Complex Graphs and Networks

Lecture 5: The small world phenomenon: average distance and diameter

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Overview of talks

- Lecture 1: Overview and outlines
- Lecture 2: Generative models - preferential attachment schemes
- Lecture 3: Duplication models for biological networks
- Lecture 4: The rise of the giant component
- Lecture 5: The small world phenomenon: average distance and diameter
- Lecture 6: Spectrum of random graphs with given degrees

"Six degree separation"

Experiments of Stanley Milgram (1967)

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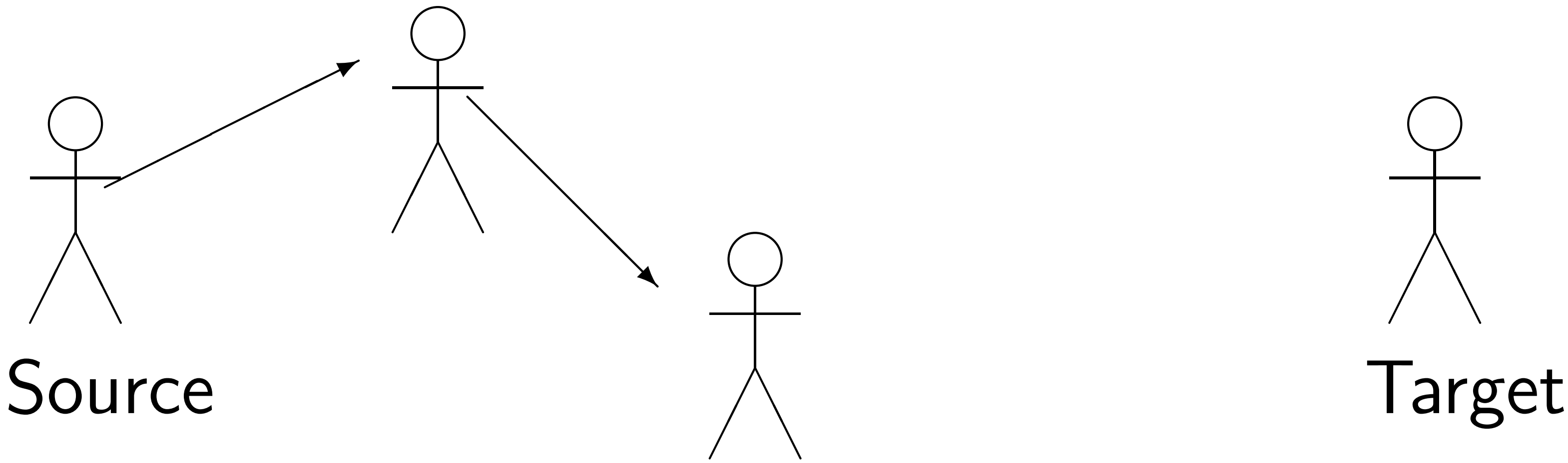

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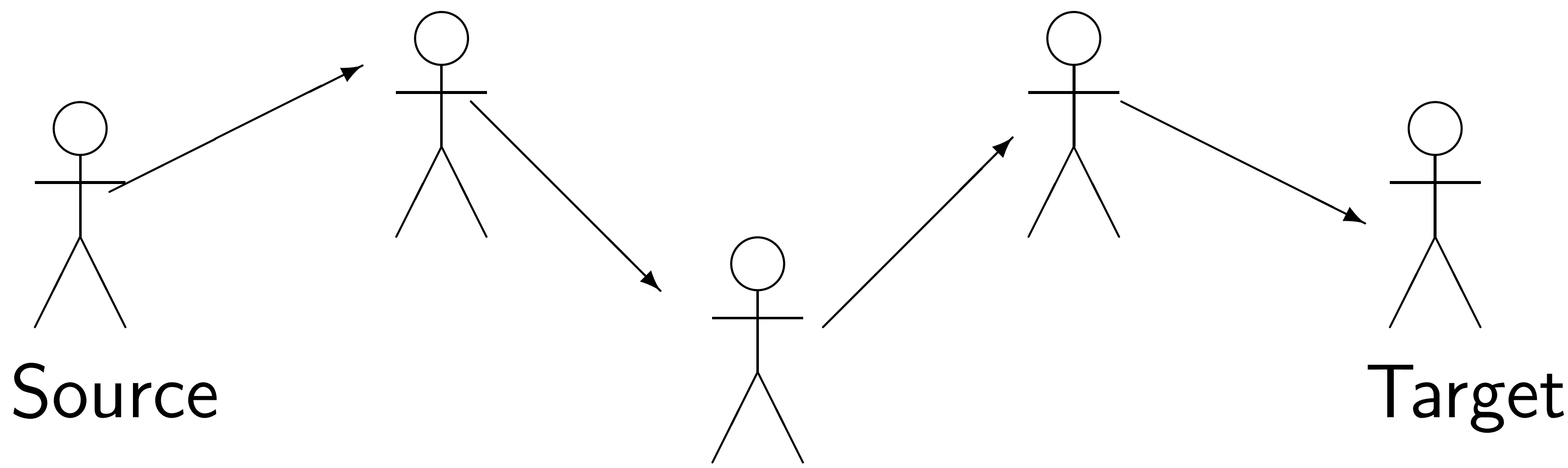

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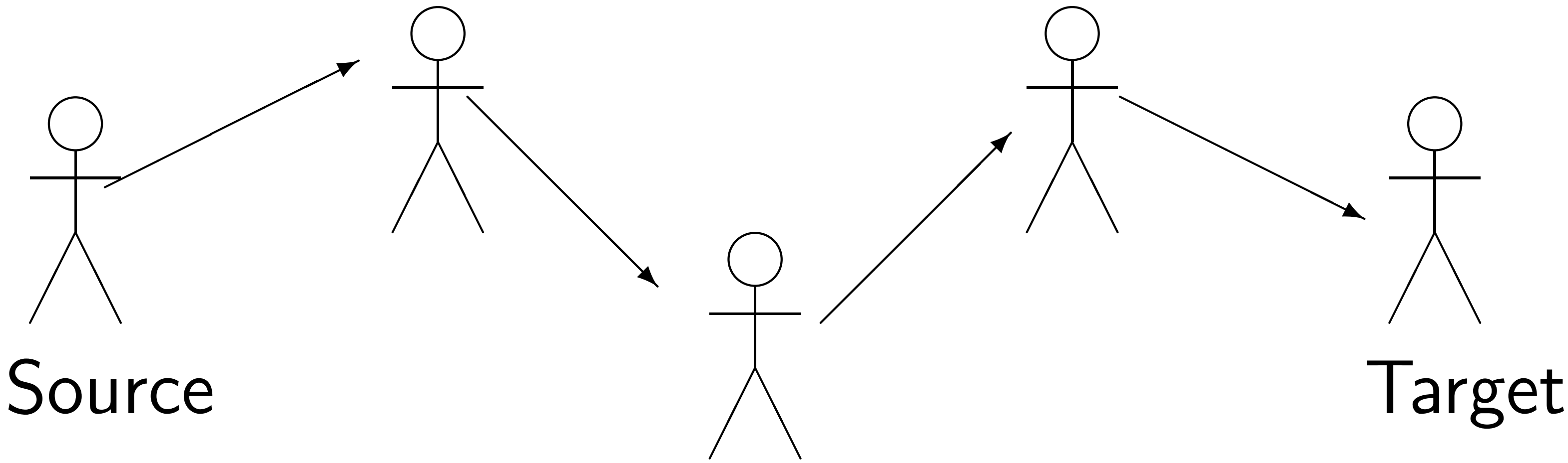

Milgram: “The average distance of the social graph is at most 6.”

Diameter and average distance

Diameter: the maximum distance $d(u, v)$, where u and v are in the same connected component.

