Constructing integral uniform flows in symmetric networks with application to the edge-forwarding index problem*

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August 29, 1998

Abstract

We study the integral uniform (multicommodity) flow problem in a graph G and construct a fractional solution whose properties are invariant under the action of a group of automorphisms $\Gamma < Aut(G)$. The fractional solution is shown to be close to an integral solution (depending on properties of Γ), and in particular becomes an integral solution for a class of graphs containing Cayley graphs. As an application we estimate asymptotically (up to additive error terms) the edge congestion of an optimal integral uniform flow (edge forwarding index) in the cube connected cycles and the butterfly.

*The research of the first author was supported by NSF grant CCR-9528228. Research of the second author was supported in part by the Hungarian NSF contracts T 016 358 and T 019 367, and by the NSF contract DMS 970 1211.

1 Introduction

The uniform concurrent multicommodity flow (uniform flow) problem [14, 17] is the problem of supplying one unit of (fractional) flow between all ordered pairs of vertices in a graph; the objective is to minimize the largest flow through any edge which is called the *congestion*. The integral version of this problem has been studied under the name *edge forwarding index* [2, 8, 9, 16], and calls for the assignment of one path per each ordered pair of vertices to minimize the congestion. The (fractional) uniform flow problem is known to be solvable in polynomial time [17], and starting from the work of Shahrokhi and Matula [17], there have been a series of papers on how to approximately solve this problem faster [1], [11], [6], and [10]. The integral version is known to be NP-hard [3, 16].

There is a need for estimating the value of the congestion, since many important graph theoretical parameters are related to the congestion. For instance the congestion of a uniform flow provides for lower bounds for the bisection width [13, 21], and expansion (isoperimetric) rates [20, 21], whereas, the congestion of an integral uniform flow, or the forwarding index, provides for lower bounds for the the crossing number [13, 21, 22, 24, 25]. Moreover, the close connection between the integral multicommodity flow problems and packet routing was discovered by Leighton, Maggs, and Rao [12], who showed the existence of a near optimal offline schedule for routing the packets on a set of paths involving a near optimal solution to the integral multicommodity flow problem.

In this paper (Section 3), we present an algorithm for constructing uniform flows which exhibit invariance under the action of a group of automorphisms of the graph (Theorem 1). The uniform flows constructed here are shown to be "near integral", in the sense that the number of paths hosting flow are bounded as a function of the order of the stabilizer of a two-tuple of vertices in the group. In particular, for a class of graphs containing Cayley graphs, the constructed flow is integral. Previously, we have been able to construct invariant uniform flows [20, 21]. Our previous methods, however, would construct uniform flows which use too many paths, and hence were very far from being an integral uniform flow. In [8, 19] the integral uniform flows were constructed only for Cayley graphs. These constructions could be shown to be edge-optimal when the underlying Cayley graphs were also edge-transitive. Our general construction in the present paper (Theorem 1) implies all these previous ad-hoc results. Moreover, for the class of orbit proportional graphs which contains all edge-transitive graphs, defined in Section 3, the algorithm is shown to construct an optimal (fractional) flow (Theorem 2). Our method also provides formulas for the congestion of the optimal uniform flow.

There is a continuing interest in computing the edge-forwarding index of specific families of graphs, especially of those which occur as architectures in parallel computing. Recently Gauyacq [5] computed closed bounds for the edge-forwarding index of star graphs and some related families, which are close enough to yield asymptotic formula when the parameter (dimension) of the graph family approaches infinity.

Sections 4 and 5 are devoted to the forwarding indices of cube connected cycles and butterflies. These quantities were not computed before in the literature. Section 4 is a technical section and is dedicated to the structure of butterflies (with wrap around) and cube connected cycles, which are among the popular architectures in parallel computing. It is shown there that the cube connected cycles is orbit proportional, whereas the butterfly is not.

Section 5 is a very important part of the paper, where we obtain asymptotic formulas for the edge-forwarding indices of cube connected cycles and butterflies. Our general construction is shown to provide for an optimal integral flow for the cube connected cycles. The fact that the cube connected cycles is orbit proportional is crucial in verifying the optimality of the construction, and the optimality result can not be proved using other techniques in the forwarding index literature [8]. We apply then a probabilistic analysis to this optimal integral flow in order to obtain asymptotic formula for the optimal congestion in n-dimensional cube connected cycles. The value of the congestion turns out to be $\frac{5}{4}n^2 2^n(1-o(1))$ asymptotically.

For the butterfly which is not orbit proportional, neither the general construction presented here, nor the previous ad-hoc constructions [8, 19] can be shown to provide for an optimal solution for the edge-forwarding index. (These constructed integral flows can be shown to have a congestion which is within a multiplicative factor of 2 from the optimal congestion.) Nonetheless, using the properties of the invariant flows and some observations from the duality of the linear programming, we construct an integral flow whose congestion is asymptotically equal to that of the optimal flow, $\frac{5}{4}n^2 2^{n-1}(1+o(1))$.

Section 6 sketches an application of our formula for the edge forwarding index of the butterfly by providing a lower bound of $(\frac{16}{125} - o(1))4^n$, on its crossing number. As far as we know our multiplicative constant is the best currently known.

This paper is based on the technical report [18] from 1991, a preliminary conference version of the paper with incomplete proofs was published in [23].

2 Definitions

Let $G =< V, E >$ be a connected simple finite graph. Let $L(p)$ denote the number of edges in the path p and let $L(i, j)$ denote the length of the *shortest* ij path. (We preserve the word *distance* for something else.) We set $L = \sum_{(i,j)\in V\times V} L(i, j)$. Let p be a path with end vertices a and b in the graph $G =< V, E >$. Then p will give rise to an oriented path from a to b; and to another from b to a. For any ordered pair of vertices $(i, j) \in V \times V$, we denote by P_{ij} the set of all oriented paths from i to j; any $p \in P_{ij}$ is termed an ij path. Let $P = \cup_{(i,j)\in V\times V} P_{ij}$ be the set of all oriented paths. Throughout this paper the term path means oriented path, unless stated otherwise. For $e \in E$, let P_e denote the collection of all paths containing e. Finally, let R^+ denote the set of non-negative real numbers.

