Towards a Katona type proof for the 2-intersecting Erdős-Ko-Rado theorem*

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Abstract

We study the possibility of the existence of a Katona type proof for the Erdős-Ko-Rado theorem for 2- and 3-intersecting families of sets. An Erdős-Ko-Rado type theorem for 2-intersecting integer arithmetic progressions and a model theoretic argument show that such an approach works in the 2-intersecting case.

1 Introduction

One of the basic results in extremal set theory is the Erdős-Ko-Rado (EKR) theorem [7]: if $\mathcal{F}$ is an *intersecting* family of k-element subsets of an n-element set (i.e. every two members of $\mathcal{F}$ have at least one element in common) and $n \geq 2k$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and this bound is attained. A similar result holds for t-intersecting k-element subsets (Wilson, [16]): if $n \geq (k-t+1)(t+1)$ and $\mathcal{F}$ is a t-intersecting family, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.

The simplest proof of the Erdős-Ko-Rado theorem is due to Katona [13]. This proof allows for a strengthening of the Erdős-Ko-Rado theorem, the Bollobás inequality [3]: If $\mathcal{F}$ is family of

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subsets of an n element set so that they pairwise intersect and none of them contains another as a subset, then

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{f_i}{\binom{n-1}{i-1}} \leq 1, \tag{1}$$

where f_i is the number of i-element sets in $\mathcal{F}$.

Péter Erdős, Ulrich Faigle and Walter Kern [8] came up with a general framework for group-theoretical proofs of Erdős-Ko-Rado type theorems and Bollobás type inequalities that generalizes the celebrated cyclic permutation proof of Katona for the classic Erdős-Ko-Rado theorem. They explicitly asked for t-intersecting generalization of their method. The present work was strongly motivated by their paper.

A Katona type proof has not been discovered yet for t-intersecting families and no Bollobás inequality is known for t-intersecting families. The present paper makes one step forward toward such extensions. We give a formal generalization of Katona's proof from the natural permutation group representation of the cyclic group to sharply t-transitive permutation groups. To make sure that the formal generalization actually works, an extra condition is needed. Then we study how this extra condition for the case $t = 2$, formulated for finite fields, can be stated for 2-intersecting integer arithmetic progressions, and then using the truth of the latter version, we show the existence of a Katona type proof for the case $t = 2$, for infinitely many pairs (n, k) by model theoretic arguments. We also study possible Bollobás inequalities.

A permutation group acting on an n-element set is *t-transitive*, if any ordered t-set of vertices is mapped to any ordered t-set of vertices by a group element, and is *sharply t-transitive* if it can be done by only a single group element. Infinite families of sharply 2- and 3-transitive permutation groups exist, but only finitely many such groups exist for $n > t \geq 4$.

For any prime power q, sharply 2-transitive permutation groups do act on q vertices, and they have been classified by Zassenhaus [17], see also [6]. One of those groups is the *affine linear group over $GF(q)$*, that is, the group of linear functions $f = ax + b : \ GF(q) \to GF(q)$ for composition with $a \neq 0$. In this paper we consider *this* sharply 2-transitive permutation group only.

The non-constant fractional linear transformations $x \to \frac{ax+b}{cx+d}$ $(a, b, c, d \in GF(q))$ form a group for composition and permute $GF(q) \cup \{\infty\}$ under the usual arithmetic rules and act sharply 3-transitively. Group elements fixing the infinity are exactly the linear transformations.

In Katona's original proof the action of a cyclic permutation group is sharply 1-transitive. Katona needed an additional fact, which is often called Katona's Lemma. As a reminder, we recall Katona's Lemma in an algebraic disguise (cf. [14, Ex. 13.28(a)]):

Lemma 1.1 *Consider the cyclic group Z_n with generator g. Assume $k \leq n/2$, and let $K = \{g, g^2, \ldots, g^k\}$. If for distinct group elements $g_1, g_2, \ldots, g_m \in Z_n$ the sets $g_i(K)$ are pairwise intersecting, then $m \leq k$.* □

The major difficulty that we face is how to find analogues of Katona's Lemma for sharply 2- and 3-transitive permutation group actions.