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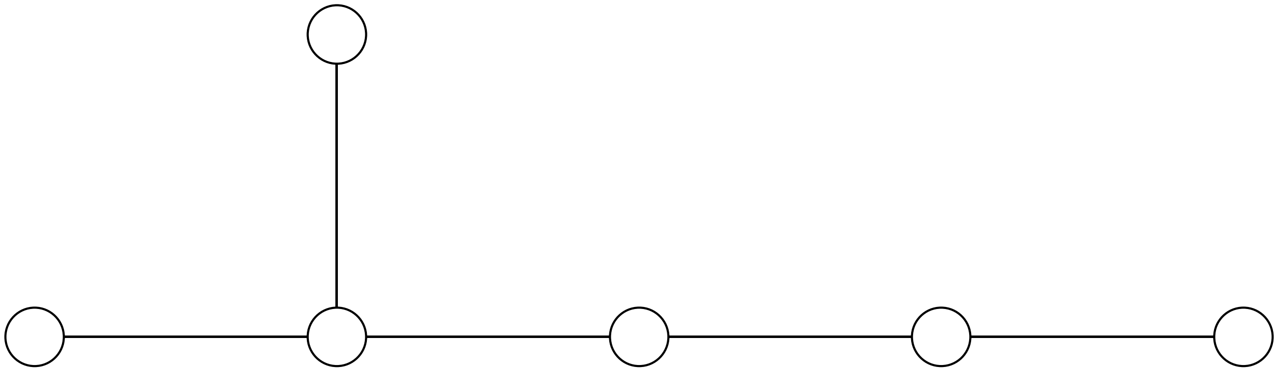

Diameter is 4. Average distance is 2.13.

Experimental results

- The Hollywood graph: $n \approx 656,065$. The average Bacon number is 2.94. The maximum Bacon number is 9.

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Many real-world graphs have small diameters comparing to its sizes.

Disadvantage of experimental methods

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- Case by case
- Inadequate information
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- Prohibitively large sizes

Questions

What is the magnitude of the diameter and the average distance with respect to the graph size?

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How to characterize these graphs?

Modelling graphs

We will use random graphs to model real-world graphs because

- Data sets are too large and dynamic for exact analysis.
- Most real-world graphs have a random or statistical nature.

Random graphs

A random graph is a set of graphs together with a probability distribution on that set.

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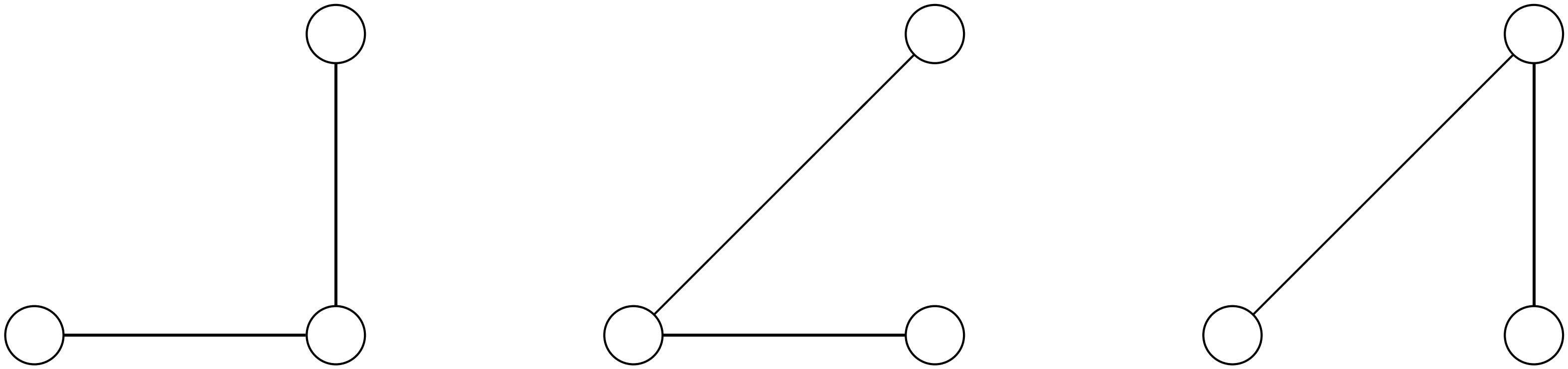
Example: A random graph on 3 vertices and 2 edges with the uniform distribution on it.



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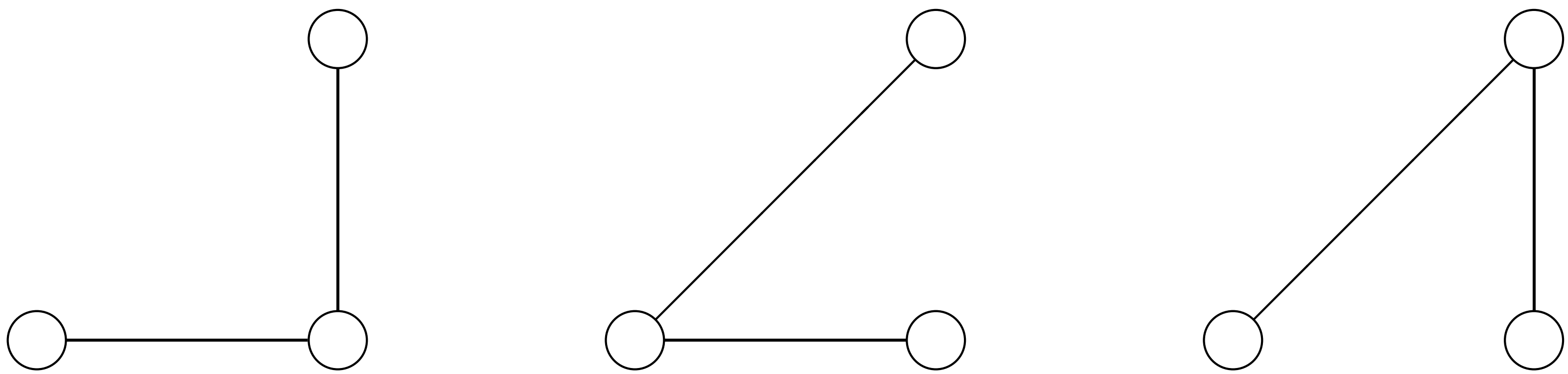

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A random graph G *almost surely* satisfies a property P, if

$$Pr(G \text{ satisfies } P) = 1 - o_n(1).$$

Erdős-Rényi model $G(n, p)$

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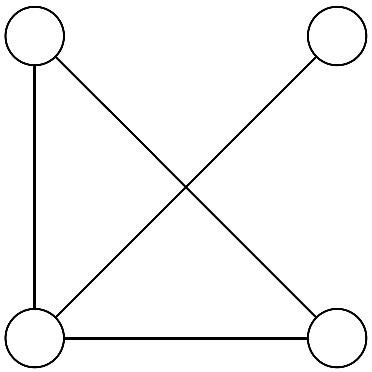

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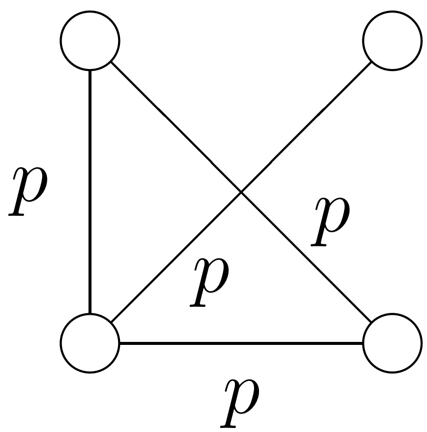

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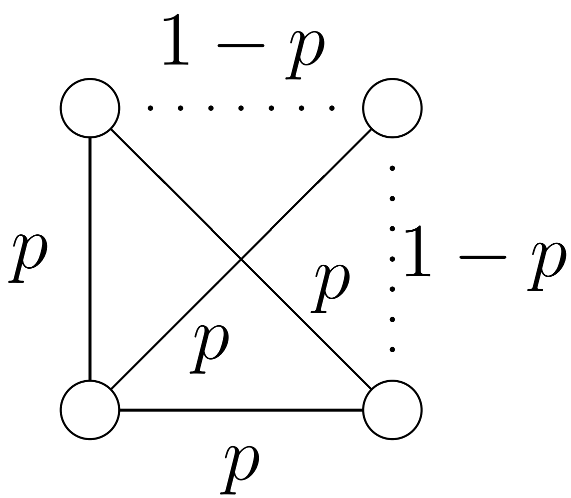

