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The Game of Nim

- **Two players:** “Left” and “Right”.
- **Game board:** a number of heaps of counters.
- **Rules:** Two players take turns. Either player can remove any positive number of counters from any one heap.
- **Ending positions:** Whoever gets stuck is the loser.
Nimbers

The game value of a heap of size \( n \) is denoted by \(*n\). It can be defined recursively as follows:

\[
* = \{0 \mid 0\}; \\
*2 = \{0,* \mid 0,*\}; \\
*3 = \{0,*,*2 \mid 0,*,*2\}; \\
\vdots \\
*n = \{0,*,*2,\ldots,* (n-1) \mid 0,*,*2,\ldots,* (n-1)\}.
\]
The game value of a heap of size $n$ is denoted by $*n$. It can defined recursively as follows:

$$
* = \{ 0 \mid 0 \};
*2 = \{ 0, * \mid 0, * \};
*3 = \{ 0, *, *2 \mid 0, *, *2 \};
\vdots
* n = \{ 0, *, *2, \cdots, *(n - 1) \mid 0, *, *2, \cdots, *(n - 1) \}. 
$$

So the previous nim game has the game value

$$
*5 + * + * + * + * + * + * + 6 + *4.
$$
Identities of nimbers

\[ *n + *n = 0. \]
Identities of nimbers

\[ *n + *n = 0. \]

\[ * + *2 + *3 = 0. \]
Identities of nimbers

\[ \ast n + \ast n = 0. \]

\[ \ast + \ast 2 + \ast 3 = 0. \]

\[ \ast + \ast 4 + \ast 5 = 0. \]
Identities of nimbers

\[*n + *n = 0.\]

\[* + *2 + *3 = 0.\]

\[* + *4 + *5 = 0.\]

\[*2 + *4 + *6 = 0.\]
Identities of nimbers

\[ *n + *n = 0. \]

\[ * + *2 + *3 = 0. \]

\[ * + *4 + *5 = 0. \]

\[ *2 + *4 + *6 = 0. \]

What is the value of \[ *3 + *5? \]
Identities of nimbers

\[ *n + *n = 0. \]

\[ * + *2 + *3 = 0. \]

\[ * + *4 + *5 = 0. \]

\[ *2 + *4 + *6 = 0. \]

What is the value of \( *3 + *5 \)?

Nimber addition: \( a + b = c \) if \( *a + *b = *c \).
Nim-Addition Table

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Nimber addition rule

**Property 1:** If \( a < 2^n \), then

\[
2^n + a = 2^n + a.
\]
Nimber addition rule

Property 1: If \( a < 2^n \), then

\[
2^n \ast a = 2^n + a.
\]

Property 2: For any \( a, b \), \( \ast a + \ast b \) is the sum of two binary numbers without carrying.
Nimber addition rule

Property 1: If \( a < 2^n \), then
\[
2^n + a = 2^n + a.
\]

Property 2: For any \( a, b \), \( *a + *b \) is the sum of two binary numbers without carrying.

Property 3: If \( a, b < 2^n \), then \( a + b < 2^n \).
Proof of Property 1

Proof: We use double induction on $n$ and $a$. 
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Initial case: $n = 0$ or $a = 0$. This is trivial.
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Inductive step: Assume \( 2^n + a' = 2^n + a' \) for all \( a' < a \), and \( 2^k + b = 2^k + b \) for all \( k < n \) and \( b < 2^k \).
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Consider the Nim game of three heaps of size $(2^n + a)$, $2^n$, and $a$. It suffices to show that this is a zero game.
Proof of Property 1

**Proof:** We use double induction on $n$ and $a$.

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**Inductive step:** Assume $2^n + a' = 2^n + a'$ for all $a' < a$, and $2^k + b = 2^k + b$ for all $k < n$ and $b < 2^k$.

Consider the Nim game of three heaps of size $(2^n + a)$, $2^n$, and $a$. It suffices to show that this is a zero game.

- If $2^n + a \rightarrow 2^n + a'$, then $a \rightarrow a'$.
- If $2^n + a \rightarrow b$ ($b < 2^n$), then $2^n \rightarrow a'$.
- If $2^n \rightarrow b$, then $2^n + a \rightarrow a + b$.
- If $a \rightarrow a'$, then $2^n + a \rightarrow 2^n + a'$.

