Monochromatic 4-term arithmetic progressions in 2-colorings of $\mathbb{Z}_n$

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Three modules

We will consider monochromatic k-term Arithmetic Progressions ($k = 3, 4, 5$) in the following:

$$\boxed{[n]} \qquad \boxed{\mathbb{Z}_p}$$

$$\boxed{\mathbb{Z}_n}$$

Monochromatic progressions

We consider the monochromatic 3-term arithmetic progression (3-APs) in a 2-coloring of first 9 integers:

$$\color{red}{1} \quad \color{red}{2} \quad \color{blue}{3} \quad \color{blue}{4} \quad \color{red}{5} \quad \color{blue}{6} \quad \color{blue}{7} \quad \color{red}{8} \quad \color{blue}{9}$$

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Here is an alternative representation using 0-1 string.

0 0 1 1 0 1 1 0 1

Van der Waerden's theorem

Theorem [Van der Waerden, 1927] For any given positive integers r and k, there is some number N such that if the integers in $[N] = \{1, 2, ..., N\}$ are colored, each with one of r different colors, then there are at least k integers in arithmetic progression all of the same color.

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The least such N is the *Van der Waerden number* $W(r, k)$.

$$W(2, 3) = 9.$$

Van der Waerden numbers

Some values of $W(r,k)$

r\ k	3	4	5	6	7
2	9	35	178	1,132	$> 3,703$
3	27	> 292	$> 2,173$	$> 11,191$	$> 48,811$
4	76	$> 1,048$	$> 17,705$	$> 91,331$	$> 420,217$
5	> 170	$> 2,254$	$> 98,740$	$> 540,025$	
6	> 223	$> 9,778$	$> 98,748$	$> 816,981$	

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Gowers [2001]

$$W(r,k) \leq 2^{2^{r^{2^{2^{k+9}}}}}$$

Monochromatic k-APs

For any fixed r and k, let $c_{r,k}$ be the greatest number such that any r-coloring of $[n]$ (for n sufficiently large) contains at least

$$(c_{r,k} + o(1))n^2$$

monochromatic k-APs.

Question: What can we say about $c_{r,k}$?

An upper bound on $c_{r,k}$

The set $[n]$ has $(\frac{1}{2(k-1)} + o(1))n^2$ k-APs. If we colors $[n]$ randomly using r colors, then each k-AP being monochromatic has the probability $\frac{1}{r^{k-1}}$.

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We have

$$c_{r,k} \leq \frac{1}{2(k-1)r^{k-1}} + o(1).$$

A lower bound on $c_{r,k}$

Let $K := W(r, k)$.

- There are about $\frac{1}{2(K-1)}n^2$ of K-APs in $[n]$.

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Putting together, we have

$$c_{r,k} \geq \frac{1}{2W(r,k)^3} > 0.$$

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In this talk, we we only consider the case using 2 colors. Write c_k for $c_{2,k}$.

History

Parillo, Robertson, and Saracino [2007] proved

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Butler, Constello, and Graham [2010] proved

$$\begin{aligned} c_4 &\le 0.0172202\cdots, \\ c_5 &\le 0.005719619\cdots. \end{aligned}$$

Our results on $[n]$

Theorem 1 [Lu-Peng 2011]:

$$\begin{aligned} c_4 &\leq \frac{1}{72} = 0.01388888\cdots, \\ c_5 &\leq \frac{1}{304} = 0.003289474\cdots. \end{aligned}$$

Monochromatic APs on $\mathbb{Z}_n$

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- Here we allow *degenerated* k-APs, i.e., $d = \frac{n}{i}$ for some $i \in \{1, \ldots, k-1\}$.
- The total number of k-APs in $\mathbb{Z}_n$ is n^2.

Definition of m_k

Let $m_k(n)$ be the largest number such that any 2-coloring of $\mathbb{Z}_n$ has at least $m_k(n)n^2$ monochromatic k-APs.