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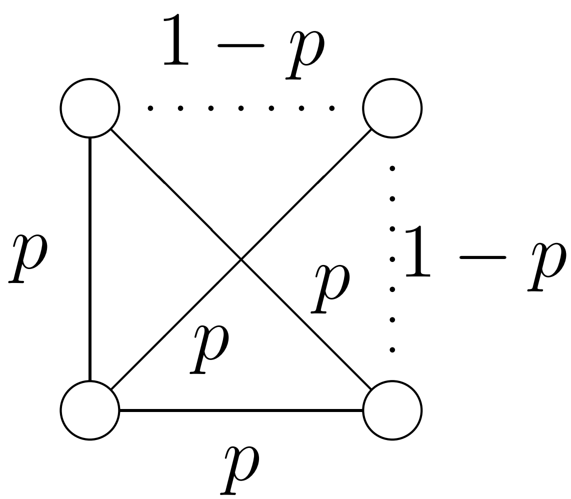
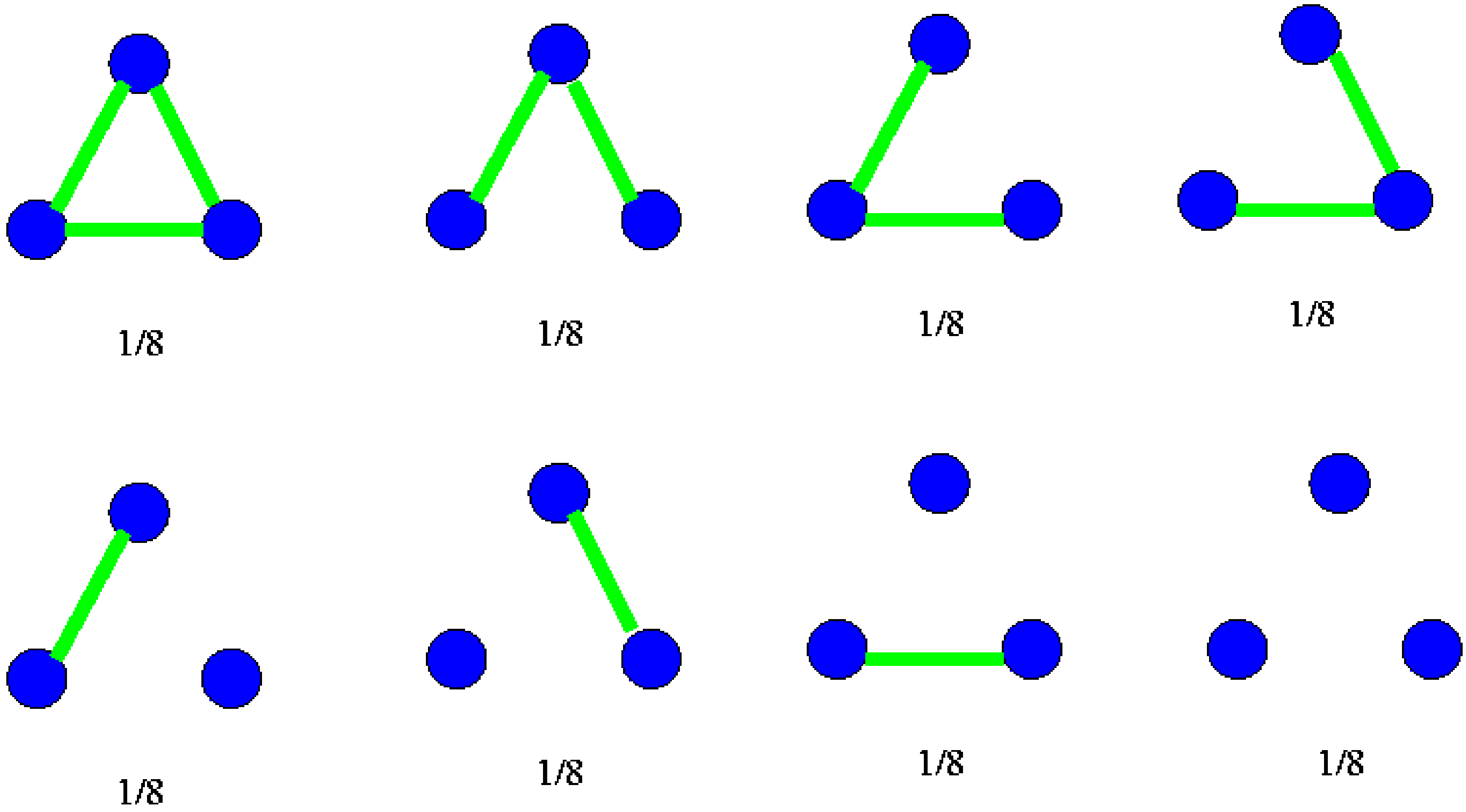


The probability of this graph is

$$p^4(1-p)^2.$$

A example: $G(3, \frac{1}{2})$



The birth of random graph theory

Paul Erdős and A. Rényi, On the evolution of random graphs
Magyar Tud. Akad. Mat. Kut. Int. Kozl. **5** (1960) 17-61.

The birth of random graph theory

ON THE EVOLUTION OF RANDOM GRAPHS

by

P. ERDÖS and A. RÉNYI
Institute of Mathematics
Hungarian Academy of Sciences, Hungary

1. Definition of a random graph

Let $E_{n,N}$ denote the set of all graphs having n given labelled vertices $V_1, V_2, \cdots,$ V_n and N edges. The graphs considered are supposed to be not oriented, without parallel edges and without slings (such graphs are sometimes called linear graphs). Thus a graph belonging to the set $E_{n,N}$ is obtained by choosing N out of the possible $\binom{n}{2}$ edges between the points $V_1, V_2, \cdots, V_n$, and therefore the number of elements of $E_{n,N}$ is equal to $\binom{\binom{n}{2}}{N}$. A random graph $\Gamma_{n,N}$ can be defined as an element of $E_{n,N}$ chosen at random, so that each of the elements of $E_{n,N}$ have the same probability to be chosen, namely $1/\binom{\binom{n}{2}}{N}$. There is however an other slightly

Evolution of $G(n, p)$

p	
0	the diameter.
$\frac{c}{n}$	$\Theta(\frac{\log n}{\log np})$
$\omega(\frac{1}{n})$	$(1 + o(1))\frac{\log n}{\log np}$
$\omega(\frac{\log n}{n})$	$\lfloor \frac{\log n}{\log np} \rfloor$ or $\lceil \frac{\log n}{\log np} \rceil$
$\vdots$	$\vdots$
$n^{-3/4}$	
	3 or 4
$n^{-2/3}$	
	2 or 3
$n^{-\frac{1}{2}}$	
	2
1	1

Diameter of $G(n,p)$

Bollobás (1985): (denser graph)

$$diam(G(n,p)) = \lfloor \frac{\log n}{\log np} \rfloor \text{ or } \lceil \frac{\log n}{\log np} \rceil \text{ if } np \gg \log n.$$

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Chung Lu, (2000) (Sparser graph)

$$diam(G(n,p)) = \begin{cases} (1+o(1))\frac{\log n}{\log np} & \text{if } np \to \infty \\ \Theta(\frac{\log n}{\log np}) & \text{if } \infty > np > 1. \end{cases}$$

Model $G(w_1, w_2, \ldots, w_n)$

Random graph model with given expected degree sequence

- n nodes with weights $w_1, w_2, \ldots, w_n$.

