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On sheets of orbit covers for classical semisimple Lie groups

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Abstract David Vogan gave programmatic conjectures about the Dixmier's map and he made two conjectures that induction may be independent of the choice of parabolic group used and the sheets of orbit data are conjugated or disjointed[1]. In our previous paper, we gave a geometric version of the parabolic induction of the geometric orbit datum (i.e. orbit covers), and proved Vogan's first conjecture for geometric orbit datum: the parabolic induction of the geometric orbit datum is independent of the choice of parabolic group. In this paper, we will prove the other Vogan's conjecture, that is, the sheets are conjugated or disjointed for classical semisimple complex groups.

Keywords: Lie group, representation of Lie group, Dixmier's map, geometric orbit data.

Vogan[1] gave the definition of induction of orbit datum. He made two conjectures : the induction may be independent of the choice of parabolic group used and the sheets of orbit data are conjugated or disjointed. Since his definition is algebraic, and is difficult to calculate, it is not easy to verify them via his algebraic definition. But restricted to geometric orbit datum, we gave a geometric version of Vogan's definitions and proved the first conjecture in ref. [2]. In this paper we will prove the other Vogan's conjecture: the sheets are conjugated or disjointed for classical semisimple complex groups. By the Jordan decomposition of orbit datum[1], we just need to prove:

Theorem 1. For all classical semisimple Lie groups, the sets of nilpotent orbit covers induced by different rigid nilpotent orbit covers are disjointed.

1 Induced nilpotent orbit cover

With the same notations as in ref. [2], here we assume u, v are both nilpotent. By ref. [3], we have $G^v/G^v_0 \cong G^\phi/G^\phi_0$, where ϕ is a standard triple of v in $\mathfrak{g}$ by Jacobson-Morozov theorem. Of course we have $L^u/L^u_0 \cong L^{\phi'}/L^{\phi'}_0$, where ϕ' is a standard triple of u in $\mathfrak{l}$[3]. The following lemma shows we can calculate the induction of nilpotent orbit cover in a better way.

Lemma 1. With notations as above, we can find suitable ϕ, ϕ' such that

1. $P^\phi = L^\phi$;

2. $L^\phi \subset L^{\phi'}$ and meets all the components of $L^{\phi'}$, so induces a surjective homomorphism which coincides with Θ_v given by

$$\Theta_v\colon L^\phi/P^\phi_0 \longrightarrow L^{\phi'}/L^{\phi'}_0.$$

Proof. Obviously. Q.E.D.

Remark 1. We also call the pair (ϕ, Π) the representation of $\widetilde{\mathcal{O}}$[2]. In the following sections, we always choose proper ϕ, ϕ' satisfying the above lemma.

Unlike orbits, which are only dependent on $\mathfrak{g}$, orbit covers are dependent on the group G. The following lemma points out the relation between orbit covers of G and that of its cover group

$\widetilde{G}$.

Lemma 2. Let $\widetilde{G}$ and G be two connected Lie groups and $\pi: \widetilde{G} \to G$ the cover map (which is also a group homomorphism). Then it induces a nature injective map:

$$\pi^*: \{\text{orbit covers of } G\} \to \{\text{orbit covers of } \widetilde{G}\}$$

defined by $\pi^*(v, \Pi) \stackrel{\sim}{=} (v, \pi^*(\Pi))$, where $\pi^*(\Pi) = \pi^{-1}(N)/\widetilde{G}_0^v$. Here N is the open subgroup of G_0^v defined Π (i.e. $\Pi = N/G_0^v$).

Proof. Obviously. Q.E.D.

Since given a Levi subgroup L of G, $\widetilde{L} = \pi^{-1}(L)$ is the Levi subgroup of $\widetilde{G}$. By Lemma 2, there is an injective map (here still denoted by π^*) from orbit covers of L to that of $\widetilde{L}$. The following lemma gives the relation between induction for G and induction for $\widetilde{G}$:

Lemma 3. With notations as above, π^* is commutative to Ind, i.e. given an orbit cover $\widetilde{\mathcal{O}}_L$ of L, we have

$$\mathrm{Ind}_{\widetilde{L}}^{\widetilde{G}}(\pi^*(\widetilde{\mathcal{O}}_L)) = \pi^*(\mathrm{Ind}_L^G(\widetilde{\mathcal{O}}_L)).$$

Proof. Obviously. Q.E.D.

Corollary 1. Every cover of orbit (of $\mathfrak{g}$) is an orbit cover of simple-connected group G. If G is not simple-connected, only part of such covers which admit a G-action on it become the orbit cover of G.

2 The induction of orbit covers for classical simple groups

First, we consider the simple groups of type A_l, this is the simplest case. By Lemma 3, we need only consider the simple-connected group $SL(n, \mathbb{C})$.

Let $G = SL(n, \mathbb{C})$ with Lie algebra $\mathfrak{g} = sl(n, \mathbb{C})$. Choose ϕ, ϕ' as Lemma 1 does. Let (ϕ', Π_L) be a representation of $\widetilde{\mathcal{O}}_L$ and (ϕ, Π_G) be a representation of $\widetilde{\mathcal{O}}_G$. Every orbit of G is 1-1 corresponding to a partition of n (an element of $\mathcal{P}$(n)). Given a partition $d = (d_1, \ldots, d_r)$, denote the corresponding orbit by $\mathcal{O}_d$, then $\pi_1(\mathcal{O}_d) = G^\phi/G_0^\phi = Z_{gcd(d_1, \ldots, d_r)}$. Given a Levi subgroup $L = S(GL(n_1, \mathbb{C}) \times \cdots \times GL(n_r, \mathbb{C}))$ with Lie algebra $\mathfrak{l}$, the semisimple part of $\mathfrak{l}$ is $[\mathfrak{l} \quad \mathfrak{l}] = sl(n_1, \mathbb{C}) \times \cdots \times sl(n_r, \mathbb{C})$. Each orbit $\mathcal{O}_L$ of L is given by partitions $(d^{(1)}, \ldots, d^{(r)})$ where $d^{(k)} \in \mathcal{P}(n_k)$ for $k = 1, \ldots, r$.