A *uniform concurrent multicommodity flow* (shortly *uniform flow*) f is a function $f : P \to R^+$, such that $\sum_{p\in P_{ij}} f(p) = 1$ for any $(i, j) \in V \times V, i \neq j$.

We call $f(p)$ the flow on path p; if $f(p) > 0$, then p is called an *active path.* We set $f(e) = \sum_{p\in P_e} f(p)$ for any edge $e = xy$, and call $f(e)$ the flow on the edge e. For a uniform flow f we denote $\max_{e\in E} f(e)$ by μ_f and call μ_f the *congestion* of f. A uniform flow f is called *integral*, if $f(p) = 1$ for any active path p. Let μ_G denote the smallest congestion achieved by a uniform flow in G. A uniform flow f in $G =< V, E >$ is *edge optimal*, if $\mu_f = \mu_G$. An edge optimal uniform flow can be computed using a node-arc form linear program [4] in polynomial time. Computing the integral versions of the multicommodity flows and uniform flows have been known to be NP-hard [3].

A *distance function* [15] on $G =< V, E >$ is a function $d : E \to R^+$ such that $d(e) > 0$ for at least one edge e. For any path $p \in P$, we define

$d(p) = \sum_{e \in p} d(e)$, moreover for any $(i,j) \in V \times V$, we define $d(i,j) = d(j,i) = \min\{d(p) : p \in P_{ij}\}$ for all $(i,j) \in V \times V$. We further assume that $d(i,i) = 0$, for any $i \in V$. We term $d(p)$ the *distance* of path p. We define the *distance congestion* μ_d for the distance function d by

$$\mu_d = \frac{\sum_{(i,j) \in V \times V} d(i,j)}{\sum_{e \in E} d(e)}.$$

3 Uniform flows and graph symmetries

Our reference to algebraic graph theory is [26]. It is well known that the set of all permutations on V constitute a group on V which is called the *automorphism group* of G. Let $Aut(G)$ denote the automorphism group of G, and let Γ be a subgroup of $Aut(G)$, then we write $\Gamma < Aut(G)$. Note that the action of any $\Gamma < Aut(G)$ on E partitions E into equivalent classes. We call each class a Γ-*edge orbit*.

A uniform flow f is called a Γ-*invariant* [20, 21] if for any $g \in \Gamma$ and any $p \in P$, we have $f(p) = f(g(p))$. Next, we show how to construct an invariant uniform flow in which the number of active paths depends on the structure of Γ, thus in certain desirable cases which includes Cayley graphs we will have integral uniform flows. Let $(a,b) \in V \times V$, we define the ab stabilizer of Γ, denoted by Γ_{ab} to be the set of all automorphisms in Γ which map a to a and b to b. Formally, $\Gamma_{ab} = \{\gamma \in \Gamma | \gamma(a) = a, \gamma(b) = b\}$. For $p \in P$, and $(a,b) \in V \times V$, let $\Gamma(p)$ and $\Gamma(a,b)$, denote $\{\gamma(p) | \gamma \in \Gamma\}$, and $\{(\gamma(a), \gamma(b)) | \gamma \in \Gamma\}$.

Theorem 1 *A Γ-invariant uniform flow f^* in G can be computed in a polynomial time of $|V|$ and $|\Gamma|$ so that the number of active paths for any pair $(a,b) \in V \times V$ is at most $|\Gamma_{ab}|$. Moreover, any active ab path p has $L(p) = L(a,b)$, and $\mu_{f^*} \leq \frac{L}{|E_1|}$, where E_1 is the smallest Γ-edge orbit.*

Proof. The action of Γ partitions $V \times V$ into equivalence classes $R^1, R^2, ..., R^k$; thus (a,b) and (c,d) are in the same equivalent class, if $c = \gamma(a)$, and $d = \gamma(b)$, for some $\gamma \in \Gamma$. Moreover, for any $(a,b) \in V \times V$, and any two shortest ab paths, p_1, p_2 define

$$p_1 \; R_{ab} \; p_2, \quad \text{iff} \quad p_2 = \gamma(p_1) \;\; \text{for some } \gamma \in \Gamma.$$

It is easily seen that, for any $(a, b) \in V \times V$, R_{ab} is an equivalence relation on the set of shortest ab paths; let R_{ab}^p denote the equivalence class containing the shortest ab path p and note that $|R_{ab}^p| \leq |\Gamma_{ab}|$.

We now describe the construction of f^* in each R^i. For $i = 1, 2, ..., k$, select a vertex pair (a_i, b_i) in R^i, and also select one shortest path p_i from a_i to b_i. Define for $i = 1, 2, ..., k$,

$$f^*(p) = \begin{cases} \frac{1}{|R_{ab}^{p_i}|}, & \text{if } p \in \Gamma(p_i), \\ 0, & \text{otherwise.} \end{cases}$$

The claim regarding the time complexity is easy to verify. Moreover, the invariance of f^*, and the claim regarding the number of active paths are direct consequence of the construction. Now let $(a, b) \in R^i, i = 1, 2, ..., k$ and note that $\sum_{p \in P_{ab}} f^*(p) = \sum_{p \in P_{a_i b_i}} f^*(p) = |R_{a_i b_i}^{p_i}| \frac{1}{|R_{a_i b_i}^{p_i}|} = 1$. Finally, the upper bound on the congestion follows by observing that any two edges in E_1 must host the same amount of flow, and applying a simple averaging argument same as in [21]. □

Note that in any edge transitive graph G, with $\Gamma = Aut(G)$, we have $E = E_1$, and indeed in this case f^* is edge optimal with $\mu_{f^*} = \frac{L}{|E|}$, since by the duality theory of linear programming [15] $\frac{L}{|E|}$ is a lower bound on the congestion of any uniform flow. Moreover, when G is a Cayley graph, $|\Gamma_{ab}| = 1$, hence f^* is an integral uniform flow (it has exactly one active path per vertex pair) and can be used for packet routing. Indeed in this case the general construction in Theorem 1 implies our previous ad-hoc results in Theorems 3.2, and 3.3 of [19] for off line computation of packet routes.