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- The graph H has probability

$$\prod_{ij \in E(H)} p_{ij} \prod_{ij \notin E(H)} (1 - p_{ij}).$$

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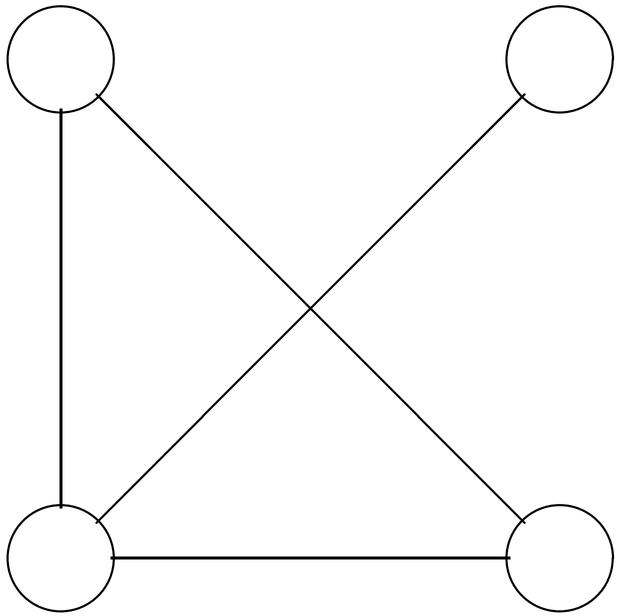
An example: $G(w_1, w_2, w_3, w_4)$



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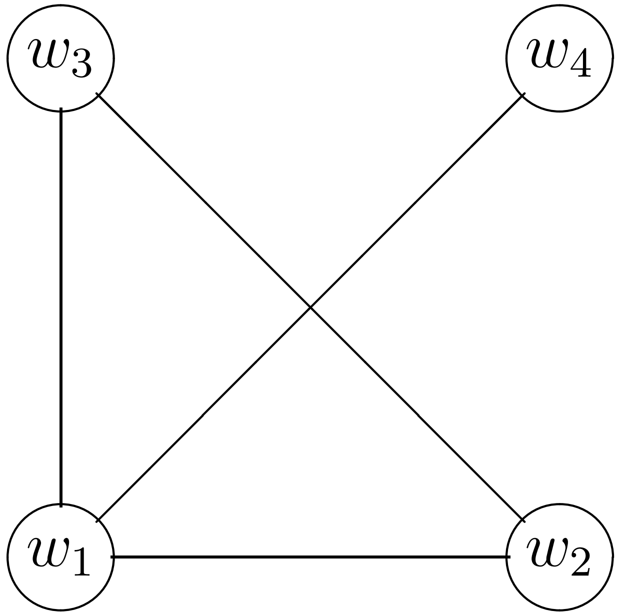

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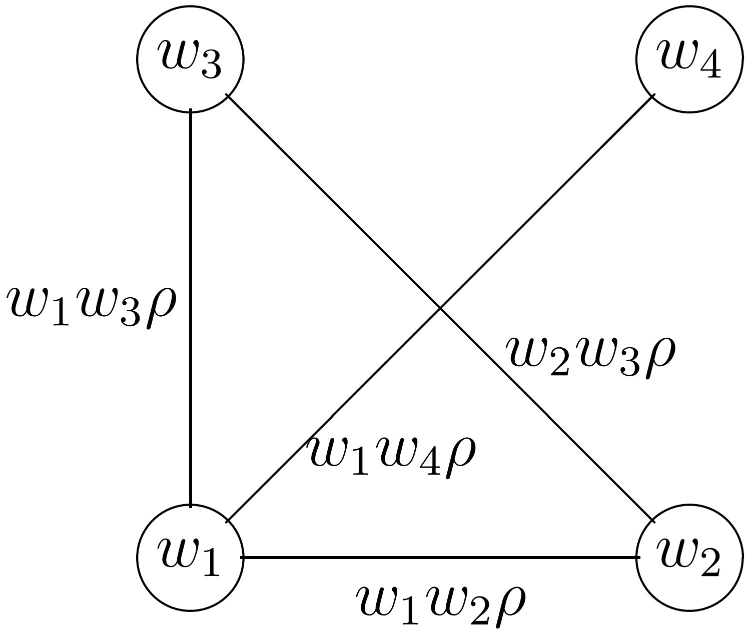

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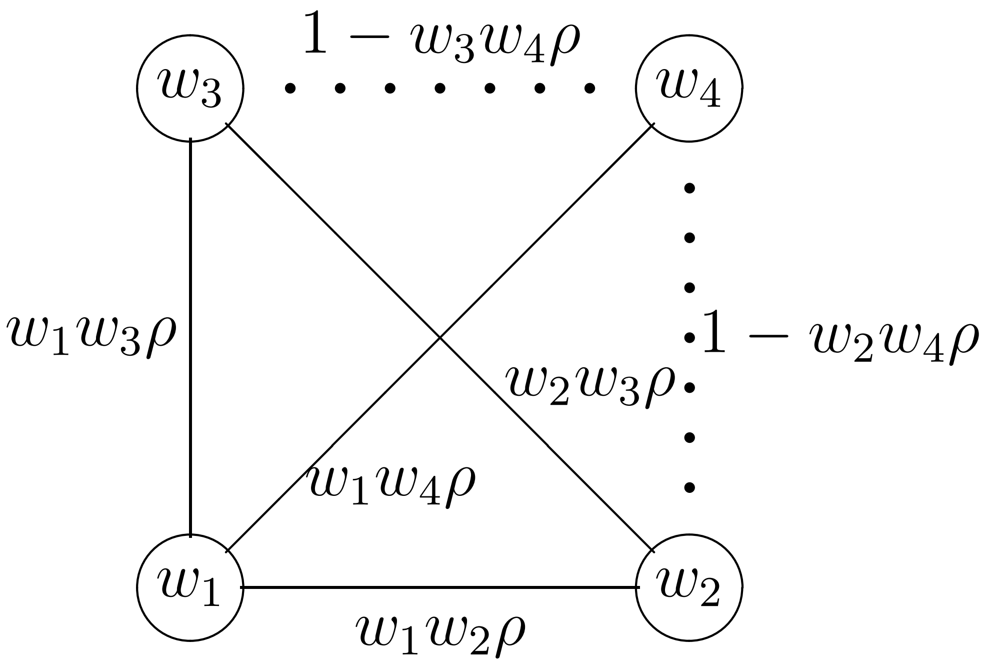

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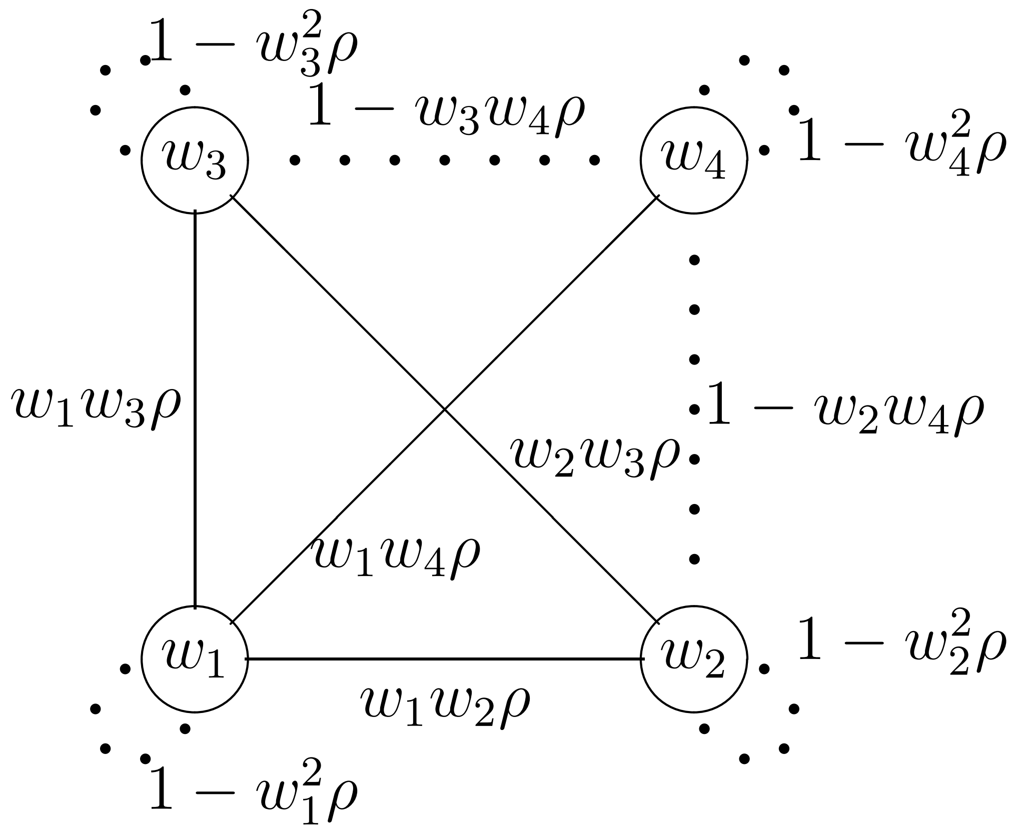