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We have

$$m_k(n) \leq \frac{1}{2^{k-1}} + o_n(1).$$

3-**APs in** $\mathbb{Z}_n$

Fact: For n sufficiently large, we have

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I.e., random colorings have asymptotically fewer number of APs.

The value $m_4(\mathbb{Z}_p)$ for prime p

Wolf [2010] proved for sufficient large prime p

$$\frac{1}{16} + o(1) \leq m_4(\mathbb{Z}_p) \leq \frac{1}{8}(1 - \frac{1}{259200}) + o(1).$$

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This result shows that random colorings are not optimal. It uses quadratic Fourier analysis and probabilistic method.

Our result on $\mathbb{Z}_p$

Theorem 2 [Lu-Peng 2011]: For sufficient large prime p, we have

$$\frac{7}{96} + o(1) \leq m_4(\mathbb{Z}_p) \leq \frac{17}{150} + o(1).$$

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- The gap $\frac{1}{8} - m_4(\mathbb{Z}_p)$ is increased from Wolf's bound 0.000000482 to 0.0116666.
- Our upper bound uses an explicit construction.

Related results

Theorem [Cameron-Cilleruelo-Serra (2003)] Any 2-coloring of a finite abelian graph with order n relatively prime to 6 has at least

$$(\frac{2}{33} + o(1))n^2$$

monochromatic APs.

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It implies for if $gcd(n,6)=1$ then

$$m_4(\mathbb{Z}_n)\geq \frac{2}{33}+o(1).$$

Lower bound on $m_4(\mathbb{Z}_n)$

Theorem 2 [Lu-Peng 2011]: If n is not divisible by 4 and large enough, then we have

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- If n is odd, then
  $m_4(\mathbb{Z}_n) \leq \frac{17}{150} + o(1) < 0.1133334 + o(1).$

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For sufficiently large n, there is a 2-coloring of $\mathbb{Z}_n$ with substantially fewer monochromatic APs than random 2-colorings have.

Block construction method

Let $n = bt$ and B be a "good" 2-coloring/bit-string in $\mathbb{Z}_b$ with x monochromatic 4-APs. Consider a 2-coloring of $\mathbb{Z}_b$ defined as follows.

$$\underbrace{BB\cdots B}_{t}$$

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In particular, if $b \mid n$, we have

$$m_4(\mathbb{Z}_n) \le m_4(\mathbb{Z}_b).$$

Block construction with extra bits

Let $n = bt + r$ $(0 < r < b)$ and B be a "good" 2-coloring/bit-string in $\mathbb{Z}_b$. How many monochromatic 4-APs in the following construction?

$$\underbrace{BB\cdots B}_{t}R$$

Here R is any bit string of length r.

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$$\underbrace{BB\cdots B}_{t}R$$

Here R is any bit string of length r.

The number of all 4-APs which pass through some bit(s) in R is $O(n)$. The major term in the number of all monochromatic 4-APs only depends on B and r.

