Math576 Combinatorial Game Theory
Homework 3 Solution

1. Simplify the following game values.
   
   (a) \{0, *, 3, 4 | 0, *, 3, 4\}
   
   (b) \(*3 + *5 + *7\)
   
   (c) \{0, 1 | 1, 2\}
   
   (d) \{-\frac{1}{2}, * | *\}
   
   (e) \{-1 + * | -1 + *\}

Solution:

\{0, *, 3, 4 | 0, *, 3, 4\} = *2, \quad \text{by Mex rule.}

\(*3 + *5 + *7 = *\), \quad \text{by nim addition}

\{0, 1 | 1, 2\} = \{1 | 1\} = 1 + * = 1*

\{-\frac{1}{2}, * | *\} = \{* | *\} = 0. \quad \text{delete dominated option}

\{1 + * | 1 + *\} = 1 + \{* | *\} = 1 + 0 = 1.

2. Find the value of the following Col game:

\[\begin{array}{ccc}
& \bullet & \bullet \\
& \bullet & \bigcirc \\
& \bigcirc & \bigcirc \\
\end{array}\]

Solution: Left colors one of the first three vertices and results three Left options:

\[\begin{array}{ccc}
& \bullet & \bigcirc \\
& \bullet & \bigcirc \\
& \bigcirc & \bigcirc \\
\end{array}\]

\[-\frac{1}{2} & -1 & 1 + (-1) = 0\]

Right can color the third or fourth vertex and results two Right options:

\[\begin{array}{ccc}
& 1 & 2 \\
\end{array}\]

Thus, this Col game value is

\{-1, -\frac{1}{2}, 0 | 1, 2\} = \{0 | 1\} = \frac{1}{2}.

3. Find the value of the following green Hackenbush game.
The final answer is 0.

4. Two players are playing the Nim game with the following heaps:

\[ 2, 5, 6, 11. \]
• What is the game value of the current position?
• What is the winning move of the first player?

**Solution:** Write the numbers in base 2 and compute the nim sum.

\[
\begin{align*}
1 0 &= 2 \\
1 0 1 &= 5 \\
1 1 0 &= 6 \\
1 0 1 1 &= 11 \\
1 0 1 0 &= 10
\end{align*}
\]

The game value is
\[\ast 2 + \ast 5 + \ast 6 + \ast 11 = \ast 10.\]

The winning move for the first player is 11 → 1, which restores the game value to 0.

5. In the White Knight game, the Knight is at position g4 with a baggage of a Nim-heap of height 1.

• What is the game value of the current position?
• What is the winning move of the first player?

**Solution:** The g4 position contributes \(\ast 2\) so the total value is
\[\ast 2 + \ast = \ast 3.\]

The first player needs to restore this value to 0. He moves the Knight to h2 so the new game value becomes
\[\ast + \ast = 0.\]