$$L^{\phi'}/L_0^{\phi'} = Z_{gcd\{d_i^k | \forall k, i\}}.$$

The induction of orbit is given by:

Let $\mathcal{O}_G = \mathrm{Ind}_L^G(\mathcal{O}_L)$. Then the partition d of $\mathcal{O}_G$ is given by : $d_i = \sum_{k=1}^r d_i^{(k)}, \forall i \geqslant 1$. The induction of orbit cover:

In the case, $G^\phi = P^\phi$. So we have a surjective map:

$$\Theta_v: G^\phi/G_0^\phi \to L^{\phi'}/L_0^{\phi'}.$$

Denote $gcd\{d_i^{(k)} \mid 1 \leqslant k \leqslant r, 1 \leqslant i\}$ by n_l and $gcd\{d_i \mid 1 \leqslant i\}$ by n_g, of course $n_l \mid n_g$ and $n_g \mid n$. Let Z_n be the center of G. Notice Z_n meets every component of both G^ϕ and $L^{\phi'}$. So there exists the following commutative diagram:

$$\begin{array}{ccc} Z_n & \stackrel{\text{id}}{\longleftrightarrow} & Z_n \\ \downarrow & & \downarrow \\ Z_{n_g} & \stackrel{\Theta_v}{\longrightarrow} & Z_{n_l} \end{array}$$

Here Z_l is the multiplication group of the primitive l-th root of unity, and all maps are natural projective maps. Then: $\Pi_G = \Theta^{-1}(\Pi_L)$.

Definition 1. An orbit cover is called rigid, if it cannot be induced from any orbit cover of any Levi subgroup.

Proposition 1. The non-conjugated sheets of $SL(n, ,C)$ are disjointed.

Proof. In fact, we can write every sheet of orbit covers in G as follows: Given an orbit cover $\widetilde{\mathcal{O}}_d$ of G, let (v, Π_G) be a representation of $\widetilde{\mathcal{O}}_G$. Let $(f_1, \ldots, f_k)$ be the dual partition of $(d_1, \ldots, d_r)$ (this partition rewritten as $\bar{f} = [\bar{f}_1^{t_1}, \ldots, \bar{f}_m^{t_m}]$, where $\bar{f}_1 > \cdots > \bar{f}_m > 0$ with multiplicity $t_1, \ldots, t_m$), and $n_g = gcd(d_1, \ldots, d_r)$, $\Pi_G = Z_{n'_g}$ and $s = n_g/n'_g$. Since $s \mid t_i, \forall\, 1 \leqslant i \leqslant m$, let

$$L = S(\overbrace{GL(s\bar{f}_1, \mathbb{C}) \times \cdots GL(s\bar{f}_1, \mathbb{C})}^{t_1/s} \times \cdots \times \overbrace{GL(s\bar{f}_m, \mathbb{C}) \times \cdots GL(s\bar{f}_m, \mathbb{C})}^{t_m/s}).$$

Let $\mathcal{O}_L$ correspond to the partition:

$$\overbrace{(s, \ldots, s)}^{f_1} \times \cdots \times \overbrace{(s, \ldots, s)}^{f_m}.$$

Now $(\mathcal{O}_L, \{1\})$ is the rigid orbit cover of L, and

$$\mathrm{Ind}_L^G(\mathcal{O}_L, \{1\}) = \widetilde{\mathcal{O}}_G.$$

Notice this construction is unique. So two non-conjugated sheets are disjointed. Q.E.D.

Now, we assume $\mathfrak{g}$ is the simple complex Lie algebra with Cartan type of B_l, C_l, and D_l. Let G be the corresponding connected algebraic group, i.e. G is one of $Sp(2n, \mathbb{C})$, $SO(n, \mathbb{C})$, $Spin(n, \mathbb{C})$, $PSp(2n, \mathbb{C})$ and $PSO(2n, \mathbb{C})$. We will give the induction of nilpotent orbit covers.

We introduce some result of ref. [3] about the induction of (co)adjoint orbit of classical $\mathfrak{g}$, suppose $\mathfrak{g} \subset sl(N, \mathbb{C})$ naturally. Then

Proposition 2. The (co)adjoint orbit of $\mathfrak{g}$ is naturally one to one corresponding to the following set $\mathcal{P}_{\pm 1}(N)$:

1. For $\mathfrak{g}$ of type B_l, $\mathcal{P}_1(N) =\{$ the partition of N such that the even parts occur even times$\}$.

2. For $\mathfrak{g}$ of type C_l, $\mathcal{P}_{-1}(N) =\{$ the partition of N such that the odd parts occur even times$\}$.

3. For $\mathfrak{g}$ of type D_l, let $\mathcal{P}_1(N)$ be the same as 1, except that the partition d is very even (it has no odd part, and all even parts occur even). Then d is corresponding to two orbits, namely $\mathcal{O}_d^{\mathrm{I}}, \mathcal{O}_d^{\mathrm{II}}$.

For any partition d, realizing the corresponding orbit $\mathcal{O}_d$ as Chapter 5 in ref. [3] does, let ϕ be a proper standard triple ϕ (by Jacobson-Morozov Theorem[3]) containing $v \in \mathcal{O}_d$ and Z_2 be the multiplication group of two elements $1, -1$. For $d \in \mathcal{P}_{-1}(N)$ (type C_l) we pick up the even parts and for $d \in \mathcal{P}_1(N)$ (type B_l and D_l) we pick up the odd parts: $\bar{d}_1 > \cdots > \bar{d}_m > 0$ with multiplicity $s_1, \ldots, s_m$, or we simply denote it by $\bar{d} = [\bar{d}_1^{s_1}, \ldots, \bar{d}_m^{s_m}]$. Let Δ'_d be the image of sign-diagonal map : $Z_2 \longrightarrow (Z_2)^m$ which sends ξ to $(\xi^{s_1}, \ldots, \xi^{s_m})$. Denote $S((Z_2)^m) := \{(\xi_1, \ldots, \xi_m) \in (Z_2)^m \mid \prod_{k=1}^m \xi_k = 1\}$, $S(\Delta'_d) := S((Z_2)^m) \cap \Delta'_d$. Then