So the second player can always reduce to the zero position and thus wins the game. Thus $2^n + a = 2^n + a$. 
Winning strategy of Nim

If the current position is non-zero, then the current player can win the game by resetting the game value to zero and wins the game.
Winning strategy of Nim

If the current position is non-zero, then the current player can win the game by resetting the game value to zero and wins the game.

Consider the Nim Game

\[ *3 + *4 + *6 + *9 \]

Note that the current position has the game value \( *8 \neq 0 \). The only winning move is \( 9 \rightarrow 1 \) so that the new game value is

\[ *3 + *4 + *6 + * = 0. \]
An **impartial game** is a game so that Left and Right have exactly the same options at any game position.
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**Sprague-Grundy theorem:** Every impartial game has a game value \( *n \) for some non-negative integer \( n \).
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**Sprague-Grundy theorem:** Every impartial game has a game value $*n$ for some non-negative integer $n$.

**The Minimal-Excluded Rule:** If $G = \{ *a, *b, *c, \ldots | *a, *b, *c, \ldots \}$ is an impartial game, then $G = *n$ where $n$ is the minimal-excluded number from the list $a, b, c, \ldots$. 
An **impartial game** is a game so that Left and Right have exactly the same options at any game position.

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**The Minimal-Excluded Rule:** If $G = \{*a,*b,*c,\ldots \mid *a,*b,*c,\ldots \}$ is an impartial game, then $G = *n$ where $n$ is the minimal-excluded number from the list $a,b,c,\ldots$

$$\{0,*,*2,*4 \mid 0,*,*2,*4\} = *3$$

since 3 is the minimal-excluded number from 0, 1, 2, 4.
**Game of White Knight**

- **Two players**: “Left” and “Right”.
- **Game board**: a chess board and a Nim-heap of size 6.
- **Rules**: Two players take turns. Either player can either move the Knight on the chess board North-West, or lose some of his belonging— to remove any positive number of chips from the Nim-heap.
- **Ending positions**: Whoever gets stuck is the loser.
Value of Knight only

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Two players: “Left” and “Right”.

Game board: several queens on a chess board.

Rules: Any number of Queens can be on the same square and each player when its her turn to move, can move any single queen an arbitrary distance North, West, North-West as indicated, even jumping over other queens.

Ending positions: Whoever gets stuck is the loser.
## Value of one Queen

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Poker-Nim

- **Two players:** “Left” and “Right”.
- **Game board:** a number of heaps of counters.
- **Rules:** Two players take turns. Either player can either remove any positive number of chips from any one heap or add to it some of his chips that he acquired in earlier moves.
- **Ending positions:** Whoever gets stuck is the loser.
If a player can win a Nim position, then he can win the Poker-Nim game as well. If the opponent add some chips to a heap, he can reverse the opponent’s move by taking the same amount of chips away for the heap.

Poker-Nim = Nim

The new move will only prolong the time of playing game but not affect who will win this game.
Consider a general game

\[ G = \{ A, B, C, \ldots \mid D, E, F, \ldots \}. \]

We say Right’s move to \( D \) is reversible if there is some move for Left from \( D \) to \( D^L \) which is at least as good for Left as \( G \) was, i.e. \( D^L \geq G \).
General reversible move

Consider a general game

\[ G = \{ A, B, C, \ldots \mid D, E, F, \ldots \} \].

We say Right’s move to \( D \) is reversible if there is some move for Left from \( D \) to \( D^L \) which is at least as good for Left as \( G \) was, i.e. \( D^L \geq G \).

Suppose

\[ D^L = \{ U, V, W, \ldots, \mid X, Y, Z, \ldots \} \].