2 Katona's proof revisited

Theorem 2.1 *Let us be given a sharply t-transitive permutation group Γ acting on a set X with $|X| = n$. Assume that there exists a $Y \subseteq X$ with $|Y| = k$ such that*

$$\text{for distinct group elements } \phi_1, \phi_2, \dots, \phi_m \in \Gamma,$$

$$\text{if for all } i, j\ |\phi_i(Y) \cap \phi_j(Y)| \geq t, \text{ then } m \leq \tfrac{k!}{(k-t)!}. \tag{2}$$

Then for any t-intersecting family $\mathcal{F}$ of k-subsets of X, $|\mathcal{F}| \leq \binom{n-t}{k-t}$.

Proof. Let us denote by S_n the set of all permutations of $|X|$. For $g \in S_n$, let $\chi_{g(Y) \in \mathcal{F}}$ be 0 or 1 according to $g(Y) \notin \mathcal{F}$ or $g(Y) \in \mathcal{F}$. We are going to count

$$\sum_{g \in S_n} \chi_{g(Y) \in \mathcal{F}} = \sum_{\phi\Gamma} \sum_{g \in \phi\Gamma} \chi_{g(Y) \in \mathcal{F}} \tag{3}$$

in two different ways (the sum $\sum_{\phi\Gamma}$ is over all cosets of Γ in G). There are $|\mathcal{F}|$ elements of $\mathcal{F}$ and each can be obtained in the form of $g(Y)$ for $k!(n-k)!$ elements $g \in S_n$. Hence

$$|\mathcal{F}| k!(n-k)! = \sum_{g \in S_n} \chi_{g(Y) \in \mathcal{F}}.$$

On the other hand, we have

$$\sum_{g \in \phi\Gamma} \chi_{g(Y) \in \mathcal{F}} \leq k!/(k-t)!,$$

since if $g_i = \phi h_i$ has the property that $g_i(Y) \in \mathcal{F}$, then for all i we have $h_i(Y) \in \{\phi^{-1}(F) : F \in \mathcal{F}\}$, and hence $\{h_i(Y) :\ i = 1, 2, \dots, m\}$ is t-intersecting and condition (2) applies to it. We have the same upper bound for the summation over any coset. To count the number of cosets note that a sharply t-transitive permutation group acting on n elements has $n!/(n-t)!$ elements. By Lagrange's Theorem the number of cosets is $\frac{n!}{n!/(n-t)!} = (n-t)!$. Combining these observations we have

$$|\mathcal{F}| k!(n-k)! \leq (n-t)! k!/(k-t)!$$

and the theorem follows. □

Note that a cyclic permutation group on n elements act sharply 1-transitively, and condition (2) is the conclusion of the Lemma 1.1 in the usual presentations of Katona's proof in texts.

3 2-intersecting arithmetic progressions

Given a field $\mathbb{F}$, let us denote by $\mathbf{1}$, $\mathbf{2}$,..., $\mathbf{k}$ the field elements that we obtain by adding the multiplicative unit to itself repeatedly.

In order to apply Theorem 2.1 for the case $t = 2$ using the affine linear group, we tried $Y = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{k}\}$, and needed the corresponding condition (2). We failed to verify directly condition (2) but we were led to the following conjecture:

Conjecture 3.1 *If $A_1, A_2, \dots, A_m$ are k-term increasing arithmetic progressions of rational numbers, and any two of them has at least two elements in common, then $m \leq \binom{k}{2}$.*

It is easy to see that Conjecture 3.1 is equivalent for rational, real and for integer arithmetic progressions, and therefore we freely interchange these versions. This conjecture is the best possible, as it is easily shown by the following example: take two distinct numbers, $x < y$, and for all $1 \le i < j \le k$ take an arithmetic progression where x is the i^{th} term and y is the j^{th} term. This conjecture is the rational version of condition (2) for $t = 2$ with $Y = \{1, 2, \ldots, k\}$ indeed. Take the linear functions $\phi_i(x) = a_i x + b_i$. If $\phi_i(Y)$ $(i \in I)$ is 2-intersecting, then $|I| \le 2\binom{k}{2} = k(k-1)$, since any arithmetic progression can be obtained in exactly two ways as an image of Y.