Proposition 3.

$$G^{\phi}/G_0^{\phi}=\begin{cases}(Z_2)^m & G=Sp(2n,\mathbb{C}),\\ S((Z_2)^m) & G=SO(N,\mathbb{C}),\\ S((Z_2)^m) \quad d \text{ is not rather odd} & G=Spin(N,\mathbb{C}),\\ \overline{S((Z_2)^m)} \quad d \text{ is rather odd} & G=Spin(N,\mathbb{C}),\\ (Z_2)^m/\Delta'_d & G=PSp(2n,\mathbb{C}),\\ S((Z_2)^m)/S(\Delta'_d) & G=PSO(2n,\mathbb{C}),\end{cases}$$

where $\overline{S((Z_2)^m)}$ is the central extension of $S((Z_2)^m)$ by $Z_2=\{\pm 1\}$, i.e. there is a short exact sequence: $1\longrightarrow Z_2\longrightarrow \overline{S((Z_2)^m)}\longrightarrow S((Z_2)^m)$, where Z_2 is the center of $\overline{S((Z_2)^m)}$. d is called rather odd if its odd parts have multiplicity one.

Because of the induction-by-stages[2], we only need to know how the fundamental induction[3] does. Let $\mathfrak{l}=gl(l,\mathbb{C})\oplus\mathfrak{g}'$, where $\mathfrak{g}'$ is classical and of the same type as that of $\mathfrak{g}$. The standard representation of $\mathfrak{g}'$ (res. $\mathfrak{g}$) is r (res. N), then $2l+r=N$. Let L be the corresponding Levi subgroup. Then

$$L=\begin{cases}GL(l,\mathbb{C})\times Sp(2r',\mathbb{C}) & Sp(2n,\mathbb{C}),\\ GL(l,\mathbb{C})\times SO(r,\mathbb{C}) & SO(N,\mathbb{C}),\\ \overline{GL(l,\mathbb{C})}\otimes_{Z_2} Spin(r,\mathbb{C}) & Spin(N,\mathbb{C}),\\ (GL(l,\mathbb{C})\times Sp(2r',\mathbb{C}))/\pm I & PSp(2n,\mathbb{C}),\\ (GL(l,\mathbb{C})\times SO(r,\mathbb{C}))/\pm I & PSO(2n,\mathbb{C}),\end{cases}\tag{1}$$

where $2r'=r$, $2n=N$. $\overline{GL(l,\mathbb{C})}\otimes_{Z_2} Spin(r,\mathbb{C})$ is double cover of $GL(l,\mathbb{C})\times SO(r,\mathbb{C})$ in $Spin(N,\mathbb{C})$. In type D_l, $r=2$ is not allowed, while $r=0$ there are two non-conjugated L. Replace $GL(l,\mathbb{C})$ by $GL(l,\mathbb{C})_I$ and $GL(l,\mathbb{C})_{II}$ in the above formula.

Now we assume that the orbit $\mathcal{O}_L$ of L has zero factor in $gl(l,\mathbb{C})$, so we can view $\mathcal{O}_L$ as the orbit of $\mathfrak{g}'$, corresponding to the partition $p=[p_1,\dots,p_r]\in\mathcal{P}_{\pm 1}(N)$. We have

Proposition 4. Let G' be the corresponding Lie group corresponding to $\mathfrak{g}'$ with the same type of G, except for $G=Spin(N,\mathbb{C})$, in this case let $G'=SO(r,\mathbb{C})$. Then $L^{\phi'}/L_0^{\phi'}=G'^{\phi}/G'^{\phi}_0$.

Proposition 5[3]**.** With notation above, let X be the Cartan type for $\mathfrak{g}$, so that $X=B,C$ or D. Put $\mathcal{O}=\mathrm{Ind}_L^G(\mathcal{O}_p)$.

1. The partition of $\mathcal{O}$ is the X-collapse q_X. If $\mathfrak{g}=so(4n,\mathbb{C})$ and q_X is very even, then $q_X=q$.

2. If $\mathfrak{g}=so(4n,\mathbb{C})$, and $q_X=q$ is very even, but $r\neq 0$, then the numeral of $\mathcal{O}$ is the same as that of $\mathcal{O}_p$.

3. If $\mathfrak{g}=so(4n,\mathbb{C})$, $q_X=q$ is very even, and $r=0$, then the numeral of $\mathcal{O}$ is the same as that of $\mathfrak{l}$ if n is even but differs from it if n is odd.

Let $d=q_X$, and define $\bar{p}=[\bar{p}_1^{t_1},\dots,\bar{p}_{m'}^{t_{m'}}]$, for the chosen part of p as we define $\bar{d}$ in (Corollary 2.4.1 in ref. [3]).

Definition 2. We call the fundamental induction:

1. of type I, if $q=q_X$ and there exists a unique j, $1\leqslant j\leqslant m$, such that $\bar{d}_j=\bar{p}_j+2$, $\bar{d}_{j+1}=\bar{p}_{j+1}=\bar{p}_j$. In this case, we have $m'=m-1$, and $t_k=\begin{cases}s_k, & k<j,\\ s_j+s_{j+1}, & k=j,\\ s_{k-1}, & k>j;\end{cases}$

2. of type II, if $q=q_X$ and not of type I, we have $m'=m$, $s_k=t_k$, $\forall 1\leqslant k\leqslant m$;

3. of type III, if $q \neq q_X$. We have $m' = m - 1$ and there exists a unique i, $1 \leqslant i \leqslant m$, such that $\bar{d}_i = \bar{p}_i + 1$ with $s_i = 2$ and $s_k = \begin{cases} t_k, & k < i, \\ 2, & k = i, \\ t_{k-1}, & k > i. \end{cases}$ for type I, define $\psi_j : (Z_2)^m \to (Z_2)^{m-1}$ by

$$\psi_j(\xi_1, \ldots, \xi_m) = \begin{cases} (\xi_1, \ldots, \overbrace{\xi_j \xi_{j+1}}^{j}, \ldots, \xi_m), & j < m, \\ (\xi_1, \ldots, \xi_{m-1}), & j = m. \end{cases}$$