Now whenever Right plays from \( G \) to \( D \), Left will reverse from \( D \) to \( D^L \). We may view it as a new game

\[ H = \{ A, B, C, \ldots \mid X, Y, Z, \ldots, E, F, \ldots \} \].
If any Right option $D$ of $G$ has itself a Left option $D^L \geq G$, then it will not affect the value of $G$ if we replace $D$ as a Right option of $G$ by all the Right options $X, Y, Z, \ldots$ of that $D^L$. 
**Proof:** It is suffice to show $G + (-H) = 0$. 
Bypassing reversible move

**Proof:** It is suffice to show $G + \langle -H \rangle = 0$.

Left plays first. If Left moves from $G$ to $A$, $B$, or $C$, Right can move from $-H$ to $-A$, $-B$, or $-C$ and win the game. If Left moves from $-H$ to $-X$, then the value is $G - X \leq D^L - X$, which Right can also win.
Now Right plays first. If Right moves other than $G$ to $D$, Left can always move the zero position and win the game. If Right moves from $G$ to $D$, then Left can move from $D$ to $D^L$. Now the value is $D^L - H$. Right cannot move from $D^L$ to $X, Y, Z$. His only hope is to move from $-H$ to $-A, -B$, or $-C$. But then Left can win since $D^L - A \geq G - A$ and Left wins $G - A$. 

continue
Bypassing Right’s reversible move:

If any Right option $D$ of $G$ has itself a Left option $D^L \geq G$, then it will not affect the value of $G$ if we replace $D$ as a Right option of $G$ by all the Right options $X, Y, Z, \ldots$ of that $D^L$. 
Bypassing Right’s reversible move:
If any Right option \( D \) of \( G \) has itself a Left option \( D^L \geq G \), then it will not affect the value of \( G \) if we replace \( D \) as a Right option of \( G \) by all the Right options \( X, Y, Z, \ldots \) of that \( D^L \).

Bypassing Left’s reversible move:
If any Left option \( C \) of \( G \) has itself a Left option \( C^R \leq G \), then it will not affect the value of \( G \) if we replace \( C \) as a Left option of \( G \) by the list of all the Left options of that \( C^R \).
Deleting dominated options

In the game

\[ G = \{ A, B, C, \ldots \mid D, E, F, \ldots \}, \]

if \( A \leq B \) we say that \( A \) is **dominated** by \( B \), and if \( D \leq E \), that \( E \) is dominated by \( D \).

**Deleting dominated options:**

It won’t affect the value of \( G \) if we delete dominated options but retain the options that dominated them.
Deleting dominated options

In the game

\[ G = \{ A, B, C, \ldots \mid D, E, F, \ldots \}, \]

if \( A \leq B \) we say that \( A \) is \textbf{dominated} by \( B \), and if \( D \leq E \), that \( E \) is dominated by \( D \).

Deleting dominated options:

It won’t affect the value of \( G \) if we delete dominated options but retain the options that dominated them.

\textbf{Proof:} Let \( K = \{ B, C \ldots \mid D, F, \ldots \} \). We show \( G + (−K) = 0 \). Any move from \( G \) to \( A \) or \( E \) are countered by those from \( −K \) to \( −B \) or \( −D \), and all other moves in either component are countered by moves to their negatives from the other.
2 Toads and 2 Frogs

The diagram represents a game with two toads and two frogs, illustrating the possible moves and outcomes in a combinatorial game theory context.
Define

\[ \uparrow = \{ 0 \mid * \} , \]
\[ \downarrow = \{ * \mid 0 \} . \]
Define

\[ \uparrow = \{ 0 \mid * \}, \]
\[ \downarrow = \{ * \mid 0 \}. \]

**Property:**

- \( -\frac{1}{2^n} \downarrow < 0 < \uparrow < \frac{1}{2^n} \) for any integer \( n > 0 \).
Define

\[ \uparrow = \{ 0 \mid * \} , \]
\[ \downarrow = \{ * \mid 0 \} . \]

**Property:**

- \[ -\frac{1}{2^n} < \downarrow < 0 < \uparrow < \frac{1}{2^n} \] for any integer \( n > 0 \).
- \[ \downarrow = - \uparrow . \]
Define

\[ \uparrow = \{0 \mid \ast\}, \]
\[ \downarrow = \{\ast \mid 0\}. \]

Property:
- \(-\frac{1}{2^n} < \downarrow < 0 < \uparrow < \frac{1}{2^n}\) for any integer \(n > 0\).
- \(\downarrow = -\uparrow\).
- Both \(\uparrow\) and \(\downarrow\) are fuzz to \(\ast\).
Define

\[ \uparrow = \{0 \mid \ast\}, \]
\[ \downarrow = \{\ast \mid 0\}. \]

Property:

