Math 576 Combinatorial Game Theory
Homework 2 solution

1. Let $C(2, n)$ be the game value of the rectangle $2 \times n$ in the Cut Cake game. Prove $C(2, n) = \lfloor \frac{n}{2} \rfloor - 1$ for all $n \geq 2$.

**Proof:** We use induction on $n$.

Initial cases, $n = 1, 2$. We have

\[
C(2, 1) = \{ | 2C(1, 1) \} = \{ | 0 \} = -1;
\]
\[
C(2, 2) = \{2C(2, 1) \mid 2C(1, 2)\} = \{-2 \mid 2\} = 0.
\]

Now we assume that $C(2, k) = \lfloor \frac{k}{2} \rfloor - 1$ for all $1 \leq k \leq n + 1$. For $n + 1$, the left option for $C(2, n + 1)$ are

\[
C(2, k) + C(2, n + 1 - k)
\]

for some $1 \leq k \leq n$. By inductive hypothesis, it has the value

\[
\lfloor \frac{k}{2} \rfloor - 1 + \lfloor \frac{n + 1 - k}{2} \rfloor - 1 = \begin{cases} 
\lfloor \frac{n+1}{2} \rfloor - 2 & \text{if } k \text{ is even} \\
\lfloor \frac{n+1}{2} \rfloor - 3 & \text{if } k \text{ is odd}
\end{cases}
\]

The maximum value of left options is $\lfloor \frac{n+1}{2} \rfloor - 2$. The right option has value

\[
2C(1, n + 1) = 2n.
\]

We have

\[
C(2, n) = \{ \lfloor \frac{n+1}{2} \rfloor - 2 | 2n\} = \lfloor \frac{n+1}{2} \rfloor - 1.
\]

The inductive proof is finished.

2. Two players are playing the cut cake game. The current game position consists of three rectangles: $8 \times 4$, $5 \times 3$, $3 \times 8$. What is the game value? If it is Left’s turn now, what is his best move?

**Solution:** Note that $C(8, 4) = -1$, $C(5, 3) = -1$, and $C(3, 8) = 3$. The game value is

\[
(-1) + (-1) + 3 = 1.
\]

One of Left’s best move is cut the third cake in half. After his move, the game value became:

\[
(-1) + (-1) + 1 + 1 = 0.
\]

Left can win.
3. Find the game value of the rectangle $990 \times 448$ in the Maundy Cake game.

**Solution:** Write

$990 \rightarrow 495 \rightarrow 165 \rightarrow 55 \rightarrow 11 \rightarrow 1$

$448 \rightarrow 224 \rightarrow 112 \rightarrow 56 \rightarrow 28 \rightarrow 14 \rightarrow 7 \rightarrow 1.$

By the theorem we proved in the lecture, this Maundy Cake game has the value

$7 + 1 = 8.$

4. For each case, find a pair of two fuzzy games $G$ and $H$ so that

- $G + H > 0.$
- $G + H = 0.$
- $G + H < 0.$
- $G + H \parallel 0.$

**Solution:** Consider the following Hackenbush games:

5. Two players are playing the Cut Cake game over an non-rectangle cake. What’s the game value of the following cake?
Solution: There are two options of Left.

Below are two options of Right.

We calculate the following game values:

\[
\begin{align*}
\{-1 \mid 1\} &= 0 & \{-2 \mid 0\} &= -1 & \{0 \mid 2\} &= 1 & \{0 \mid 2\} &= 1
\end{align*}
\]

Therefore, the game value of the original board is

\[
\{(-1) + 0, (-1) + 0 \mid 0 + 1, 1 + 0\} = \{-1 \mid 1\} = 0.
\]