For $G =< V, E >$ and $\Gamma < Aut(G)$, let $\{E_1, E_2, ..., E_k\}$ be the set of Γ-orbits of E. We say that G is Γ-*orbit proportional* (or orbit proportional when the context is clear) if for all $(i, j) \in V \times V$, any ij path p with $L(p) = L(i, j)$ and any ij path q, we have

$$|q \cap E_m| \geq |p \cap E_m|, \ m = 1, 2, ..., k.$$

In order to have examples, we note that any edge transitive graph is orbit proportional with respect to its automorphism group and any tree is orbit proportional with respect to the trivial group.

We previously proved the following Theorem [20].

Theorem 2 *Assume that $G =< V, E >$ is Γ-orbit proportional. Let $\hat{f}$ be a Γ-invariant uniform flow on G such that every ij active path p has $L(p) = L(i, j)$. Then we have:*

(i) $\hat{f}$ is edge optimal in G.

(ii) Assume that $\{E_1, E_2, ..., E_t\}$ is the set of Γ-orbits of E and for any $i = 1, 2, ..., t$, let d_i be a distance function with $d_i(e) = 1$, if $e \in E_i$, and $d_i(e) = 0$, otherwise. Then $\mu_G = \mu_{\hat{f}} = \max_i \mu_{d_i}$.

Observe that the flow f^* constructed in Theorem 1, satisfies the condition of Theorem 2, and hence when G is orbit proportional f^* is edge optimal. Indeed, the Theorems allow to estimate the optimal congestion of f^* in an orbit proportional graph. For instance for Q_k (k-dimension cube), which is edge transitive and hence orbit proportional, Theorems 1, 2 give $\mu_{f^*} = 2^k$. Moreover, since Q_k is a Cayley graph [26], for any vertex pair a, b, Γ_{ab} is the identity, and thus f^* is an optimal integral uniform flow. Finally, as we have shown in [20] the class of vertex transitive orbit proportional graphs is closed under Cartesian product. Hence, the class of orbit proportional graphs for which f^* is edge optimal is fairly large.

4 The structure of cube connected cycles and butterfly

It is well known that the cube connected cycles and the butterfly (with wrap-around) are Cayley graphs with the same underlying group Γ which is the *wreath product* of Z_n and Z_2 but with different generating sets. We exploit this structure in the following. Let $N = \{0, 1, 2, ..., n-1\}$ and $\Theta_n = \{g_{W,i} : W \subseteq N, i \in N\}$. For $i, j \in N$, let $i \oplus j$ denote $i+j$ modulo n. For $U \subseteq N$ and $i \in N$, let $U \oplus i = \{j \oplus i : j \in U\}$. Set $V \triangle U = (V \cup U) \setminus (V \cap U)$. Now Θ_n is a group with identity $g_{\emptyset,0}$ and operations

$$g_{W,t}g_{U,i} = g_{W\triangle(U\oplus t),i\oplus t} \quad \text{and} \quad g^{-1}_{W,t} = g_{W\oplus(n-t),n-t}.$$

The n-dimensional cube-connected cycles CC_n is a Cayley graph over Θ_n with the generating set

$$H = \{g_{\emptyset,1}, g_{\{0\},0}\}.$$

We term the edges produced by the first generator *cyclic edges* and the edges produced by the second generator *cubic edges*. It is easy to see that $CC_n =< V, E >$, where

$$V = \{(W, i) : W \subseteq N, i \in N\}$$

and $(W, i)(U, j) \in E$ if $i = j = W \bigtriangleup U$ (cubic edges in dimension i) or if ($i - j \equiv 1 \mod n$ or $j - i \equiv 1 \mod n$) and $U = V$ (cyclic edges).

Let BB_n denote the n-dimensional butterfly with wrap-around. It is well known that BB_n is a Cayley graph over Θ_n with the generating set $H = \{g_{\emptyset,1}, g_{\{0\},n-1}\}$. We term the edges arising from the first generator *cyclic edges* and the edges arising from the second generator *cubic edges*. It is easy to see that $BB_n =< V, E >$, where

$$V = \{(X, i) : X \subseteq N, i \in N\},$$

and $(X, i)(Y, j) \in E$ if $X = Y$ and ($i - j \equiv 1 \mod n$ or $j - i \equiv 1 \mod n$) (cyclic edges) or $X \bigtriangleup Y = i$ and $j \equiv i - 1 \mod n$ (cubic edges in dimension i).

Let C_n be the cycle on the vertex set $N = \{0, 1, \ldots, n-1\}$ with the edge set $\{0, 1\}, \{1, 2\}, ..., \{n-1, 0\}$. Define C_n^+ be C_n with one loop added at every vertex. For any walk w in CC_n or BB_n, let $Cyclic(w)$ and $Cubic(w)$ denote the multiset of cyclic edges and the multiset of cubic edges, respectively, in w. Any i, $j \in N$ ($i \neq j$) split C_n (C_n^+) into two edge disjoint ij paths. We refer to these paths as *left side* and *right side*, where the vertices of the left side precede, and the vertices of the right side follow i in the cyclic order of N. For convenience, we assume that the right side is the short side and has length $L(i, j)$.

Let i and j be two vertices of C_n^+ (C_n) and $T \subseteq N$. A *gap* induced by T is any ab path p such that (i) $a, b \in T \cup \{i, j\}$, (ii) no intermediate vertex of p is in $T \cup \{i, j\}$. The *length* of any gap is the number of edges in this gap. A *gap* induced by $F \subseteq E(C_n)$ is the gap induced by the set of vertices of edges of F, such that the gap does not use edges of F. For $i \neq j$, it makes sense to speak about gaps on the left side and gaps on the right side.