For type III, define $\varphi_i : (Z_2)^{m-1} \to (Z_2)^m$ by

$$\varphi_i(\xi_1, \ldots, \xi_{m-1}) = (\xi_1, \ldots, \overset{i}{1}, \ldots, \xi_{m-1}).$$

Also define Δ'_p as Δ'_d. Since

$$\begin{aligned} &\psi_j(\Delta'_d) = \Delta'_p, \\ &\varphi_i(\Delta'_p) = \Delta'_d, \\ &\psi_j((S(Z_2)^m)) = S((Z_2)^{m-1}), \\ &\varphi_i(S((Z_2)^{m-1})) = S(Z_2)^m, \\ &\psi_j(S((Z_2)^m)/S(\Delta'_d)) = S((Z_2)^{m-1})/S(\Delta'_p), \\ &\varphi_i(S((Z_2)^{m-1})/S(\Delta'_p)) = S((Z_2)^m)/S(\Delta'_d), \end{aligned}$$

these induce the following maps (simply we denote them by ψ_j and φ_i):

$$\begin{array}{llcll} \psi_j\colon & (Z_2)^m/\Delta'_d & \longrightarrow & (Z_2)^{m-1}/\Delta'_p & \text{surjective}, \\ \varphi_i\colon & (Z_2)^{m-1}/\Delta'_p & \longrightarrow & (Z_2)^m/\Delta'_d & \text{injective}, \\ \psi_j\colon & S((Z_2)^m) & \longrightarrow & S((Z_2)^{m-1}) & \text{surjective}, \\ \varphi_i\colon & S((Z_2)^{m-1}) & \longrightarrow & S((Z_2)^m) & \text{injective}, \\ \psi_j\colon & S((Z_2)^m)/S(\Delta'_d) & \longrightarrow & S((Z_2)^{m-1})/S(\Delta'_p) & \text{surjective}, \\ \varphi_i\colon & S((Z_2)^{m-1})/S(\Delta'_p) & \longrightarrow & S((Z_2)^m)/S(\Delta'_d) & \text{injective}. \end{array}$$

For $G = Spin(N, \mathbb{C})$ and d is rather odd, type III does not exist. But in this case, it becomes complex by the following lemma:

Lemma 4. Let $G = \overline{GL(n, \mathbb{C})}$ be a two-folder cover of $GL(n, \mathbb{C})$. The rigid orbit G other than the zero orbit cover occurs when $n = 2k$ and $d = 2^k$, the non-zero rigid orbit cover is the 2-folder cover of $\mathcal{O}_d$.

Proof. Obviously. Q.E.D.

Proposition 6. For $G = Spin(N, \mathbb{C})$ and d is rather odd, consider the Levi subgroup $L = \overline{GL(l, \mathbb{C})} \otimes_{Z_2} Spin(r, \mathbb{C})$, if $l = 2k$, and $\mathcal{O}_L$ has $\mathcal{O}_{[2^k]}$ factor in $gl(l, \mathbb{C})$ and $\mathcal{O}_p$ factor in $Spin(r, \mathbb{C})$. Suppose p is also rather odd. Then we have the following two more fundamental inductions:

1. of type IV, there exists a unique j such that $\bar{d}_j = \bar{p}_j + 4$, $\bar{d}_{j+1} = \bar{p}_{j+1}$. In this case, we have $m' = m$, $s_k = t_k$, $\forall 1 \leqslant k \leqslant m$.

2. of type V, we have $m' = m - 2$ and there exists a unique i, such that $\bar{d}_i = p_i + 3$ and $\bar{d}_{i+1} = p_i + 1$ with p_i even.

For type IV, $L^{\phi'}/L_0^{\phi'} \cong G^{\phi}/G_0^{\phi}$.

For type V,

$$\begin{aligned}
L^{\phi'}/L_0^{\phi'} &\cong \overline{Z_2^{m-2}} \\
&\cong \left\{ \begin{array}{c} \text{the product of} \\ \text{even terms of } F_k \end{array} \middle| \begin{array}{ll} F_k^2 = (-1)^{\bar{p}_k(\bar{p}_k+1)/2}, & 1 \leqslant k \leqslant m-2, \\ F_kF_j = -F_jF_k, & k \neq j. \end{array} \right\}, \\
G^{\phi}/G_0^{\phi} &\cong \overline{Z_2^{m}} \\
&\cong \left\{ \begin{array}{c} \text{the product of} \\ \text{even terms of } E_k \end{array} \middle| \begin{array}{ll} E_k^2 = (-1)^{\bar{d}_k(\bar{d}_k+1)/2}, & 1 \leqslant k \leqslant m, \\ E_kE_j = -E_jE_k, & k \neq j. \end{array} \right\},
\end{aligned}$$

(see 14.3 in ref. [4]).

There is a nature injective map $\tau_i : L^{\phi'}/L_0^{\phi'} \longrightarrow G^{\phi}/G_0^{\phi}$ given by:

$$\tau_i(F_k) = \begin{cases} E_k, & k < i, \\ E_{k+2}, & k \geqslant i, \end{cases}$$

which is a group isomorphism. Now for any subgroup Π_L of $L^{\phi'}/L_0^{\phi'}$, we define Π_G as the subgroup generated by $\tau_i(\Pi_L)$ and E_iE_{i+1}. Up to the conjugation of G^{ϕ}/G_0^{ϕ}, Π_G is well-defined (independent of the choice of ± 1 in $\pm E_k$) and have the following properties:

1. $-1 \in \Pi_G$ if and only if $-1 \in \Pi_L$.
2. If $-1 \in \Pi_L$, then $\Pi_G/\{\pm 1\} = \psi_{i+1}^{-1}(\varphi_i(\Pi_i))$.

Now, we can prove the following theorem.