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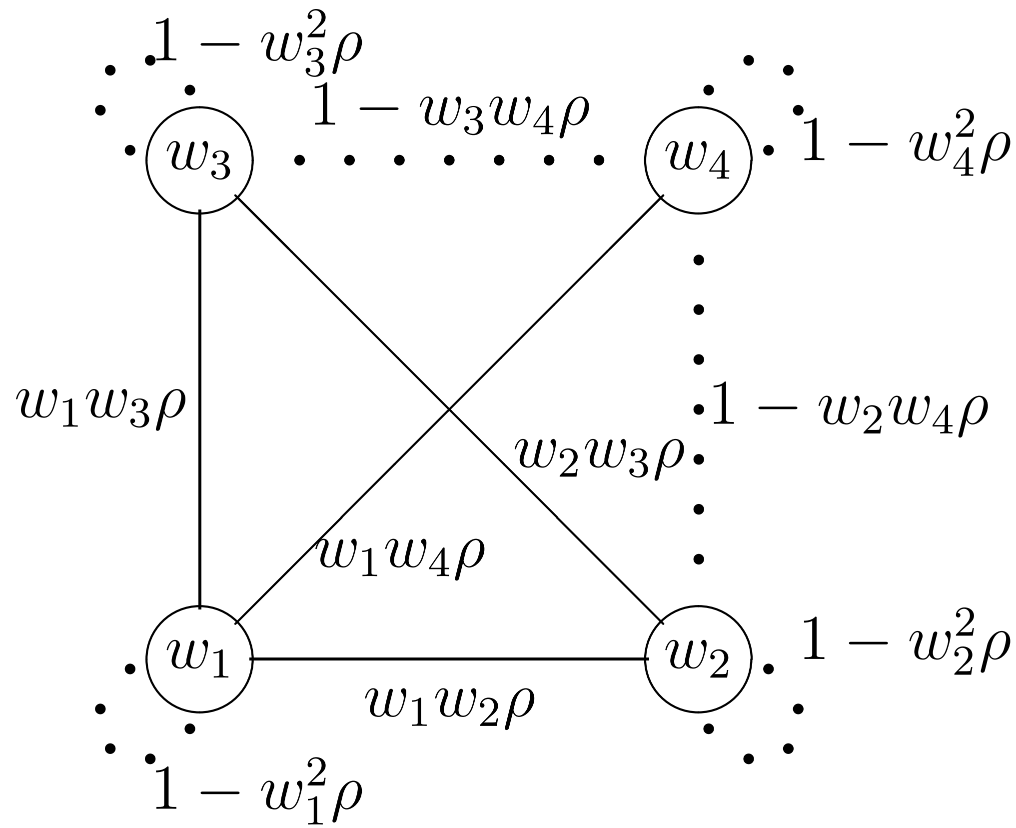
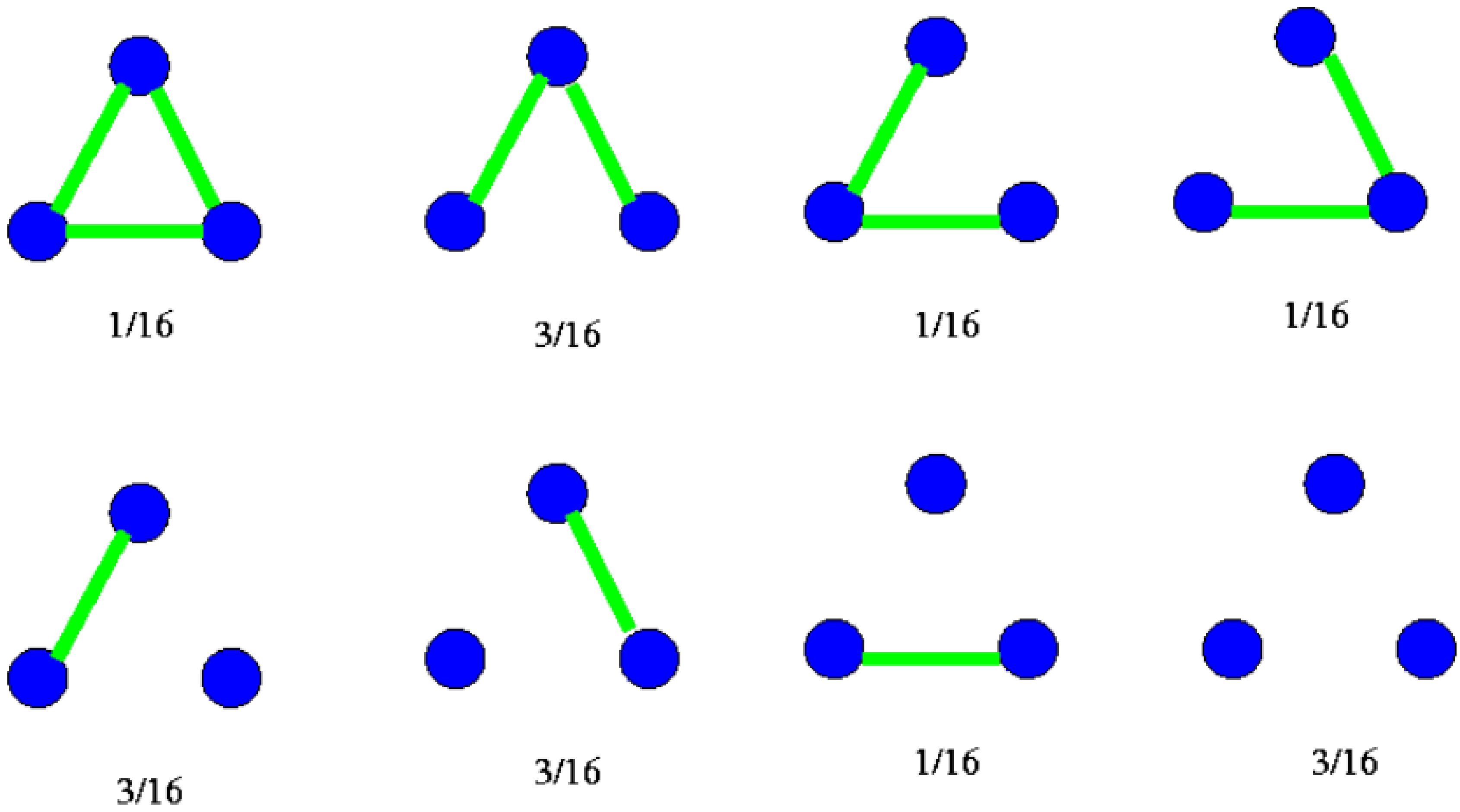


The probability of the graph is

$$w_1^3 w_2^2 w_3^2 w_4 \rho^4 (1 - w_2 w_4 \rho) \times (1 - w_3 w_4 \rho) \prod_{i=1}^{4} (1 - w_i^2 \rho).$$

A example: $G(1,2,1)$



Loops are omitted here.