We analyze the structure of shortest paths in CC_n first. Let $X = (W, i)$ be a vertex of CC_n; the projection of X on C_n^+ is the vertex i. This projection can be extended to the edges and therefore to the walks of CC_n in the following fashion: the cyclic edges of CC_n are projected to the edges of C_n^+, whereas the cubic edges of CC_n are projected to the loops of C_n^+. Given two vertices $X_1 = (U, i)$ and $X_2 = (W, j)$ in CC_n, it is convenient to analyze the structure of any X_1X_2 walk p in CC_n by projecting it on C_n^+ to get an ij walk q in C_n^+. Notice that $L(p) = L(q)$, since each edge or loop contributes one to the length of the walk in C_n^+. Let $X_1 = (W, i)$ and $X_2 = (U, j)$ be two distinct vertices of CC_n. Any loop of an ij walk in C_n^+ at a vertex $a \in N$ is called an *essential loop* if $a \in W \triangle U$, otherwise the loop is *non-essential*. An ij walk w in C_n^+ is called an *essential walk*, if w has the following properties: (i) every essential loop of C_n^+ is traversed by w exactly once, and (ii) every non-essential loop of C_n^+ is traversed by w an even number of times.

Lemma 1 *Let $X_1 = (W, i)$ and $X_2 = (U, j)$ be two distinct vertices of CC_n and p be a shortest X_1X_2 path in CC_n that is projecting to a walk q of C_n^+. The following hold:*

(i) Any X_1X_2 walk in CC_n contains an odd number of edges from each dimension $i \in W \triangle U$, and an even number of edges from any dimension $j \notin W \triangle U$.

(ii) Any essential ij walk in C_n^+ is the projection of an X_1X_2 walk in CC_n.

(iii) $|Cubic(p)| = |U \triangle W|$, with one cubic edge in each dimension $i \in U \triangle W$.

(iv) q contains any edge of C_n^+ at most twice.

(v) Assume that $i \neq j$ and let l_1 and l_2 be the lengths of the longest gaps induced by $U \triangle W$ on the right side and the left side of C_n^+, respectively, then, $|Cyclic(p)| = n + \min\{L(i, j) - 2l_1, n - L(i, j) - 2l_2\}$.

(vi) Assume that $i = j$ and let l be the length of the longest gap induced by $U \triangle W$ on C_n^+. Then, $|Cyclic(p)| = \min\{n, 2n - 2l\}$.

Proof. (i) is easy to verify. (ii) can be shown employing induction on $|U \triangle W|$, but we omit the details. To prove (iii), assume to the contrary that the path p violates (iii). Consider the projection of p on C_n^+ which is an ij walk q. Now, delete all non-essential loops in q. Next, for any essential loop of C_n^+ which is traversed more than once by q, delete unnecessary occurrences of the loop so that finally this loop is traversed only once by the walk. The length of the resulting walk is strictly less than length of q and satisfies the conditions of part (ii). Thus there is an X_1X_2 path p' in CC_n

