Math576 Combinatorial Game Theory
Homework 4 Solution

1. Simplify the following game values.
   (a) \{0,\downarrow 0,\ast,\uparrow\}
   (b) \{\ast | 0,\ast\}
   (c) \{\downarrow,0 | 0,\uparrow\ast\}
   (d) \{\uparrow\uparrow|\}
   (e) \{0,\ast | \ast2,\uparrow\}

   Solution:
   \{0,\downarrow 0,\ast,\uparrow\} = \{0 | 0,\ast\} = \downarrow *, \text{ deleting dominated option.}
   \{\ast | 0,\ast\} = \downarrow \text{ bypassing right reversible move } *.\n   \{\downarrow,0 | 0,\uparrow\ast\} = \{0 | 0,\uparrow\ast\} = *, \text{ adding gifted horse } \uparrow\ast \text{ to } * = \{0,0\}.
   \{\uparrow\uparrow|\} = \{0,\uparrow\uparrow|\} = 3, \uparrow \text{ is a gifted horse to } \{0 | \uparrow\} = 3, \uparrow \ast .
   \{0,\ast | \ast2,\uparrow\} = \{0,\ast | \ast2\} = \{0 | \ast2\} = \uparrow \ast3, \ast \text{ is the gift horse for } \uparrow \ast3.

2. Prove that \{\ast2|0\} = \downarrow \ast3.
   Proof \downarrow \ast3 = \{\ast | 0\} + \{0,\ast,\ast2 | 0,\ast,\ast2\}
   = \{\ast2,\downarrow,\ast,\downarrow \ast2 | \ast3,\downarrow,\ast,\downarrow \ast2\}
   = \{\ast2 | \downarrow,\ast,\downarrow \ast2\} \text{ deleting dominated options}
   = \{\ast3|0\} \text{ deleting two gift horse options or bypassing them.}

3. Two player play the Kayles game. There are three blocks of pins with sizes 7, 8, 9 respectively. What is the game value? What is the winning move for the first player?
   Solution: The game value is
   \[\ast2 + \ast + \ast4 = \ast7.\]
   The first player can win by knocking down the second pins in the block of 9, i.e, breaking into two blocks of size 1 and size 7. Now the game value is
   \[\ast2 + \ast + (\ast + \ast2) = 0.\]
   So this player wins.

4. Find the nim sequence for the substraction game \(S(2,3,6,8)\). What is the period of this nim sequence?
   Solution: The substraction game \(S(2,3,6,8)\) has the nim sequence
   \[00112031220312\]
   with period 14.
5. Explain what rules is for the Take-and-Break Games .34, then find the nim sequence.

Solution: The Take-and-Break Game .34 means

- The player can take away the heap consisting of one bean.
- The player can take away one bean away from the top of the heap more than one bean.
- The player can take away 2 beans from any heap more than 4 beans and split the remaining of the heap into two non-empty heaps.

The nim sequence is

\[ 0.10120103121203 \]

with period 8 starting at \( n = 7 \).