- \[ -\frac{1}{2^n} < \downarrow < 0 < \uparrow < \frac{1}{2^n} \]
  for any integer \( n > 0 \).
- \( \downarrow = -\uparrow \).
- Both \( \uparrow \) and \( \downarrow \) are fuzz to \( \ast \).
- \( \{\uparrow \downarrow\} = \{\uparrow \mid 0\} = \{0 \mid \downarrow\} = \{0 \mid 0\} = \ast \).
Define

\[ \uparrow = \{ 0 \mid * \}, \]
\[ \downarrow = \{ * \mid 0 \}. \]

**Property:**

- \(- \frac{1}{2^n} < \downarrow < 0 < \uparrow < \frac{1}{2^n}\) for any integer \( n > 0 \).
- \( \downarrow = - \uparrow \).
- Both \( \uparrow \) and \( \downarrow \) are fuzz to \( * \).
- \( \{ \uparrow \mid \downarrow \} = \{ \uparrow \mid 0 \} = \{ 0 \mid \downarrow \} = \{ 0 \mid 0 \} = * \).

The last equality can be proved by showing \( \downarrow \) (or) \( \uparrow \) is a Right’s (or Left’s) reversible move for \( G = \{ \uparrow \mid \downarrow \} \). This can be done by showing \( G + * = 0 \).
Define $\uparrow^* = \uparrow + \ast$. We have

\[
\uparrow^* = \uparrow + \ast \\
= \{0 \mid \ast\} + \{0 \mid 0\} \\
= \{\uparrow + 0, 0 + \ast \mid \uparrow + 0, \ast + \ast\} \\
= \{\uparrow, \ast \mid 0, \uparrow\} \\
= \{\uparrow, \ast \mid 0\} \quad \text{remove dominated option} \\
= \{0, \ast \mid 0\} \quad \text{bypassing reversible move.}
\]
Define $\uparrow^* = \uparrow + *$. We have

$$
\uparrow^* = \uparrow + *
= \{0 \mid *\} + \{0 \mid 0\}
= \{\uparrow + 0, 0 + * \mid \uparrow + 0, * + *\}
= \{\uparrow, * \mid 0, \uparrow\}
= \{\uparrow, * \mid 0\} \quad \text{remove dominated option}
= \{0, * \mid 0\} \quad \text{bypassing reversible move.}
$$

The blue flower has value $\uparrow^*$. 
Childish Hackenbush

- **Two players:** “Left” and “Right”.
- **Game board:** blue-red graphs connected to the ground.
- **Rules:** Two players take turns. Right deletes one red edge provided deleting this red edge does not result any piece disconnected from the ground. Left does the similar move but deletes one blue edge.
- **Ending positions:** Whoever gets stuck is the loser.
Childish Hackenbush Values

Diagram showing various patterns and values related to Hackenbush games.
Childish Hackenbush Values

\[ \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{7}{8} \]

\[ \frac{1}{2} \quad \frac{1}{4} \quad \frac{7}{8} \quad \frac{15}{16} \quad \frac{11}{12} \]

\[ * \quad \{1|0\} \quad 1^* \quad 1^\perp \quad \{\frac{1}{2}|1\} \]
Define

\[ \uparrow = \uparrow + \uparrow, \]
\[ \uparrow* = \uparrow + \uparrow + \ast. \]
Define

\[ \uparrow = \uparrow + \uparrow, \]
\[ \uparrow* = \uparrow + \uparrow + \uparrow* . \]

We can show \( \uparrow* > 0! \)
Define

\[ \uparrow = \uparrow + \uparrow, \]
\[ \uparrow^* = \uparrow + \uparrow + \dagger. \]

We can show \( \uparrow^* > 0! \)

Similarly we can define \( \downarrow \) and \( \downarrow^* \). We have \( \downarrow^* < 0 \).
Whereabouts of $\uparrow^*$ and $\ast n$

$\uparrow^*$ is fuzz to 0 and $\uparrow$, is greater than $\downarrow$ and less than 3. $\uparrow$. 
Whereabouts of $\uparrow^* \text{ and } *n$

$\uparrow^*$ is fuzz to 0 and $\uparrow\uparrow$, is greater than $\downarrow\downarrow$ and less than 3. $\uparrow$.