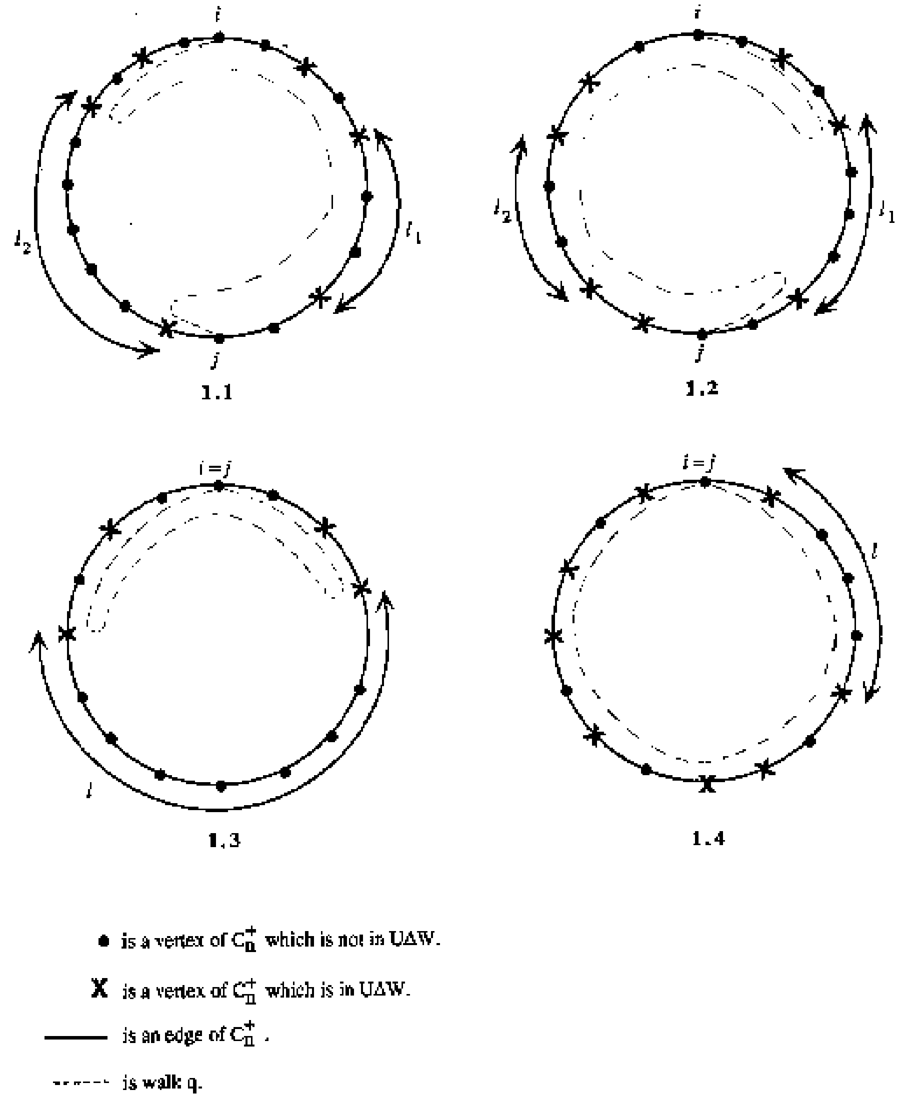

Fig. 1 : Configurations of the ij walk q in Lemma 1

with $L(p') < L(p)$, a contradiction. To prove (iv), observe that q is a shortest ij walk among all ij walks in C_n^+ (in terms of the number of edges) which goes through all essential loops. Since q can be viewed as a topologically continuous curve, it has one of the two configurations illustrated in Figures 1.1 and 1.2, if $i \neq j$. Similarly, q has one of the two configurations illustrated in Figures 1.3 and 1.4, if $i = j$. This proves (iv). Finally, to show (v) observe from Figures 1.1 and 1.2 that the portion of C_n which is not traversed by q is a gap induced by $U \triangle W$. However, q can be viewed as an ij shortest walk in C_n containing all vertices in $U \triangle W$; thus, this gap is a longest gap. Now if this gap is located on the left side of C_n, then,

$$|Cyclic(p)| + l_1 = n + L(i,j) - l_1,$$

otherwise,

$$|Cyclic(p)| + l_2 = n + (n - L(i,j)) - l_2,$$

it follows that

$$|Cyclic(p)| = n + \min\{L(i,j) - 2l_1, n - L(i,j) - 2l_2\}.$$

The proof of (vi) is derived using Figures 1.3 and 1.4 in a similar fashion to (v). $\square$

Now we continue with the structure of shortest paths in the butterfly. A walk w in C_n is called a *labeled walk*, if the edges in w are labeled cubic or cyclic. If an edge is contained more than once in w, we allow different labels at different occurrences of the edge. Let $X = (W, i)$ be a vertex of BB_n; the projection of X on C_n is defined to be the vertex i of C_n. Given two vertices $X_1 = (U, i)$ and $X_2 = (W, j)$ in BB_n, it is convenient to analyze the structure of any X_1X_2 walk p in BB_n by projecting it on C_n to get a labeled ij walk q in C_n. Any edge of q which is the projection of a cyclic edge of p is labeled *cyclic*, any edge of q which projection of a cubic edge of p is labeled *cubic*. Let $X_1 = (W, i), X_2 = (U, j)$ be two distinct vertices in BB_n; an edge $(i, i{\oplus}1)$ in C_n is called *essential*, if $i{\oplus}1 \in W \triangle U$. A labeled ij walk w in C_n is called an *essential walk*, if it has the following properties: (i) every essential edge is assigned the cubic label exactly once in w, and (ii) the number of occurrences of any non-essential edge with cubic label in w is even. Note that an essential walk w can use an essential edge e several times with cyclic label, as long as e is labeled cubic in w only once.

Lemma 2 *Let $X_1 = (W, i)$ and $X_2 = (U, j)$ be two distinct vertices of BB_n. Assume p is a shortest X_1X_2 path in BB_n projecting to a walk q of C_n. The following hold:*

(i) Let w be the projection of any X_1X_2 walk in BB_n to C_n, then, the number of occurrences of any non-essential edge with cubic label in w is even.

(ii) Any ij essential walk w in C_n is the projection of an X_1X_2 walk in BB_n.

(iii) q is a shortest ij essential walk in C_n.

(iv) q does not use any edge of C_n more than twice.

(v) Assume that $i \neq j$ and let l_1 (l_2) be the lengths of the longest gaps induced by the set of edges $\{(m, m\oplus 1) : m\oplus 1 \in U \triangle W\}$ on the right (left) side. Then, $L(p) = L(q) = n + \min(L(i,j) - 2l_1, n - L(i,j) - 2l_2)$.

(vi) Assume that $i = j$, and let l be the length of the longest gap induced by the set of edges $\{(m, m\oplus 1) : m\oplus 1 \in U \triangle W\}$ on C_n, then $L(p) = \min\{n, 2n - 2l\}$.

Proof. The proof is like the proof of Lemma 1 and is therefore omitted. $\square$

Lemma 3 *CC_n is Θ_n-orbit proportional, while BB_n is not.*

Proof. Note that the edge orbits of CC_n under Θ_n are the set of cyclic edges and the set of cubic edges. Let $X_1 = (W, i)$ and $X_2 = (U, j)$ be two vertices of CC_n. By Lemma 1(iii) any X_1X_2 path p with $L(p) = L(X_1, X_2)$ must have $|Cubic(p)| = |U \triangle W|$. Now assume that p' is any X_1X_2 path in CC_n; by Lemma 1(i) p' must have an odd number of cubic edges from each dimension $i, i \in U \triangle W$ and even number of cubic edges from any dimension $i, i \notin U \triangle W$. Thus, $|Cubic(p')| \geq |Cubic(p)|$. Assume to the contrary that, $|Cyclic(p')| < |Cyclic(p)|$, and consider q' the projection of p' on C_n^+. Then, q' can be converted to an essential ij walk $\hat{q}$ in C_n^+ by applying the method in Lemma 1(iii) to remove the unnecessary essential and non-essential loops. For the path $\hat{p}$ whose projection is $\hat{q}$ we have, $|Cubic(\hat{p})| = |U \triangle W| =$

$|Cubic(p)|$, and $|Cyclic(\hat{p})| = |Cyclic(p')| < |Cyclic(p)|$. It follows that $L(\hat{p}) < L(p)$, a contradiction.

To show that BB_n is not orbit proportional, take BB_4, $X_1 = (\{2,4\},1)$, $X_2 = (\emptyset, 2)$. Consider two shortest X_1X_2 paths p_1 and p_2,

$$p_1 : (\{2,4\},1)\text{cubic}(\{4\},2)\text{cyclic}(\{4\},3)\text{cubic}(\emptyset,4)\text{cyclic}(\emptyset,3)\text{cyclic}(\emptyset,2)$$

$$p_2 : (\{2,4\},1)\text{cubic}(\{4\},2)\text{cubic}(\{3,4\},3)\text{cubic}(\{3\},4)\text{cyclic}(\{3\},3)\text{cubic}(\emptyset,2).$$

Observe that p_1 has 2 cubic and 3 cyclic edges, while p_2 has 1 cyclic and 4 cubic edges; thus BB_4 is not Θ_n-orbit proportional.□

Lemma 4 *Assume $U \subseteq N$ is chosen randomly with uniform distribution. Then,*

(i) $Prob(\Big||U| - \frac{n}{2}\Big| \leq n^{2/3}) = 1 - o(1)$.