Theorem 2. With notation above, let $\widetilde{\mathcal{O}}_d = \mathrm{Ind}_L^G(\widetilde{\mathcal{O}}_p)$ with corresponding representations (ϕ, Π_G) and (ϕ', Π_L), then we have the following induction table (table 1).

Here $\overline{H}$ is the Z_2-central extension of H, i.e. there is an exact sequence : $1 \to Z_2 \to \overline{H} \to H \to 1$ where $Z_2 = \{\pm 1\}$ is the central subgroup. If $H' \subset H$, $\overline{H'}$ stands for the inverse image of H' in H.

3 Sheets of classical semisimple groups

Definition 3. Given a Levi subgroup L of G and a rigid orbit cover $\widetilde{\mathcal{O}}_L$ of L, let $\mathfrak{l}^L = \{v \in \mathfrak{l} \mid L \cdot v = v\}$. A **sheet** attached to $(L, \widetilde{\mathcal{O}}_L)$ is $\{\mathrm{Ind}_L^G(v_s + \mathcal{O}_L, \Pi_L) \mid v_s \in \mathfrak{l}^L\}$.

By Jordan decomposition[1], we know every orbit cover of G belongs to some sheets. In this section, we will prove Theorem 1. If two sheets attached to $(L_1, \widetilde{\mathcal{O}}_{L_1})$ and $(L_2, \widetilde{\mathcal{O}}_{L_2})$ intersect at $\widetilde{\mathcal{O}}_G$ (with Jordan decomposition: $(L, v_s, \widetilde{\mathcal{O}}_L)$), then $\mathrm{Ind}_{L_1}^G(v_s + \widetilde{\mathcal{O}}_{L_1}) = \mathrm{Ind}_{L_2}^G(v_s + \widetilde{\mathcal{O}}_{L_2}) = \mathrm{Ind}_L^G(v_s + \mathcal{O}_L, \Pi_L)$. Now, $L_1, L_2 \subset G^{v_s} = L$ by ref. [1], we have $\widetilde{\mathcal{O}}_L = \mathrm{Ind}_{L_1}^L(\widetilde{\mathcal{O}}_{L_1}) = \mathrm{Ind}_{L_2}^L(\widetilde{\mathcal{O}}_{L_2})$, then $\mathrm{Ind}_{L_1}^G(\widetilde{\mathcal{O}}_{L_1}) = \mathrm{Ind}_{L_2}^G(\widetilde{\mathcal{O}}_{L_2})$. Hence, if we show that any two different (here mean not conjugated in G) sheets cannot share the same nilpotent orbit cover, then we can prove that the different sheets are disjoint.

First we consider the Levi-subgroups of simple groups. By sec. 2, except for the case $G = Spin(N, \mathbb{C})$ and d is rather odd, if $\widetilde{\mathcal{O}}_L$ is rigid, $\mathcal{O}$ has to have zero on the factor of $\sum gl(\bullet, \mathbb{C})$ in $\mathfrak{l}$. By induction-by-stages[2], we can decompose Ind_L^G into the product of a series of the fundamental induction. What we do in the above section is the last step of the fundamental induction for the simple groups (table 1). Now we will list the middle step of the fundamental induction (table 2) for the simple groups, then show that different sheets have different nilpotent orbit covers.

Let $\mathfrak{g}$ be the Lie algebra of G, then $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}'$ (where $\mathfrak{m} = \sum gl(\bullet, \mathbb{C})$ and $\mathfrak{g}'$ is the simple Lie algebra of type B_l, C_l or D_l). Consider a maximal Levi subalgebra $\mathfrak{l} = \mathfrak{m} + gl(l, \mathbb{C}) + \mathfrak{g}''$ of $\mathfrak{g}$.

Table 1 Induction of orbit cover

G	Type	G^{ϕ}/G_0^{ϕ}	$L^{\phi'}/L_0^{\phi'}$	Π_G
	I	$(Z_2)^m$	$(Z_2)^{m-1}$	$\psi_j^{-1}(\Pi_L)$
$Sp(2n,\mathbb{C})$	II	$(Z_2)^m$	$(Z_2)^m$	Π_L
	III	$(Z_2)^m$	$(Z_2)^{m-1}$	$\varphi_i(\Pi_L)$
$SO(N,\mathbb{C})$	I	$S((Z_2)^m)$	$S((Z_2)^{m-1})$	$\psi_j^{-1}(\Pi_L)$
$Spin(N,\mathbb{C})$ for d	II	$S((Z_2)^m)$	$S((Z_2)^m)$	Π_L
is not rather odd	III	$S((Z_2)^m)$	$S((Z_2)^{m-1})$	$\varphi_i(\Pi_L)$
	I	$\overline{S((Z_2)^m)}$	$S((Z_2)^{m-1})$	$\overline{\psi_j^{-1}(\Pi_L)}$
$Spin(N,\mathbb{C})$ for d	II	$\overline{S((Z_2)^m)}$	$S((Z_2)^m)$	$\overline{\Pi_L}$
	IV	$\overline{S((Z_2)^m)}$	$\overline{S((Z_2)^m)}$	Π_L
is rather odd	V	$\overline{S((Z_2)^m)}$	$\overline{S((Z_2)^{m-2})}$	Π_G
	I	$(Z_2)^m/\Delta'_d$	$(Z_2)^{m-1}/\Delta'_p$	$\psi_j^{-1}(\Pi_L)$
$PSp(2n,\mathbb{C})$	II	$(Z_2)^m/\Delta'_d$	$(Z_2)^m/\Delta'_p$	Π_L
	III	$(Z_2)^m/\Delta'_d$	$(Z_2)^{m-1}/\Delta'_p$	$\varphi_i(\Pi_L)$
	I	$S((Z_2)^m)/S(\Delta'_d)$	$S((Z_2)^{m-1})/S(\Delta'_p)$	$\psi_j^{-1}(\Pi_L)$
$PSO(2n,\mathbb{C})$	II	$S((Z_2)^m)/S(\Delta'_d)$	$S((Z_2)^m)/S(\Delta'_p)$	Π_L
	III	$S((Z_2)^m)/S(\Delta'_d)$	$S((Z_2)^{m-1})/S(\Delta'_p)$	$\varphi_i(\Pi_L)$