For $n \geq 2$. $*n$ is greater than $\downarrow\downarrow$ and less than $\uparrow\uparrow$. 
Up in Toads-Frogs

\[ \{0|\uparrow\}\]

\[ \text{TTF} \quad \text{TF} \quad \text{TTFF} \quad \text{TF} \quad \text{TF} \]

\[ \text{0} \]

\[ \text{0} \]

\[ \text{TFT} \quad \text{TF} \quad \text{TF} \quad \text{FF} \quad \text{TF} \quad \text{FF} \]

\[ \text{0} \]

\[ \text{TFT} \quad \text{TTFF} \quad \text{TF} \quad \text{TTFF} \quad \text{TF} \quad \text{FF} \]

\[ \text{0} \]

\[ \text{TFF} \quad \text{TF} \quad \text{FF} \quad \text{TF} \quad \text{FF} \quad \text{TF} \]

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Interesting identity

\[ \{\uparrow\uparrow\} = \{0 \mid \uparrow\} = \uparrow^*. \]

Bypassing Left’s reversible move \( \uparrow \), we get

\[ \{\uparrow\uparrow\} = \{0 \mid \uparrow\}. \]
Interesting identity

\[
\{\uparrow|\uparrow\} = \{0 \mid \uparrow\} = \uparrow^*.
\]

Bypassing Left’s reversible move \(\uparrow\), we get

\[
\{\uparrow|\uparrow\} = \{0 \mid \uparrow\}.
\]

Now let \(X = \{0 \mid \uparrow\}\). Consider the sum of Games:

\[
X + \downarrow + \downarrow^* = \{0 \mid \uparrow\} + \{* \mid 0\} + \{0 \mid 0, *\}.
\]

Left first: if \(X \rightarrow 0\) then the value is \(*\) and Right wins. If Left plays on \(\downarrow\) or \(\downarrow^*\), then Right \(X \rightarrow \uparrow\) and Right wins.

Right first: if \(X \rightarrow \uparrow\) then the value is \(\downarrow^*\) so Right wins. If Right plays on \(\downarrow\) or \(\downarrow^*\), then Left responses on \(\downarrow^*\) or \(\downarrow\) respectively. The game value becomes \(X\) or \(X + *\). Both of them are positive so Left wins. Thus \(X + \downarrow^* = 0\).
Here are the simplest forms for Ups and Stars.

\[
\begin{array}{lll}
3. \downarrow = \{\downarrow \ast | 0\} & 3. \downarrow \ast = \{\downarrow | 0\} & 3. \downarrow \ast n = \{\downarrow \ast m | 0\} \\
\downarrow = \{\downarrow \ast | 0\} & \downarrow \ast = \{\downarrow | 0\} & \downarrow \ast n = \{\downarrow \ast m | 0\}
\end{array}
\]

\[
\begin{array}{lll}
\downarrow = \{\ast | 0\} & \downarrow \ast = \{0 | 0, \ast\} & \downarrow \ast n = \{\ast m | 0\}
\end{array}
\]

\[
\begin{array}{lll}
0 = \{\|\} & \ast = \{0 | 0\} & \uparrow \ast n = \{0 | \ast m\}
\end{array}
\]

\[
\begin{array}{lll}
\uparrow = \{0 | \ast\} & \uparrow \ast = \{0, \ast | 0\} & \uparrow \ast n = \{0 | \uparrow \ast m\}
\end{array}
\]

\[
\begin{array}{lll}
\uparrow = \{0 | \uparrow \ast\} & \uparrow \ast = \{0 | \uparrow\} & \uparrow \ast n = \{0 | \uparrow \ast m\}
\end{array}
\]

Here \( n \geq 2 \) and \( m = n + 1 \).
It does not affect the value of $G$ if we add a new Left option $H$ provided $H \prec G$, or a new right option $\bar{H}$ provided $\bar{H} \succ G$. 
It does not affect the value of $G$ if we add a new Left option $H$ provided $H \triangleleft G$, or a new right option $\bar{H}$ provided $\bar{H} \triangleright G$.