(ii) $Prob($ length of the longest gap induced by U on $C_n < \log^2 n) = 1 - o(1)$.

(iii) Let E' be a random subset of edges of C_n chosen with the probability 2^{-n}. Then

$$Prob(\text{ length of the longest gap induced by } E' \text{ on } C_n < \log^2 n) = 1 - o(1).$$

(iv) Assume that p is any ij path in C_n. Then

$$Prob(||p \cap U| - \frac{L(p)}{2}| \leq n^{2/3}) = 1 - o(1).$$

Proof. (i) follows from the normal convergence of the binomial distribution. Conditioning on the event $||U| - \frac{n}{2}| \leq n^{2/3}$, we overestimate the probability in (ii) if we take $n/2 - O(n^{2/3})$ independent samples from the vertices of C_n and ask for the probability of obtaining at least one gap of length at least $\log^2 n$. This modified probability is at most

$$n\left(1 - \frac{\log^2 n}{n}\right)^{n/2 - O(n^{2/3})} = o(1).$$

To verify (iii) observe that $Prob(\exists \text{gap} \geq x) \leq 2^{-x}$. Now, let $x = 1 + \log^2 n$ to finish the proof of (iii). Finally, to prove (iv) observe that

$$Prob(||p \cap (U \triangle W)| - L(p)/2| > n^{2/3}) = \frac{2\sum_{t > L_{i,j}/2 + n^{2/3}} \binom{L(i,j)}{t}}{2^{L(i,j)}} = o(1).$$

$\square$

5 Optimal integral uniform flows in CC_n and BB_n

To estimate the congestion of an optimal integral flow in CC_n and BB_n, we will use probabilistic methods. It should be noted that although the tools involve usage of probability, the final outcome is completely deterministic and does not involve probability.

Theorem 3 *For $CC_n =< V, E >$, there exists an edge optimal integral uniform flow f, such that $\mu_f = \frac{5}{4} n^2 2^n (1 - o(1))$.*

Proof. Since CC_n is a Cayley graph, our construction in Theorem 1 gives an integral uniform flow. By Lemma 3 CC_n is orbit proportional, hence by Theorem 2(i) the flow f is edge optimal. To evaluate μ_f we use Theorem 2(ii). Define, $d_1(e) = 0$, if e is cyclic and $d_1(e) = 1$, if e is cubic. Similarly, define $d_2(e) = 0$, if e is cubic, and $d_2(e) = 1$, if e is cyclic.

By Theorem 2(ii) we have, $\mu_f = \max(\mu_{d_1}, \mu_{d_2})$. Note that, for $X_1 = (U, i) \in V$ and $X_2 = (W, j) \in V$, by of Lemma 1(iii) we have, $d_1(X_1, X_2) = U \triangle W$. It is easy to see that $\sum_{U \subseteq N} \sum_{W \subseteq N} |U \triangle W| = \frac{n}{2} 4^n$ and hence that

$$\sum_{(X_1, X_2) \in V \times V} d_1(X_1, X_2) = \frac{n^3}{2} 4^n, \tag{1}$$

thus, $\mu_{d_1} = \frac{\sum_{(X_1, X_2) \in V \times V} d_1(X_1, X_2)}{\sum_{e \in E} d_1(e)} = \frac{n^2 4^n n/2}{n 2^{n-1}} = n^2 2^n$. Let $X_0 = (\emptyset, 0) \in V$ and $X = (U, i) \in V$, and assume that p is a shortest $X_0 X$ path. Orbit proportionality implies $d_2(X_0, X) = |Cyclic(p)|$. By Lemma 1(v)-(vi),

$$d_2(X_0, X) = |Cyclic(p)| \leq n + L(0, i). \tag{2}$$

In order to study the distribution of distances $d_j(X_0, X)$, we think about X as a random variable and use facts from probability theory (the normal convergence) to estimate the distribution. Next consider any vertex $X = (U, i)$, such that U is selected randomly with the probability 2^{-n}; we refer to X as a random vertex. For any random vertex $X = (U, i)$, by Lemma 4(i), $||U| - n/2| = o(n)$ with probability $1 - o(1)$. It follows from Lemma 4(ii), that U does not induce any gaps longer than $\log^2 n$ on C_n^+ with probability $1 - o(1)$. Therefore by Lemma 1(v)-(vi) we have,

$$d_2(X_0, X) = |Cyclic(X_0, X)| = (n + L(0, i))(1 - o(1)) \tag{3}$$

with the probability $1 - o(1)$. It follows from (2) and (3) that,

$$\sum_{X=(W,i)\in V} d_2(X_0, X) = (1 - o(1)) \sum_{X=(W,i)\in V} (n + L(0, i)). \tag{4}$$

(The sums in (4) are taken over all vertices!) It is easy to verify that, $\sum_{X=(W,i)\in V}(n + L(0, i)) = \frac{5}{4}n^2 2^n(1 - o(1))$, therefore, $\sum_{X\in V} d_2(X_0, X) = \frac{5}{4}n^2 2^n(1 - o(1))$. It easily follows from the vertex transitivity of CC_n that

$$\sum_{(X_1,X_2)\in V\times V} d_2(X_1, X_2) = n2^n \sum_{X\in V} d_2(X_0, X) = \frac{5}{4}n^3 4^n((1 - o(1)). \tag{5}$$

However, $\mu_{d_2} = \sum_{(X_1,X_2)\in V\times V} d_2(X_1, X_2)/(n2^n) = \frac{5}{4}n^2 2^n(1 - o(1)) \geq \mu_{d_1}$, for large n. Therefore, $\mu_f = \frac{5}{4}n^2 2^n(1 - o(1))$. □

Since BB_n is not Θ_n-orbit proportional, the construction of Theorem 1 only gives an integral approximate solution. (Our results in [19] can be used to show that the congestion is within a multiplicative factor of 2 from the optimal.) We will now present an algorithm which computes an integral flow with asymptotically optimal congestion. The key point behind our near-optimal uniform flow for BB_n is a collection of shortest paths, which uses each cyclic edge and each cubic edge about the same times. We note that the complexity of the algorithm is $O(n^3 4^n)$.