Notations

For $G = G(w_1, \ldots, w_n)$, let

- $d = \frac{1}{n}\sum_{i=1}^n w_i$
- $\tilde{d} = \frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n w_i}$.
- The volume of S: $\mathrm{Vol}(S) = \sum_{i\in S} w_i$.
- The k-th volume of S: $\mathrm{Vol}_k(S) = \sum_{i\in S} w_i^k$.

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We have

$$\tilde{d} \geq d$$

"$=$" holds if and only if $w_1 = \cdots = w_n$.

Results

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For $G(n,p)$, $\tilde{d} = d = np$. These results are consistent to results for $G(n,p)$.

Admissible condition

(i) $\log \tilde{d} \ll \log n$.

(ii) $d > 1 + \epsilon$. $w_i > \epsilon$ for all but $o(n)$ vertices.

(iii) $\exists$ a subset U:

$$\mathrm{vol}_2(U) = (1 + o(1))\mathrm{vol}_2(G) \gg \mathrm{vol}_3(U) \frac{\log \tilde{d} \log\log n}{\tilde{d} \log n}.$$

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Example: Power law graphs with $\beta > 3$ and $G(n, p)$.

Strongly admissible condition

(i') $\log \tilde{d} = O(\log d)$.

(ii) $d > 1 + \epsilon$. $w_i > \epsilon$ for all but $o(n)$ vertices.

(iii') $\exists$ a subset U: $\mathrm{Vol}_3(U) = O(\mathrm{Vol}_2(G))\frac{\tilde{d}}{\log \tilde{d}}$, and $\mathrm{Vol}_2(U) > d\mathrm{Vol}_2(G)/\tilde{d}$.

Example: Power law graphs with $\beta > 3$ and $G(n, p)$.

Lower bound

- Random graph $G(w_1, \ldots, w_n)$
- u, v: two vertices

With probability at least $1 - \frac{w_u w_v}{\tilde{d}(\tilde{d}-1)} e^{-c}$,

$$d(u,v) \geq \lfloor \frac{\log \mathrm{vol}(G) - c}{\log \tilde{d}} \rfloor.$$

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It implies the average distance is at least

$$(1 - o(1)) \frac{\log n}{\log \tilde{d}}.$$

Proof of lower bound

- P_j: the set of all possible pathes from u to v with length j in K_n.
- For any $\pi = uv_{i_1} \ldots v_{i_{j-1}} v \in P_j$, the probability that π is not a path of G is exactly

$$1 - w_u w_v w_{i_1}^2 \cdots w_{i_{j-1}}^2 \rho^j.$$

- For any $\pi \in P_j$, "π is not a path of G" is a monotone decreasing graph property. FKG inequality applies. (You can treat them as independent events).

Proof of lower bound

$$Pr(d(u,v) \geq k) \;\geq\; \prod_{j=1}^{k-1} \prod_{i_1 \ldots i_{j-1}} (1 - w_u w_v w_{i_1}^2 \cdots w_{i_{j-1}}^2 \rho^j)$$

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&\approx e^{-w_u w_v \sum_{j=1}^{k-1} \rho^j (\sum_{i=1}^n w_i^2)^{j-1}}
\end{aligned}
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&\approx e^{-w_u w_v \sum_{j=1}^{k-1} \rho^j (\sum_{i=1}^n w_i^2)^{j-1}} \\
&\approx e^{-w_u w_v \rho((\sum_i w_i^2 \rho)^k - 1)/(\sum_i w_i^2 \rho - 1)}
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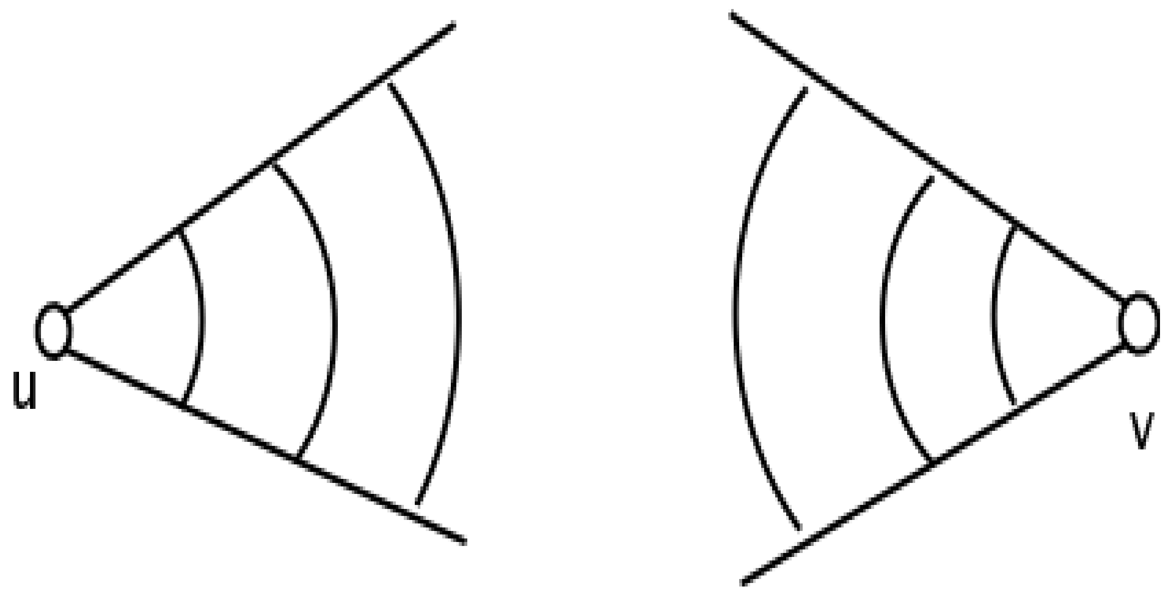
\begin{aligned}
Pr(d(u,v) \geq k) &\geq \prod_{j=1}^{k-1} \prod_{i_1 \ldots i_{j-1}} (1 - w_u w_v w_{i_1}^2 \cdots w_{i_{j-1}}^2 \rho^j) \\
&\approx \prod_{j=1}^{k-1} e^{-w_u w_v \rho^j \sum_{w_1,\ldots,w_{j-1}} w_1^2 \cdots w_{j-1}^2} \\
&\approx e^{-w_u w_v \sum_{j=1}^{k-1} \rho^j (\sum_{i=1}^{n} w_i^2)^{j-1}} \\
&\approx e^{-w_u w_v \rho((\sum_i w_i^2 \rho)^k - 1)/(\sum_i w_i^2 \rho - 1)} \\
&\geq 1 - \frac{w_u w_v}{\tilde{d}(\tilde{d}-1)} e^{-c}
\end{aligned}
$$

Here we choose $k = \lfloor \frac{\log \mathrm{vol}(G) - c}{\log \tilde{d}} \rfloor$. □

Upper bound

To construct a path from u to v, expand u and v's neighborhoods simultaneously.



The neighborhood of S:

$$\Gamma(S) = \{v \ : \ v \sim u \in S \text{ and } v \notin S\}.$$

Neighborhood expansion

Lemma 1: *In a random graph* $G(w_1, \ldots, w_n)$, *for any two subsets* S *and* T *of vertices, we have*

$$\mathrm{vol}(\Gamma(S) \cap T) \geq (1 - 2\epsilon)\mathrm{vol}(S)\frac{\mathrm{vol}_2(T)}{\mathrm{vol}(G)}$$

with probability at least $1 - e^{-c}$, *provided* $\mathrm{vol}(S)$ *satisfies*

$$\frac{2c\mathrm{vol}_3(T)\mathrm{vol}(G)}{\epsilon^2\mathrm{vol}_2^2(T)} \leq \mathrm{vol}(S) \leq \frac{\epsilon\mathrm{vol}_2(T)\mathrm{vol}(G)}{\mathrm{vol}_3(T)}$$

Early neighborhood expansion

Lemma 2: *Suppose that G is admissible. For any fixed vertex v in the giant component, if $\tau = o(\sqrt{n})$, then there is an index $i_0 \leq c_0\tau$ so that with probability at least $1 - \frac{c_1\tau^{3/2}}{e^{c_2\tau}}$, we have*

$$\mathrm{vol}(\Gamma_{i_0}(v)) \geq \tau$$

where c_i's are constants depending only on c and d.
Proof will be omitted.