Let L be the corresponding Levi subgroup (L has the same form as that of G except additional GL-factor $GL(l,\mathbb{C})$). Suppose that the dimension of the standard representation of $\mathfrak{g}'$ (res. $\mathfrak{g}''$) is N (res. r). $p\in\mathcal{P}_{\pm}(r)$ defines an orbit $\mathcal{O}_p$ of $\mathfrak{l}$ with zero factor outside $\mathfrak{g}''$. Let $\widetilde{\mathcal{O}}_d=\mathrm{Ind}_L^G(\widetilde{\mathcal{O}}_p)$, here $d\in\mathcal{P}_{\pm}(N)$ as above section and p. Let

$$M=\begin{cases}\overline{GL(\bullet,\mathbb{C})}\otimes_{Z_2}\cdots\otimes_{Z_2}\overline{GL(\bullet,\mathbb{C})}, & G=Spin(\bullet,\mathbb{C}),\\ GL(\bullet,\mathbb{C})\times\cdots\times GL(\bullet,\mathbb{C}), & \text{otherwise.}\end{cases}$$

We have the following fundamental induction (table 2).

Table 2 Fundamental induction of orbit cover with zero in GL-factor(2)

G	Type	G^{ϕ}/G_0^{ϕ}	$L^{\phi'}/L_0^{\phi'}$	Π_G
	I	$(Z_2)^m$	$(Z_2)^{m-1}$	$\psi_j^{-1}(\Pi_L)$
$M\times Sp(2n,\mathbb{C})$	II	$(Z_2)^m$	$(Z_2)^m$	Π_L
	III	$(Z_2)^m$	$(Z_2)^{m-1}$	$\varphi_i(\Pi_L)$
$M\times SO(N,\mathbb{C})$	I	$S((Z_2)^m)$	$S((Z_2)^{m-1})$	$\psi_j^{-1}(\Pi_L)$
$M\otimes_{Z_2}Spin(N,\mathbb{C})$	II	$S((Z_2)^m)$	$S((Z_2)^m)$	Π_L
	III	$S((Z_2)^m)$	$S((Z_2)^{m-1})$	$\varphi_i(\Pi_L)$
	I	$(Z_2)^m/\Delta'_d$	$(Z_2)^{m-1}/\Delta'_p$	$\psi_j^{-1}(\Pi_L)$
$(M\times Sp(2n,\mathbb{C}))/\pm I$	II	$(Z_2)^m/\Delta'_d$	$(Z_2)^m/\Delta'_p$	Π_L
	III	$(Z_2)^m/\Delta'_d$	$(Z_2)^{m-1}/\Delta'_p$	$\varphi_i(\Pi_L)$
	I	$S((Z_2)^m)/S(\Delta'_d)$	$S((Z_2)^{m-1})/S(\Delta'_p)$	$\psi_j^{-1}(\Pi_L)$
$(M\times SO(2n,\mathbb{C}))$	II	$S((Z_2)^m)/S(\Delta'_d)$	$S((Z_2)^m)/S(\Delta'_p)$	Π_L
	III	$S((Z_2)^m)/S(\Delta'_d)$	$S((Z_2)^{m-1})/S(\Delta'_p)$	$\varphi_i(\Pi_L)$

This table is obtained by Proposition 6. In the rest of this section, we simply talk $L_1\sim L_2$ means L_1 is conjugated to L_2 in G, $\widetilde{\mathcal{O}}_{L_1}\sim\widetilde{\mathcal{O}}_{L_2}$ means $(L_1,\widetilde{\mathcal{O}}_{L_1})$ is conjugated to $(L_2,\widetilde{\mathcal{O}}_{L_2})$ in G,

$L_1 \preceq L_2$ means $\exists L_1'$ and L_2' such that $L_1 \sim L_2' \subset L_2' \sim L_2$. Now, whenever $L_1 \preceq L_2$, $\mathrm{Ind}_{L_1}^{L_2}(\widetilde{\mathcal{O}}_{L_1})$ is defined by $\mathrm{Ind}_{L_1'}^{L_2'}(\widetilde{\mathcal{O}}_{L_1'})$ at the meaning of "$\sim$".

Remark 2. Given two orbit covers $\widetilde{\mathcal{O}}_1$ and $\widetilde{\mathcal{O}}_2$ of the same L, $\widetilde{\mathcal{O}}_1 \sim \widetilde{\mathcal{O}}_2$ does not mean $\widetilde{\mathcal{O}}_1$ is conjugated to $\widetilde{\mathcal{O}}_2$ in L, but they have the same partition. This case only occurs in that L is of type $D_{2l'}$ and G is of type B_l or D_{2l+1}, where $\widetilde{\mathcal{O}}_1$ and $\widetilde{\mathcal{O}}_2$ are corresponding to the same very even partition.

Corollary 2. If Ind_L^G is the fundamental induction, then for any $\Pi_G \in \mathrm{Im}(\mathrm{Ind}_L^G)$, there exist a unique Π_l such that $\mathrm{Ind}_L^G(\Pi_L) = \Pi_G$. So we can define $\left(\mathrm{Ind}_L^G\right)^{-1}(\Pi_G) = \Pi_L$.

Now, we consider the Levi-subgroups of classical semisimple groups.

Lemma 5. Given two different fundamental inductions satisfying: $\mathrm{Ind}_{L_1}^G(\widetilde{\mathcal{O}}_{L_1}) = \mathrm{Ind}_{L_2}^G(\widetilde{\mathcal{O}}_{L_2})$, then there exist a Levi subgroup L and orbit cover $\widetilde{\mathcal{O}}_L$ such that $L \preceq L_1$, $L \preceq L_2$ and

$$\begin{aligned} &\widetilde{\mathcal{O}}_{L_1} \sim \mathrm{Ind}_L^{L_1}(\widetilde{\mathcal{O}}_L), \\ &\widetilde{\mathcal{O}}_{L_2} \sim \mathrm{Ind}_L^{L_2}(\widetilde{\mathcal{O}}_L). \end{aligned}$$

Proof. First: Let G be a simple group.