Butterfly Flow Algorithm

INPUT: $< V, E >= BB_n$

OUTPUT: An integral uniform flow f.

Let $X_0 \leftarrow (\emptyset, 0)$ and compute a shortest X_0X path q_{X_0X} for every $X \neq X_0$.

For all $X = (A, i) \in V$, $X \neq X_0$ **Do**

Denote by W_{X_0X} the $0i$ walk in C_n which is the projection of q_{X_0X}. (Recall that $L(0, i)$ is the length of a shortest $0i$ path on C_n.)

Case

$L(0, i) < \frac{n}{\sqrt{8}}$: Consider any non-essential edge e which appears in W_{X_0X} with cubic label. Notice that e must appear twice with cubic label in W_{X_0X} and change the label of both occurrences of e in W_{X_0X} to cyclic. Denote this new labeled walk in C_n by W'_{X_0X}. By Lemma 2(i), W'_{X_0X} is a $0i$ essential walk and let h_{X_0X} be an X_0X path in BB_n which projects to W'_{X_0X}.

$L(0, i) > \frac{n}{\sqrt{8}}$: Consider any non-essential edge e which appears twice in W_{X_0X} with cyclic label; change the label of both occurrences of e in W_{X_0X} to cubic. Denote this new walk in C_n by W'_{X_0X}. By Lemma 2(i), W'_{X_0X} is a $0i$ essential walk and let h_{X_0X} be an X_0X path in BB_n which projects to W'_{X_0X}.

EndCase

EndFor

Extend the set of paths $S = \{h_{X_0X} : X \in V, \quad X \neq X_0\}$ to a Θ_n-invariant integral flow using the action of Θ_n. That is, compute $\Theta_n(S)$.

End.

Theorem 4 *The Butterfly Flow Algorithm constructs an integral uniform flow f in $BB_n =< V, E >$ with $\mu_f = \frac{5}{4}n^2 2^{n-1}(1 + o(1))$, which is asymptotically optimal for large n.*

Proof. It is easy to verify that the last step of the algorithm produces a flow f which is integral Θ_n-invariant using shortest paths. Assume that $X = (U, i)$ is a random vertex of BB_n, that is, U is selected randomly with the probability 2^{-n}, when i is arbitrary. Let $X_0 = (\emptyset, 0)$. Consider the X_0X path q_{X_0X} computed at the initial step of the algorithm. Let W_{X_0X} be the projection of q_{X_0X} on C_n, then by Lemma 2(iv)-(v) the portion of C_n which is not traversed by W_{X_0X} must be a (longest) gap. Since U is selected randomly, by Lemma 4(iv) the length of this gap is at most $\log^2 n$ with probability $1 - o(1)$. Therefore this gap is located with probability

$1 - o(1)$ on the shorter side of C_n. This is illustrated in Figure 2,

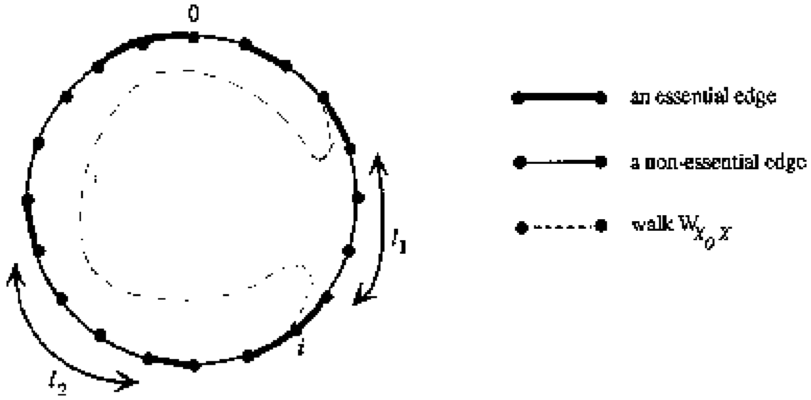

Fig. 2 : Illustration of W_{X_0X} for a random vertex of $X = (U, i)$ of BB_n

in which we have assumed that the right side of the C_n is the shorter side. By Lemma 2, any edge e, which is located on the shorter side of C_n and is contained in W_{X_0X} will appear in W_{X_0X} exactly twice. Also, any edge e located on the long side will appear in W_{X_0X} exactly once. Next we estimate (with probability $1 - o(1)$) the number of essential edges labeled cubic, the number of essential edges labeled cyclic, and the total number of non-essential edges in W_{X_0X}. These values are easily estimated using Lemma 2, Lemma 4, and the topological properties of W_{X_0X} illustrated in Figure 2, and are recorded in the following Table.

	long side of C_n	short side of C_n
Number of essential edges labeled cubic in W_{X_0X}	$\frac{1}{2}(n - L(0, i)) - o(n)$	$\frac{1}{2}L(0, i) - o(n)$
Number of essential edges labeled cyclic in W_{X_0X}	0	$\frac{1}{2}L(0, i) - o(n)$
Total number of non-essential edges in W_{X_0X}	$\frac{1}{2}(n - L(0, i)) - o(n)$	$\frac{1}{2}L(0, i) - o(n)$

Table 1. *Distribution of essential and non-essential edges of W_{X_0X} on different sides of C_n*