Time to stop neighborhood expansion

Lemma 3: *For any two disjoint subsets* S *and* T *with* $\mathrm{vol}(S)\mathrm{vol}(T) > c\mathrm{vol}(G)$, *we have*

$$Pr(d(S,T) > 1) < e^{-c}$$

where $d(S,T)$ *denotes the distance between* S *and* T.

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Proof:

$$\begin{aligned} Pr(d(S,T) > 1) &= \prod_{v_i \in S, v_j \in T} (1 - w_i w_j \rho) \\ &\leq e^{-\mathrm{vol}(S)\mathrm{vol}(T)\rho} \\ &< e^{-c}. \end{aligned}$$

Sketched proof of the theorem

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- By lemma 2, there is a $i_0 \leq C\epsilon \frac{\log n}{\log \tilde{d}}$ satisfying almost surely

$$\mathrm{vol}(\Gamma_{i_0}(v)) \geq \epsilon \frac{\log n}{\log \tilde{d}}.$$

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- By lemma 1, almost surely $\mathrm{vol}(\Gamma_i(u))$ grows roughly by a factor of $(1-2\epsilon)\tilde{d}$.

Proof continues

- Therefore, almost surely, for some $i = (\frac{1}{2} + o(1))\frac{\log n}{\log \tilde{d}}$,

$$\mathrm{vol}(\Gamma_i(u)) \geq \sqrt{\mathrm{vol}(G)\log n}.$$

Proof continues

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- Similarly, with probability $1 - o(1)$, for some $j = (\frac{1}{2} + o(1))\frac{\log n}{\log \tilde{d}}$,

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- Almost surely $\Gamma_i(u)$ and $\Gamma_j(v)$ are connected. Thus

$$d(u,v) \leq i + j + 1 = (1 + o(1))\frac{\log n}{\log \tilde{d}}. \qquad \square$$

A large deviation inequality

Lemma 4: *Let* $X_1, \ldots, X_n$ *be independent random variables with*

$$Pr(X_i = 1) = p_i, \qquad Pr(X_i = 0) = 1 - p_i$$

For $X = \sum_{i=1}^n a_i X_i$, *we have* $E(X) = \sum_{i=1}^n a_i p_i$ *and we define* $\nu = \sum_{i=1}^n a_i^2 p_i$. *Then we have*

$$Pr(X < E(X) - \lambda) \;\leq\; e^{-\lambda^2/2\nu}$$

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With probability $1 - e^{-c}$*,*

$$X > E(X) - \sqrt{2c\nu}.$$

Proof of Lemma 1

X_j: the indicated random variable for $v_j \in T \cap \Gamma(S)$.

$$\begin{aligned} Pr(X_j = 1) &= 1 - \prod_{v_i \in S} (1 - w_i w_j \rho) \\ &\geq \mathrm{vol}(S) w_j \rho - \mathrm{vol}(S)^2 w_j^2 \rho^2. \end{aligned}$$

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Since $\mathrm{vol}(\Gamma(S) \cap T) = \sum_{v_j \in T} w_j X_j$, the expected value of $\mathrm{vol}(\Gamma(S) \cap T)$ is at least $\mathrm{vol}(S)\mathrm{vol}_2(T)\rho - \mathrm{vol}(S)^2 \mathrm{vol}_3(T)\rho^2$.

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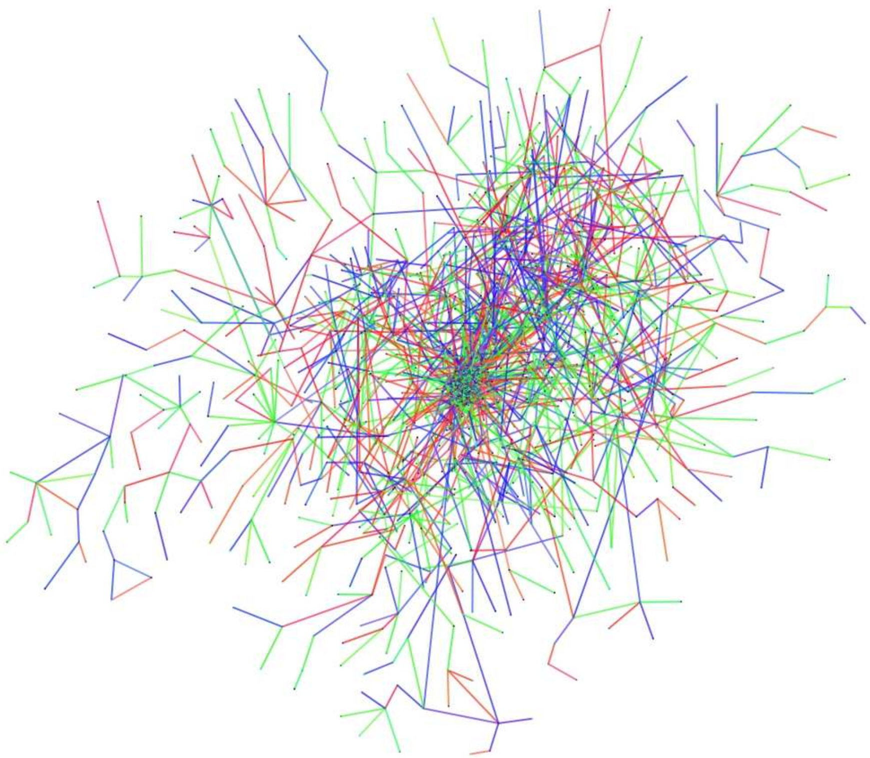
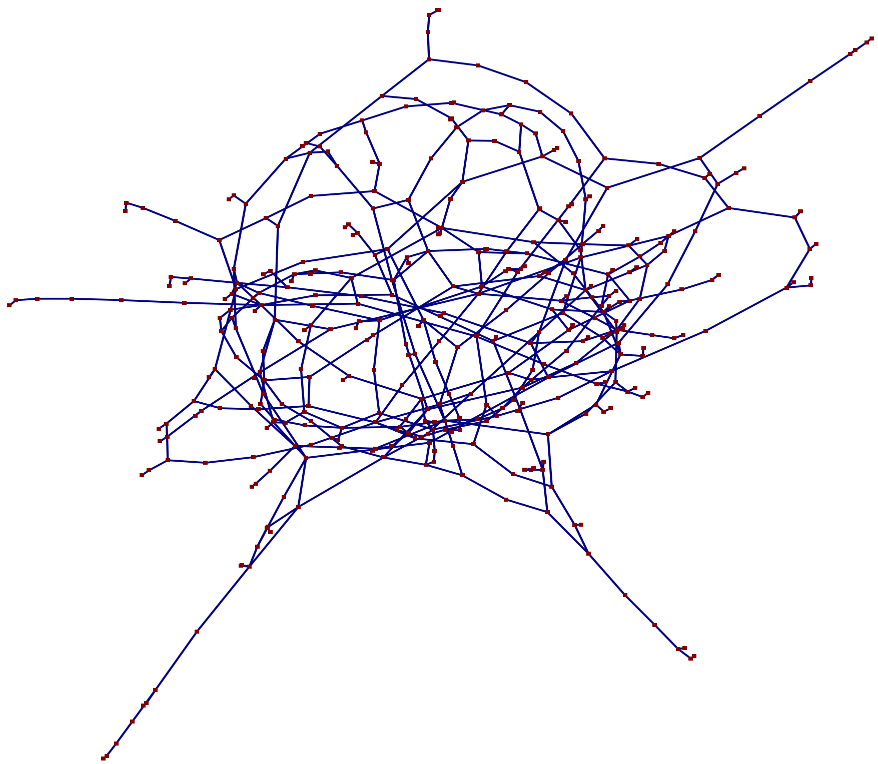
by the assumption. □

Non-admissible graph versus admissible graph



A random subgraph of the Collaboration Graph.