Consider a special case : two induction types are type I and type II (so i, j are defined) and $i = j$ or $i = j + 1$. Since $\varphi_i(\bullet) = \psi_j^{-1}(\bullet)$ never happens, by $\mathrm{Ind}_{L_1}^G(\widetilde{\mathcal{O}}_{L_1}) = \mathrm{Ind}_{L_2}^G(\widetilde{\mathcal{O}}_{L_2})$ we know this special case does not occur here. We construct L and $\mathcal{O}_L$ as follows:

Suppose that the partition of $\mathcal{O}_{L_1}$ (res.$\mathcal{O}_{L_2}$) is $p^{(1)}$ (res. $p^{(2)}$), let $\mathfrak{l} = \mathfrak{m} + gl(l_1, \mathbb{C}) + gl(l_2, \mathbb{C}) + \mathfrak{g}_{N-2l_1-2l_2}$ with L the corresponding Levi subgroup of G. Let $p \in \mathcal{P}_{\pm}(N - 2l_1 - 2l_2)$ given by

$$\text{If } \mathrm{Ind}_{L_2}^G \text{ is of type I, II} \quad \text{let} \quad p_k = \begin{cases} p_k^{(1)} - 2, & 1 \leqslant k \leqslant l_2, \\ p_k^{(1)}, & k > l_2. \end{cases}$$

$$\text{If } \mathrm{Ind}_{L_2}^G \text{ is of type III} \quad \text{let} \quad p_k = \begin{cases} p_k^{(1)} - 2, & 1 \leqslant k \leqslant l_2 - 1, \\ p_k^{(1)} - 1, & k = l_2, l_2 + 1, \\ p_k^{(1)}, & k > l_2 + 1. \end{cases}$$

Because the special case does not happen, it is easy to show that the definition of p symmetrically depends on L_1 and L_2. Define $\mathcal{O}_L = \mathcal{O}_p$. Let ϕ, ϕ', ϕ'', ϕ''' be a proper standard triple in G, L_1, L_2 and L respectively. Consider the following commutative diagram:

Induction diagram

$$\begin{tikzcd}
& {L_1}^{\phi'}/{L_1}_0^{\phi'} \arrow[dr, dash] & \\
L^{\phi'''}/L_0^{\phi'''} \arrow[ur, dash] \arrow[dr, dash] & & G^{\phi}/G_0^{\phi} \\
& {L_2}^{\phi''}/{L_2}_0^{\phi''} \arrow[ur, dash] &
\end{tikzcd}$$

on each side we label the induction map as table 3 showing.

Now table 3 (if needed, exchange L_1 and L_2) shows the complete possibility of maps label on each side. Let (ϕ', Π_{L_1}), (ϕ'', Π_{L_2}) be the representation of $\widetilde{\mathcal{O}}_{L_1}$, $\widetilde{\mathcal{O}}_{L_2}$ respectively. It is easy to verify that $\Pi_{L_1} \in \mathrm{Im}(\mathrm{Ind}_L^{L_1})$, $\Pi_{L_2} \in \mathrm{Im}(\mathrm{Ind}_L^{L_2})$, by Corollary 2, we can define

$$\begin{aligned}
\Pi &= (\mathrm{Ind}_L^{L_1})^{-1}(\Pi_{L_1}) \\
&= (\mathrm{Ind}_L^{L_1})^{-1}(Ind_{L_1}^{G})^{-1}(\Pi_G) \\
&= (\mathrm{Ind}_L^{G})^{-1}(\Pi_G) \\
&= (\mathrm{Ind}_L^{L_2})^{-1}(Ind_{L_2}^{G})^{-1}(\Pi_G) \\
&= (\mathrm{Ind}_L^{L_2})^{-1}(\Pi_{L_2})
\end{aligned}$$

then $\mathrm{Ind}_L^{L_k}(\widetilde{\mathcal{O}}_L) \sim \widetilde{\mathcal{O}}_{L_k}$, $k = 1, 2$.

Table 3 Maps of the diagram above

Type pair		$L \to L_1$	$L_1 \to G$	$L \to L_2$	$L_2 \to G$
I, I	$j_2 > j_1$	ψ_{j_2-1}	ψ_{j_1}	ψ_{j_1}	ψ_{j_2}
I, I	$j_2 = j_1$	ψ_{j_2-1}	ψ_{j_1}	ψ_{j_1-1}	ψ_{j_2}
III, III	$i_2 > i_1$	φ_{i_2-1}	φ_{i_1}	φ_{i_1}	φ_{i_2}
III, III	$i_2 = i_1$	φ_{i_2-1}	φ_{i_1}	φ_{i_1-1}	φ_{i_2}
I, III	$j > i$	φ_i	ψ_j	ψ_{j-1}	φ_i
I, III	$i > j+1$	φ_{i-1}	ψ_j	ψ_j	φ_i
I, II		$\cong$	ψ_j	ψ_j	$\cong$
II, III		φ_i	$\cong$	$\cong$	φ_i

Second: For semisimple groups.

First, assume that G is simply connected and $G{=}G^1{\times}G^2$ (where G^1 and G^2 are simple groups), then $L_1 = L_1^1 \times L_2^1$ and $L_2{=}L_1^2{\times}L_2^2$ (where L_j^i is a Levi subgroup of G^i). From the above, we can get the Levi subgroups L^i of L_j^i for $i, j = 1, 2$. Now let $L{=}L^1{\times}L^2$, then L is a Levi subgroup of L^1 and L^2. By table 3 and the proof of Proposition 1, we can prove that for G, we have the following commutative diagram.

$$\begin{tikzcd}[column sep=small]
& (P_1\cap L)^{\phi_1}/(P_1\cap L)_0^{\phi_1} \arrow[dl, "\tilde{\theta}_1"'] \arrow[r, hook] & L_1^{\phi_1}/(L_1)_0^{\phi_1} & P_1^{\phi_1}/(P_1)_0^{\phi_1} \arrow[l, "\theta_1"'] \arrow[dr] & \\
L^{\phi_3}/L_0^{\phi_3} & & & & G^{\phi}/G_0^{\phi} \\
& (P_2\cap L)^{\phi_2}/(P_2\cap L)^{\phi_2} \arrow[ul, "\tilde{\theta}_2"] \arrow[r, hook] & L_2^{\Phi_2}/(L_2)_0^{\phi_2} & P_2^{\phi_2}/(P_2)_0^{\phi_2} \arrow[l, "\theta_2"'] \arrow[ur] &
\end{tikzcd}$$

In general, we can also assume G to be simple connected by sec. 1. Then we still can get the Levi subgroup L and the above diagram, and prove the diagram is commutative by induction. From the commutative diagram, we can get the lemma easily. Q.E.D.