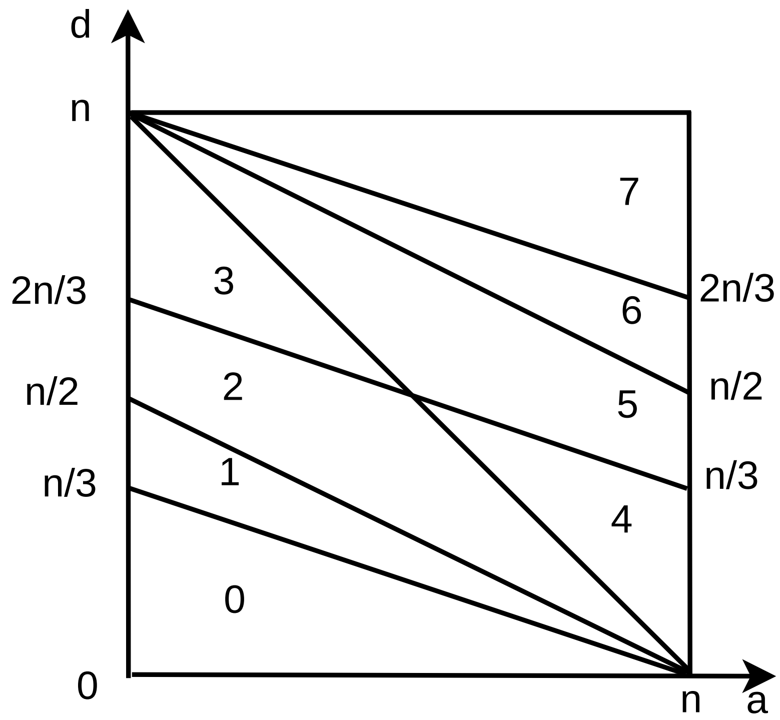
Classes of 4-APs

We divide the set of all 4-APs in $\mathbb{Z}_n$ into 8 classes.

type	meaning
0	$a < a+d < a+2d < a+3d < n$
1	$a < a+d < a+2d < n < a+3d < 2n$
2	$a < a+d < n < a+2d < a+3d < 2n$
3	$a < a+d < n < a+2d < 2n < a+3d < 3n$
4	$a < n < a+d < a+2d < a+3d < 2n$
5	$a < n < a+d < a+2d < 2n < a+3d < 3n$
6	$a < n < a+d < 2n < a+2d < a+3d < 3n$
7	$a < n < a+d < 2n < a+2d < 3n < a+3d < 4n$

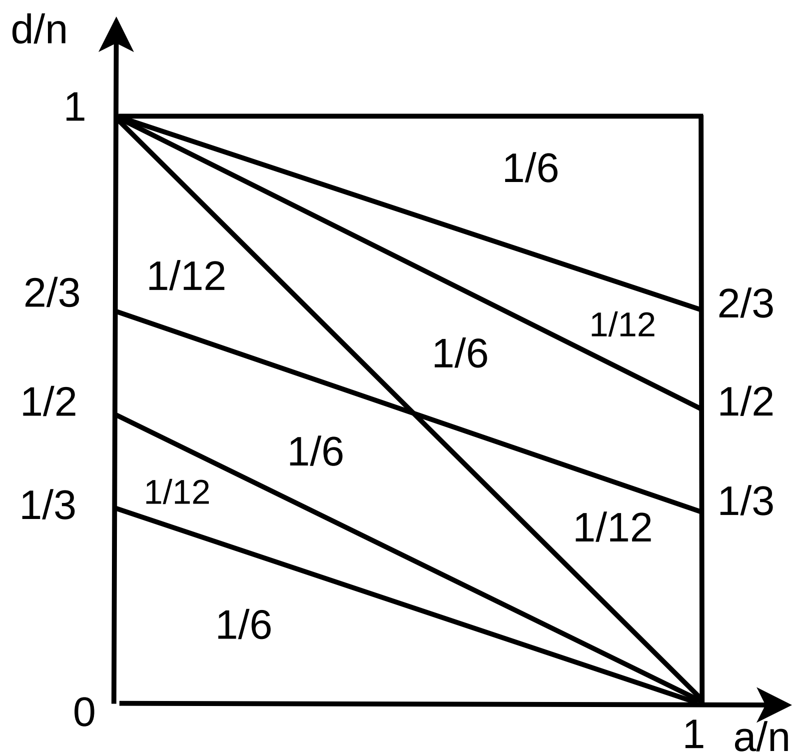
A graphical view of 8 classes

Every 4-AP $a, a+d, \ldots, a+(k-1)d$ is determined by a pair (a, d). The 8 classes can be viewed as 8 regions shown below.



The number of 4-APs

The number of 4-APs in each class is proportional to the area a_i of the corresponding i-th region as shown below.



