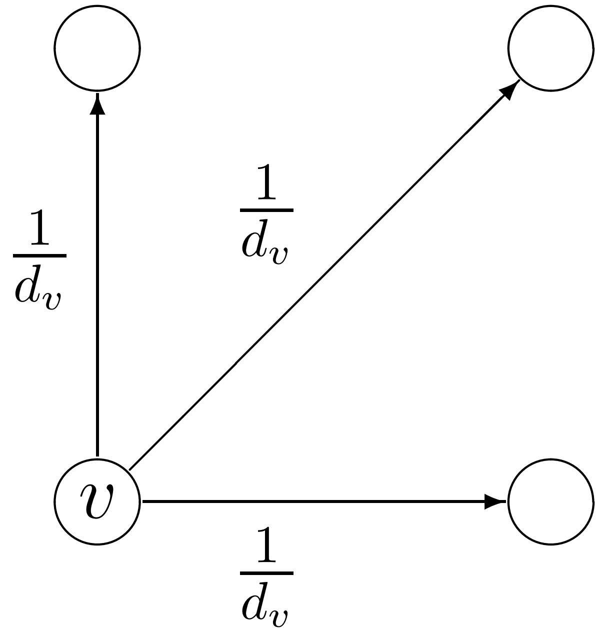
$$\begin{aligned} |\mu_i(G) - \sqrt{w_i}| &\leq |\mu_i(G) - \mu_i(G_1)| + o(\sqrt{w_i}) \\ &\leq |\mu_1(G(\bar{S})) + \mu_1(G(\bar{T})) + \mu_1(G_2) + o(\sqrt{w_i}) \\ &= o(\sqrt{w_i}). \end{aligned}$$

Laplacian spectrum

Random walks on a graph G:

$$\pi_{k+1} = AD^{-1}\pi_k.$$

$$AD^{-1} \sim D^{-1/2}AD^{-1/2}.$$



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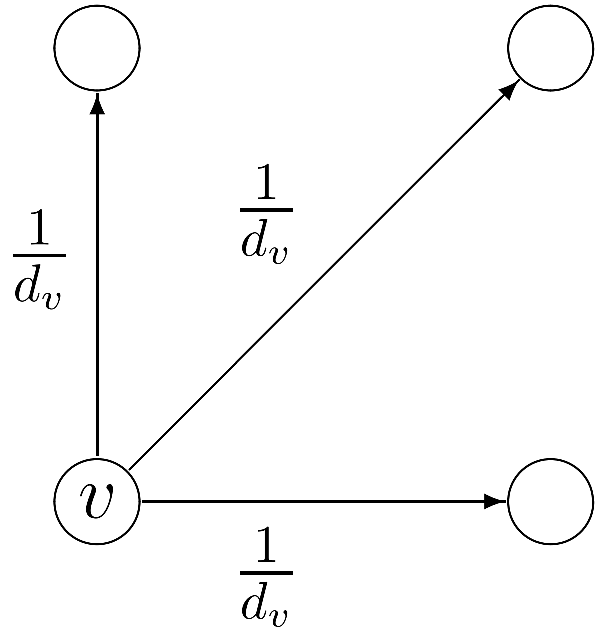
$$AD^{-1} \sim D^{-1/2}AD^{-1/2}.$$



Laplacian spectrum

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2$$

are the eigenvalues of $L = I - D^{-1/2}AD^{-1/2}$.