Lemma 6. If $\widetilde{\mathcal{O}}_L$ is a rigid orbit cover of L, $\mathrm{Ind}_{L_1}^G$ is a fundamental induction and $\mathrm{Ind}_L^G(\widetilde{\mathcal{O}}_L) = \mathrm{Ind}_{L_1}^G(\widetilde{\mathcal{O}}_{L_1})$, then $L \preceq L_1$ and

$$\widetilde{\mathcal{O}}_{L_1} \sim \mathrm{Ind}_L^{L_1}(\widetilde{\mathcal{O}}_L).$$

Proof. Decompose $\mathrm{Ind}_{L_1}^G$ into the product of some fundamental induction. We use induction on the number k of the fundamental inductions which occur in $\mathrm{Ind}_{L_1}^G$, if $k = 1$, by Lemma 5 and the rigid condition of $\widetilde{\mathcal{O}}_L$, it is obviously true. For general k, we can use Lemma 5 again and again, to reduce it to the case $k = 1$. Q.E.D.

Theorem 3. If $\widetilde{\mathcal{O}}_{L_1}$ (res. $\widetilde{\mathcal{O}}_{L_2}$) is a rigid orbit cover of L_1 (res. L_2), and $\mathrm{Ind}_{L_1}^G(\widetilde{\mathcal{O}}_{L_1}) = \mathrm{Ind}_{L_2}^G(\widetilde{\mathcal{O}}_{L_2})$, then $L_1 \sim L_2$ and

$$\widetilde{\mathcal{O}}_{L_1} \sim \widetilde{\mathcal{O}}_{L_2}.$$

Proof. Decompose $\mathrm{Ind}_{L_1}^G(\widetilde{\mathcal{O}}_{L_1})$ into the product of fundamental induction. Suppose the last one is $\mathrm{Ind}_{G'}^G(\widetilde{\mathcal{O}}_{G'})$, by Lemma 6, $L_2 \preceq G'$ and $\widetilde{\mathcal{O}}_{G'} \sim \mathrm{Ind}_{L_2}^{G'}(\widetilde{\mathcal{O}}_{L_2})$, we do it again and again, finally we get $L_2 \preceq L_1$ and $\widetilde{\mathcal{O}}_{L_1} \sim \mathrm{Ind}_{L_2}^{L_1}(\widetilde{\mathcal{O}}_{L_2})$, but $\widetilde{\mathcal{O}}_{L_1}$ is rigid. Hence, $L_1 \sim L_2$ and $\widetilde{\mathcal{O}}_{L_1} \sim \widetilde{\mathcal{O}}_{L_2}$. Q.E.D.

Remark 3. Since G is $Spin(N,\mathbb{C})$, there exists another class of sheet: Let $L = \overline{GL(l_1,\mathbb{C})} \otimes_{Z_2} \cdots \otimes_{Z_2} \overline{GL(l_s,\mathbb{C})} \otimes_{Z_2} Spin(r,\mathbb{C})$ and all l_i even, $\widetilde{\mathcal{O}}_L$ has factor $\widetilde{\mathcal{O}}_{[2^{l_i/2}]}$ (the 2-folder cover of $\mathcal{O}_{[2^{l_i/2}]}$) in $\overline{GL(l_i,\mathbb{C})}$, and rigid rather odd $\widetilde{\mathcal{O}}_p$ in $Spin(r,\mathbb{C})$ such that $-1 \notin \Pi$. These classes of sheets all induce to rather odd $\widetilde{\mathcal{O}}_d$ with $-1 \notin \Pi$ and *vice versa*. The fundamental induction is type IV or V.

On the other hand, for d is not rather odd or d is rather odd but $-1 \in \Pi$, we can view $\widetilde{\mathcal{O}}_d$ as the orbit cover of $SO(N,\mathbb{C})$. Under this identification, the parabolic induction of the orbit cover in $Spin(N,\mathbb{C})$ and $SO(N,\mathbb{C})$ is the same.

Corollary 3. For group $Sp(2n,\mathbb{C}), SO(n,\mathbb{C}), Spin(n,\mathbb{C}), Psp(2n,\mathbb{C})$ and $PSO(2n,\mathbb{C})$, the different sheets are disjoint.

Remark 4. From this and the above sections, we can give all rigid orbit covers of classical semisimple Lie groups. We will present them in another paper (in fact, we also can give the rigid orbit covers of exception simple Lie groups[5]).

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References

1. Vogan, D., Dixmier algebras, sheets, and representation theory, in Actes de colloque en l'honneur de Jacques Dixmier, Progress in Math. 92, Boston: Birkhäuser, 1990, 333—397.
2. Liang, K., Hou, Z., Geometric orbit datum and orbit covers, Science in China, Ser. A, 2001, 44(11): 1413—1419.
3. Collingwood, D. H., McGovern, W. M., Nilpotent Orbits in Semisimple Lie Algebras, New York: Van Nostrand Reinhold, 1993.
4. Lustig, G., Intersection cohomology complexes on a reductive group, Inv. Math., 1984, 75: 205—272.
5. Liang, K., Lu, L., The sheets and rigid orbit covers of exception Lie groups, Chinese Science Bulletin, 1998, 43(20): 1702—1707.