If $L(0, i) < n/\sqrt{8}$, the **Case** statement in the algorithm guarantees that any non-essential edge of W_{X_0X} which is located on the short side will be labeled cyclic in W'_{X_0X}. (Notice that any non-essential edge e of W_{X_0X} which is located in the left side appears exactly once in W_{X_0X}. Thus, by Lemma 2(i), e must have been labeled cyclic in W_{X_0X}. Consequently the label of e does not change.) Therefore, employing the last two rows of Table, we will get

$$|Cyclic(h_{X_0X})| = |Cyclic(W_{X_0X})| = ((n/2 + L(0,i))(1 - o(1)), \qquad (6)$$

with probability $1 - o(1)$. Similarly, using the first row of Table , we have

$$|Cubic(h_{X_0X})| = |Cubic(W_{X_0X})| = n/2 - o(n) \qquad (7)$$

with probability $1-o(1)$. Now assume that $L(0,i) > n/\sqrt{8}$, then, the **Case** statement in the algorithm guarantees that any non-essential edge of W_{X_0X} which is located on the short side of C_n will have a cubic label in W'_{X_0X}. Using rows one and three in Table , it is easily shown that with probability $1 - o(1)$

$$|Cubic(h_{X_0X})| = |Cubic(W'_{X_0X})| = n/2 + L(0,i) - o(n),$$

whereas using rows two and three,

$$|Cyclic(h_{X_0X})| = |Cyclic(W'_{X_0X})| = n/2 - o(n), \qquad (8)$$

with probability $1 - o(1)$. Next, we claim that

$$\sum_{X\in V} |Cyclic(h_{X_0X})| = (2-o(1))2^n\{\sum_{l=0}^{\lfloor \frac{n}{\sqrt{8}}\rfloor}(\frac{n}{2}+l)+\sum_{l=\lceil \frac{n}{\sqrt{8}}\rceil}^{\lfloor \frac{n}{2}\rfloor}\frac{n}{2}\} = \frac{5}{4}n^2 2^{n-1}(1+o(1)), \qquad (9)$$

$$\sum_{X\in V} |Cubic(h_{X_0X})| = (2-o(1))2^n\{\sum_{l=0}^{\lfloor \frac{n}{\sqrt{8}}\rfloor}\frac{n}{2}+\sum_{l=\lceil \frac{n}{\sqrt{8}}\rceil}^{\lfloor \frac{n}{2}\rfloor}(\frac{n}{2}+l)\} = \frac{5}{4}n^2 2^{n-1}(1+o(1)). \qquad (10)$$

We now justify (9) and leave (10) to the reader. Consider a random vertex $X = (U,i)$ and let $l = L(0,i)$. If $l < n/\sqrt{8}$, we can count with high accuracy $|Cyclic(h_{X_0X})|$ using (6); likewise, if $l > n/\sqrt{8}$, we can count with high accuracy $|Cyclic(h_{X_0X})|$ using (8). Now observe that there are 2^n choices for the random U, and typically 2 choices for a vertex at distance l from the vertex 0 on C_n. This justifies the existence of two sums and in (9). The evaluation of the sums is just algebra. Our estimates in (9) went through for random vertices. However, the number of cyclic and cubic edges for atypical vertices is not too large either, since the diameter of the butterfly is $O(n)$. The contribution of the neglected case $i = 0$ is negligible. Denote p_{XY} the unique active XY path in f, use the fact that the orbits of Θ_n are the set of cyclic edges and the set of cubic edges to obtain

$$CU = \sum_{(X,Y)} |Cubic(p_{XY})| = n2^n \sum_{X \in V} |Cubic(h_{X_0 X})| = \frac{5}{2} n^3 4^{n-1}(1 + o(1)),$$

$$CY = \sum_{(X,Y)} |Cyclic(p_{XY})| = n2^n \sum_{X \in V} |Cyclic(h_{X_0 X})| = \frac{5}{2} n^3 4^{n-1}(1 + o(1)).$$

Since f is Θ_n-invariant by construction, the value of f on any cyclic or cubic edge is

$$\frac{CY}{n2^n} = \frac{5}{4} n^2 2^{n-1}(1 + o(1)), \quad \text{and} \quad \frac{CU}{n2^n} = \frac{5}{4} n^2 2^{n-1}(1 + o(1)),$$

and $\mu_f = \frac{5}{4} n^2 2^{n-1}(1 + o(1))$. The identically one distance function d yields

$$\sum_{(X_1,X_2) \in V \times V} d(X_1, X_2) = \sum_{(X_1,X_2) \in V \times V} L(X_1, X_2) = CY + CU. \tag{11}$$

Consequently, $\mu_d = \frac{\sum_{(X_1,X_2) \in V \times V} d(X_1,X_2)}{\sum_{e \in E} d(e)} = \frac{CU+CY}{2n2^n} = \frac{5}{4} n^2 2^{n-1}(1 + o(1))$. This verifies the asymptotic edge optimality of f, since by duality theory of linear programming [15] μ_d is a lower bound on the congestion of an optimal flow. □

6 A crossing number result

For the reading of this section we assume that the reader is familiar with the concepts of crossing number, graph embedding [13] and randomized rounding through our paper [21].

Theorem 5 *The crossing number of the butterfly BB_n is at least $(\frac{16}{125} - o(1))4^n$.*

Proof. Let $N = n2^n$, and let $cr(G)$ denote the crossing number of any graph G. One can easily apply the randomized rounding technique (like the proof of Theorem 5.2 in our paper [21]) to obtain an embedding of the complete graph K_N into BB_n with congestion μ which is $(1/2 + o(1))$ times the edge forwarding index of BB_n: for every unordered pair of vertices u, v flip a coin independently to decide if you include the uv or the vu path from the integral uniform flow in the embedding. Using standard results on the crossing number from [13] and [21] $cr(BB_n) \geq \frac{cr(K_N)}{\mu^2} - 16n2^{n-1}$. Finally, use from [27] that for the crossing number of the complete graph K_N we have $cr(K_N) \geq (\frac{1}{80} - o(1))N^4$. Merging all these results we obtain $cr(BB_n) \geq (\frac{16}{125} - o(1))4^n$. □

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