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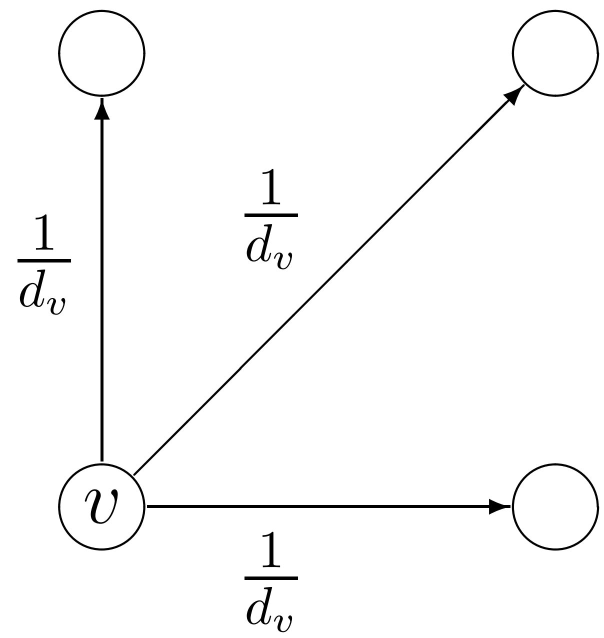

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are the eigenvalues of $L = I - D^{-1/2}AD^{-1/2}$.
The eigenvalues of AD^{-1} are $1, 1-\lambda_1, \ldots, 1-\lambda_{n-1}$.

Spectral Radius

Let

- $w_{min} = \min\{w_1, \ldots, w_n\}$
- $d = \frac{1}{n}\sum_{i=1}^{n} w_i$
- $g(n)$ — a function tending to infinity arbitrarily slowly.

Chung, Vu, and Lu (2003)
If $w_{\min} \gg \log^2 n$, then almost surely the Laplacian spectrum λ_i's of $G(w_1, \ldots, w_n)$ satisfy

$$\max_{i \neq 0} |1 - \lambda_i| \leq (1 + o(1))\frac{4}{\sqrt{d}} + \frac{g(n)\log^2 n}{w_{\min}}.$$

Approximation

$$M = D^{-1/2} A D^{-1/2} - \phi_0 \phi_0'$$

where

$$\phi_0 = \frac{1}{\sqrt{\sum_{i=1}^n d_i}} (\sqrt{d_1}, \ldots, \sqrt{d_n})'.$$

$$C = W^{-1/2} A W^{-1/2} - \chi \chi'$$

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- M has eigenvalues $0, 1 - \lambda_1, \ldots, 1 - \lambda_{n-1}$, since $M = I - L - \phi_0^* \phi_0$ and $L\phi_0 = 0$.

Results on spectrum of C

Chung, Vu, and Lu (2003)

We have

- If $w_{\min} \gg \sqrt{d}\log^2 n$, then

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- If $w_{\min} \gg \sqrt{d}\log^2 n$, then

$$\|C\| = (1 + o(1))\frac{2}{\sqrt{d}}.$$

- If $w_{\min} \gg \sqrt{d}$, the eigenvalues of C follow the semi-circle distribution with radius $r \approx \frac{2}{\sqrt{d}}$.

Proof

Wigner's high moment method:

$$\|C\| \leq [\mathrm{Trace}(C^{2k})]^{\frac{1}{2k}}.$$

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First we will bound $E(\text{Trace}(C^{2k}))$.

$$\begin{aligned} E(\text{Trace}(C^{2k})) &= \sum_{i_1, i_2, \ldots, i_{2k}} E(c_{i_1 i_2} c_{i_2 i_3} \cdots c_{i_{2k-1} i_{2k}} c_{i_{2k} i_1}) \\ &= \sum_{l \ge 1} \sum_{I_l} \prod_{h=1}^{l} E(c_{e_h}^{m_h}) \end{aligned}$$

$I_k = \{$ closed walks of length $2k$ which use l different edges $e_1, \ldots, e_l$ with corresponding multiplicities $m_1, \ldots, m_l.\}$

Proof continues

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We have

$$E(\text{Trace}(C^{2k})) \leq \sum_{l=1}^{l} W_{l,k} \frac{\rho^l}{w_{min}^{2k-2l}}.$$

Here $W_{l,k}$ denotes the set of closed good walks on K_n of length $2k$ using exactly l different edges.