**Proof:** Consider the game

$$\{G^L, H \mid G^R\} + \{-G^R \mid -G^L\}.$$  

If Left moves to $H$, then the game value is $H - G \triangleleft 0$ so Right wins. Any other move can be reversed to 0 by the next player. So this is a zero game. Thus

$$\{G^L, H \mid G^R\} = G.$$
Since $\{0 \mid \uparrow\} = \uparrow^*$, so it is fuzz to $\uparrow$, $\uparrow\uparrow$, and $3. \uparrow$. Those options are gift horses for left. We have

$$\{0 \mid \uparrow\} = \{0, \uparrow\uparrow\} = \{\uparrow\uparrow\}. \quad \{0 \mid \uparrow\} = \{0, 3.\uparrow\uparrow\} = \{3.\uparrow\uparrow\}.$$  

Thus,

$$\{3.\uparrow\uparrow\} = \{\uparrow\uparrow\} = \{\uparrow\uparrow\} = \{0 \mid \uparrow\} = \uparrow^*.$$
Since \( \{ 0 \mid \uparrow \} = \uparrow * \), so it is fuzz to \( \uparrow, \uparrow, \) and 3. \( \uparrow. \) Those options are gift horses for left. We have

\[
\{ 0 \mid \uparrow \} = \{ 0, \uparrow \uparrow \} = \{ \uparrow \uparrow \}.
\]

\[
\{ 0 \mid \uparrow \} = \{ 0, 3. \uparrow \uparrow \} = \{ 3. \uparrow \uparrow \}.
\]

Thus,

\[
\{ 3. \uparrow \uparrow \} = \{ \uparrow \uparrow \} = \{ \uparrow \uparrow \} = \{ 0 \mid \uparrow \} = \uparrow *.
\]

Note that the above arguments fail for 4. \( \uparrow. \) In fact

\[
\{ 4. \uparrow \uparrow \} > \uparrow *.
\]
Toad versus Frog

One toad meet one frog:

\[ \{ d - 2 \mid d + 2 \} \text{ where } d = b - a \text{ is the difference.} \]
Toad versus Frog

One toad meet one frog:

\[
\begin{align*}
&\{d - 2 \mid d + 2\} \text{ where } d = b - a \text{ is the difference. In general, if there is } c \text{ spaces between two creatures, we have }

&f(c, d) = \{f(c - 1, d - 1) \mid f(c - 1, d + 1)\}.
\end{align*}
\]
Toad versus Frog

One toad meet one frog:

\[
\{d - 2 \mid d + 2 \}
\]

where \(d = b - a\) is the difference. In general, if there is \(c\) spaces between two creatures, we have

\[
f(c, d) = \{f(c - 1, d - 1) \mid f(c - 1, d + 1)\}.
\]
An example

Who wins the following game? What are Left’s winning moves?

[Diagram of a grid with frogs on it]
An example

Who wins the following game? What are Left’s winning moves?

\[ 1 + * + \left( -\frac{1}{2} \right) + 0 + \left( -\frac{1}{2} \right) = *. \]

First player wins by moving in the second lane.
Theorem: The Toads-and-Frogs game \((TF)^x T \boxplus (TF)^n F\) has the value \(n. \uparrow + (n + 1).\star\) for all \(n \geq 0\), independent of \(x\).
**Theorem:** The Toads-and-Frogs game \((TF)^xT\Box(TF)^nF\) has the value \(n \uparrow + (n + 1)_*\) for all \(n \geq 0\), independent of \(x\).

**Proof:** We will prove by induction on \(n\). Initial case \(n = 0\), it is easy to show that the value is \(*\) (by another induction on \(x\)).
More Toads-and-Frogs

**Theorem:** The Toads-and-Frogs game $(TF)^xT\square(\overline{TF})^nF$ has the value $n. \uparrow + (n + 1). \ast$ for all $n \geq 0$, independent of $x$.

**Proof:** We will prove by induction on $n$. Initial case $n = 0$, it is easy to show that the value is $\ast$ (by another induction on $x$).

Inductive step:

$$(TF)^xT\square(\overline{TF})^nF = \{ 0 \mid (TF)^{x+1}T\square(\overline{TF})^{n-1}F \}$$

$$= \{ 0 \mid (n - 1). \uparrow + n. \ast \}$$

$$= n. \uparrow + (n + 1). \ast.$$
Another example

Who wins the following game? What are Left’s winning moves?

[Diagram of a game board with frogs arranged in a grid]
Another example

Who wins the following game? What are Left’s winning moves?

The value are

\[ * + 0 + \uparrow + \uparrow = \uparrow * . \]

Left wins. Left’s winning move is moving in the first lane.