A lemma

For $0 \leq i \leq 7$, write i as a bit-string $x_1x_2x_3$. Let c_i be the number of sequences in B of form

$$a, a+d+x_1r, a+2d+x_2r, a+3d+x_3r.$$

Then the number of monochromatic 4-APs in $BB\cdots BR$ is

$$\sum_{i=0}^{7} a_i c_i t^2 + O(t).$$

In particular, we have

$$m_4(\mathbb{Z}_n) \leq \sum_{i=0}^{7} a_i \frac{c_i}{b^2}.$$

A good 2-coloring in $\mathbb{Z}_{20}$

In $\mathbb{Z}_{20}$, consider the 2-coloring given by

$$1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0.$$

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For $r = 1$, we have

type	0	1	2	3	4	5	6	7
d_i	36	50	50	50	50	50	50	36

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For $r = 1$, we have

type	0	1	2	3	4	5	6	7
d_i	36	50	50	50	50	50	50	36

This implies

$$m_4(\mathbb{Z}_{20k+1}) \leq \frac{17}{150} + o(1).$$

In fact, the same bound works for all odd r .

A special 2-coloring of $\mathbb{Z}_{11}$

In $\mathbb{Z}_{11}$, consider the 2-coloring given by

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Property: B_{11} contains no non-degenerate monochromatic 4-APs of $\mathbb{Z}_{11}$.

A recursive construction

Given a 2-coloring B_t of $\mathbb{Z}_t$, define $B_{11} \ltimes B_t$ to be the following 2-coloring of $\mathbb{Z}_{11t}$

$$\underbrace{B_{11} B_{11} \cdots B_{11}}_{t},$$

where t $*$'s are replaced by B_t.

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Property:

$$m_4(\mathbb{Z}_{11t}) \leq \frac{10 + m_4(\mathbb{Z}_t)}{121}.$$

The limit

$$m_4(\mathbb{Z}_{11^s}) \leq \frac{1}{12} + \frac{1}{12 \times 11^{2s-1}}.$$

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$$\underline{\lim}_{n\to\infty} m_4(\mathbb{Z}_n) \leq \frac{1}{12}.$$

Corollary:

$$c_4 \leq \frac{1}{6} \underline{\lim}_{n\to\infty} m_4(\mathbb{Z}_n) \leq \frac{1}{72}.$$

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We conjecture that the equality holds.

k-APs (for $k > 4$)

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Corollary:

$$c_5 \leq \frac{1}{8} \varliminf_{n \to \infty} m_5(\mathbb{Z}_n) \leq \frac{1}{304}.$$

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Bulter [2011+]: found a coloring in $\mathbb{Z}_{47}$ for 6-AP and a coloring in $\mathbb{Z}_{77}$ for 7-AP. Thus,

$$c_6 \leq \frac{1}{480} \text{ and } c_6 \leq \frac{71}{71706}.$$

Lower bounds on $m_4(\mathbb{Z}_n)$

Main idea for Lower bounds:

- If n is not divisible by 4, then we are able to extend Wolf's proof to $\mathbb{Z}_n$. At the same time, we capture the patterns which are thrown away in Wolf's paper. Finally, we use heuristic search to show those patterns should appear with positive density.

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- If n is divisible by 4, then Wolf's proof can not be extended to $\mathbb{Z}_n$. However, we are able to extend Cameron-Cilleruelo-Serra's argument to get a lower bound.

Questions

- **Conjecture:** For $k \geq 4$,

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- Is $c_4 = \frac{1}{72}$?
- Is $c_5 = \frac{1}{304}$?
- Determine $m_4(\mathbb{Z}_p)$ for large p. Currently, we have

$$0.072916667\cdots \leq m_4(\mathbb{Z}_p) \leq 0.113333333\cdots.$$

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- **Conjecture:** For $k \geq 4$,

$$c_k = \frac{1}{2(k-1)} \varliminf_{n\to\infty} m_k(\mathbb{Z}_n).$$

- Is $c_4 = \frac{1}{72}$?
- Is $c_5 = \frac{1}{304}$?
- Determine $m_4(\mathbb{Z}_p)$ for large p. Currently, we have

$$0.072916667\cdots \leq m_4(\mathbb{Z}_p) \leq 0.113333333\cdots.$$

- Good lower bound for c_4?