Proof continues

$$|W_{l,k}| \leq n(n-1)\ldots(n-l)\binom{2k}{2l}\binom{2l}{l}\frac{1}{l+1}(l+1)^{4(k-l)}.$$

If $w_{min} \gg \sqrt{d}\log^2 n$, $W_{k,k}\rho^k \approx n(\frac{2}{\sqrt{d}})^{2k}$ is the main term in the previous sum.

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$$E(\text{Trace}(C^{2k})) = (1+o(1))n(\frac{2}{\sqrt{d}})^{2k}.$$

Proof continues

By Markov's inequality, we have

$$Pr(\|C\| \geq (1+\epsilon)\frac{2}{\sqrt{d}}) \;=\; Pr(\|C\|^{2k} \geq (1+\epsilon)^{2k}(\frac{2}{\sqrt{d}})^{2k})$$

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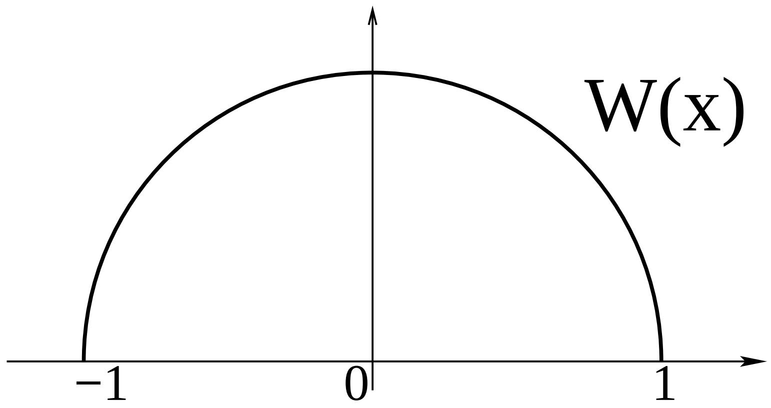
The proof of the semicircle law

Let $W(x)$ be the cumulative distribution function of the unit semicircle.



$$\int_{-1}^{1} x^{2k} dW(x) = \frac{(2k)!}{2^{2k} k!(k+1)!}$$

$$\int_{-1}^{1} x^{2k+1} dW(x) = 0$$

The proof of the semicircle law

Let $C_{\text{nor}} = (\frac{2}{\sqrt{d}})^{-1}C$. Let $N(x)$ be the number of eigenvalues of C_{nor} less than x and $W_n(x) = n^{-1}N(x)$ be the cumulative distribution function.

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For every $k \ll \log n$,

$$\int_{-\infty}^{\infty} x^{2k} dW_n(x) = \frac{1}{n} E(\text{Trace}(C_{\text{nor}}^{2k})) = \frac{(1+o(1))(2k)!}{2^{2k}k!(k+1)!},$$

$$\int_{-\infty}^{\infty} x^{2k+1} dW_n(x) = \frac{1}{n} E(\text{Trace}(C_{\text{nor}}^{2k+1})) = o(1).$$

Thus, $W_n(x) \longrightarrow W(x)$ (in probability) as $n \to \infty$.

Summary

For the random graph with given expected degree sequence $G(w_1, w_2, \ldots, w_n)$, we proved that

- The largest eigenvalue μ_1 is essentially the maximum of $\sqrt{m}$ and $\tilde{d}$, if they are apart by at least a factor of $\log^2 n$.

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References

- Fan Chung, Linyuan Lu and Van Vu, The spectra of random graphs with given expected degrees, *Proceedings of National Academy of Sciences*, **100**, No. 11, (2003), 6313-6318.
- Fan Chung, Linyuan Lu, and Van Vu, Eigenvalues of random power law graphs, *Annals of Combinatorics*, **7** (2003), 21–33.

Overview of talks

- Lecture 1: Overview and outlines
- Lecture 2: Generative models - preferential attachment schemes
- Lecture 3: Duplication models for biological networks
- Lecture 4: The rise of the giant component
- Lecture 5: The small world phenomenon: average distance and diameter
- Lecture 6: Spectrum of random graphs with given degrees