A Connected component of $G(n, p)$ with $n = 500$ and $p = 0.002$.

A random power law graph

For $\beta > 2$, $d > 1$, and $m >> d$, a random power law graph with the exponent β, the average degree d, and the maximum degree m is defined as $G(w_{i_0}, \ldots, w_{n+i_0-1})$ where

- $c = \frac{\beta-2}{\beta-1} d n^{\frac{1}{\beta-1}}$
- $i_0 = n(\frac{d(\beta-2)}{m(\beta-1)})^{\beta-1}$
- $w_i = c i^{-\frac{1}{\beta-1}}$, for $i_0 \leq i < n + i_0$.

Power law graphs with β in $(2, 3)$

Chung, Lu (2002)

- Examples: the WWW graph, Collaboration graph, etc.

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The diameter is $\Theta(\log n)$, while the average distance is $O(\log \log n)$.

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Small distance Between any pair of nodes, there is a short path.

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$$\text{A hybrid model} = \text{a local graph} + \text{a random power law graph}$$

Local connectivity

For two fixed integers $k \geq 2$ and $l \geq 2$, a graph L is said to be "locally (k, l)-connected" if for any edge uv, there are at least k edge-disjoint paths with length at most l joining u to v (including the edge uv).

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By this definition, the union of two locally (k, l)-connected graphs is locally (k, l)-connected.

The maximum locally (k, l)-connected subgraph H is the union of all locally (k, l)-connected subgraphs of G.

Algorithm(k, l):

For each edge $e = uv$, check whether there are k edge-disjoint paths with length at most l connecting u and v in the current graph G. If not, delete the edge e from G. Then iterate the procedure until no edge can be removed.

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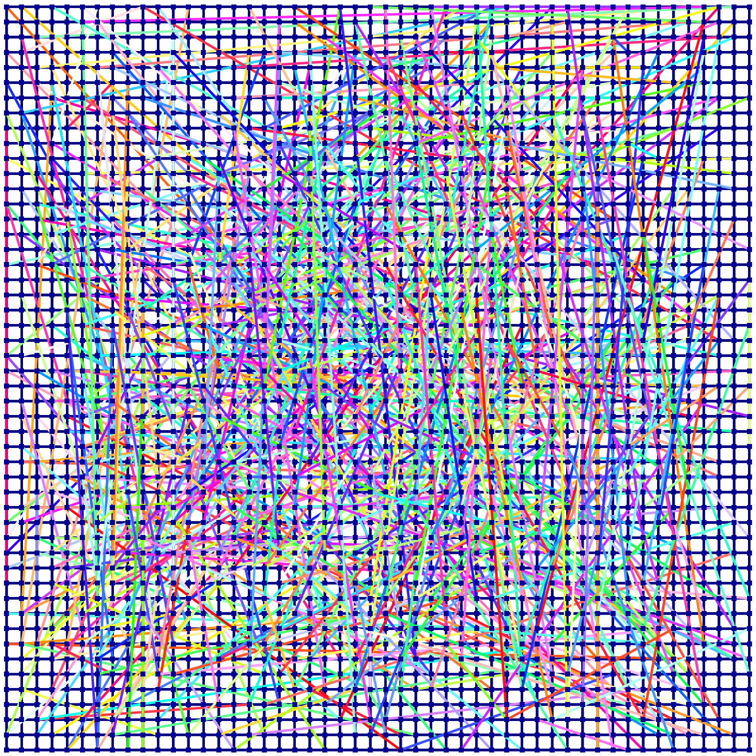
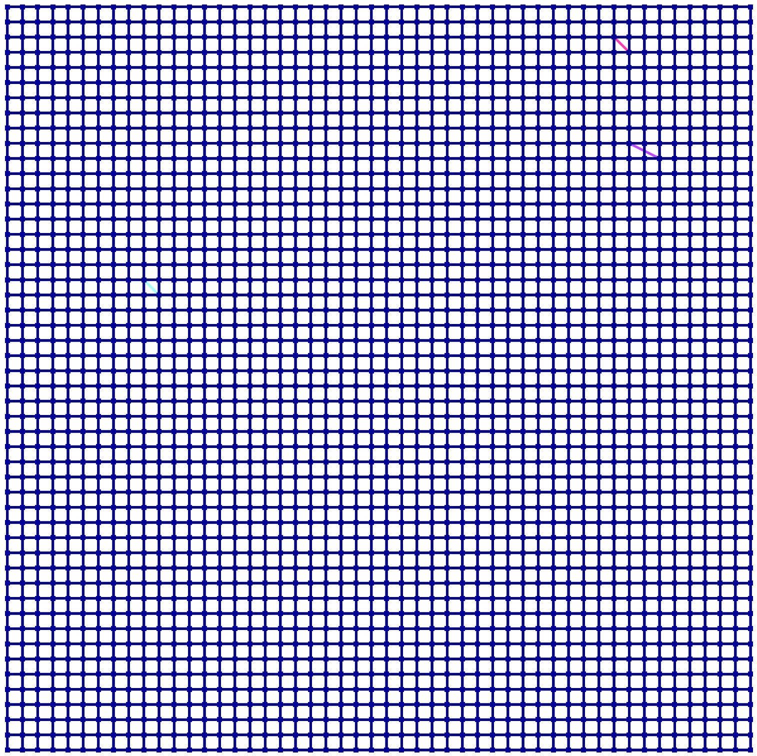
Theorem: *For any graph G,* Algorithm(k, l) *finds the unique maximum locally (k, l)-connected subgraph regardless of the order of edges chosen.*

Recovering the local graph

A hybrid graph, which contains the grid graph $C_{50} \Box C_{50}$ as the local graph, and 528 additional random edges.





The local graph is almost perfect recoverd after applying the algorithm with $k = l = 3$.

Hybrid graph model $H(n, \beta, d, m, L)$

- n: the number of vertices.

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The hybrid graph is the union of the local graph L and the random power law graph with parameter n, β, d, and m.

Result 1

Chung Lu *For any fixed constants* M, $k \geq 3$, *and* $l \geq 2$, *suppose* L *is a connected and locally* (k, l)*-connected graph with degrees bounded by* M. *Let* L' *be the maximum locally* (k, l)*-connected subgraph in the hybrid graph* $H(n, \beta, d, m, L)$ *with the maximum degree* m *satisfying* $m = o(n^{\frac{1-1/(2k)}{l+1}})$. *Then the following holds:*

1. $L \subset L'$. *The expected number of edges in* $L' \setminus L$ *is small, i.e.,* $e(L') - e(L) = O(m) = o(n^{\frac{1-1/(2k)}{l+1}})$.

Continue

2. *Almost surely, for all vertices v, the degree of v in L' can increase at most by 1 if $l \geq 3$ (and by 2 if $l = 2$).*

$$d_{L'}(v) \leq \begin{cases} d_L(v) + 2 & \text{if } l = 2; \\ d_L(v) + 1 & \text{if } l \geq 3. \end{cases}$$

Continue

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3. *The diameter $D(L')$ of L' is almost surely $(1 + o(1))D(L)$ if the diameter $D(L)$ is sufficiently large.*

Diameter and average distance

Chung Lu (2004) For a hybrid graph $H(n, \beta, d, m, L)$, almost surely, we have

Case $\beta > 3$, the average distance is $(1 + o(1))\frac{\log n}{\log \tilde{d}}$ and the diameter is $O(\log n)$.

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Case $\beta = 3$**,** the average distance is $O(\log n / \log \log n)$ and the diameter is $O(\log n)$.

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Overview of talks

- Lecture 1: Overview and outlines
- Lecture 2: Generative models - preferential attachment schemes
- Lecture 3: Duplication models for biological networks
- Lecture 4: The rise of the giant component
- Lecture 5: The small world phenomenon: average distance and diameter
- Lecture 6: Spectrum of random graphs with given degrees