There is a deep result in number theory, the *Graham Conjecture*, which is relevant for us: If $1 \le a_1 < \cdots < a_n$ are integers, then $\max_{i,j} \frac{a_i}{\gcd(a_i, a_j)} \ge n$. The Graham Conjecture was first proved for n sufficiently large by Szegedy [15], and recently even cases of equality were characterized for all n by Balasubramanian and Soundarajan [1].

How many distinct differences a set of pairwise 2-intersecting integer arithmetic progressions of length k can have? The Graham Conjecture immediately implies that the answer is at most $k-1$ differences. Indeed, assume that the distinct differences are $d_1, d_2, \ldots, d_l$. Consider two arithmetic progressions of length k, the first with difference d_i, the second with difference d_j. The distance of two consecutive intersection points of these two arithmetic progressions is exactly $\mathrm{lcm}[d_i, d_j]$. This distance, however, is at most $(k-1)d_i$ and likewise is at most $(k-1)d_j$. From here simple calculation yields

$$l \le \max_{i,j} \frac{d_i}{\gcd(d_i, d_j)} = \max_{i,j} \frac{\mathrm{lcm}[d_i, d_j]}{d_j} \le k - 1.$$

It is obvious that at most $k-1$ pairwise 2-intersecting length k integer arithmetic progressions can have the same difference. (The usual argument to prove Lemma 1.1 also yields this.) Therefore, instead of the conjectured $\binom{k}{2}$, we managed to prove $(k-1)^2$.

Kevin Ford has proven most of Conjecture 3.1 [9]:

Theorem 3.1 *Conjecture 3.1 holds if k is prime or $k > e^{10200}$.*

This opened up the way to the following argument which starts with the following straightforward lemmas. Their proofs are left to the Reader.

Lemma 3.1 *Given a natural number k, the following statement $\Upsilon(k)$ can be expressed in the first-order language of fields:*

> *"The characteristic of the field $\mathbb{F}$ is at least k, and for all $\phi_1, \phi_2, \ldots, \phi_{k(k-1)+1} :$ $\mathbb{F} \to \mathbb{F}$ linear functions if $|\phi_u(\{\mathbf{1},\ \mathbf{2}, \ldots,\ \mathbf{k}\}) \cap \phi_v(\{\mathbf{1},\ \mathbf{2}, \ldots,\ \mathbf{k}\})| \ge 2$ for all $1 \le u < v \le k(k-1)+1$, then the $k(k-1)+1$ linear functions are not all distinct."* □

Lemma 3.2 *Let $\mathbb{F}$ be a field and $Y = \{a+\mathbf{1}b, a+\mathbf{2}b, \ldots, a+\mathbf{k}b\} \subset \mathbb{F}$ an arithmetic progression with k distinct elements. If Y has two elements in common with some subfield $\mathbb{K}$ of $\mathbb{F}$ then $Y \subset \mathbb{K}$.* □

Recall that if $\mathbb{F}$ is a field then the *prime field*, $\mathbb{P}$, of $\mathbb{F}$ is the smallest nontrivial subfield of $\mathbb{F}$. When the characteristic if $\mathbb{F}$ is a prime $p > 0$ then the prime field of $\mathbb{F}$ is $\mathbb{P} = GF(p)$, the finite field of order p. When the characteristic of $\mathbb{F}$ is 0 then the prime field is $\mathbb{P} = \mathbb{Q}$, the field of rational numbers. Note that the theory of fields of characteristic 0 is not finitely axiomatizable.

Lemma 3.3 *The statement $\Upsilon(k)$ is true in some field $\mathbb{F}$ if and only if it is true in the prime field $\mathbb{P}$ of $\mathbb{F}$.*

Proof. As $\mathbb{P}$ is a subfield of $\mathbb{F}$ it is clear that if $\Upsilon(k)$ is true in $\mathbb{F}$ then it is true in $\mathbb{P}$. Now assume that $\Upsilon(k)$ is true in $\mathbb{P}$. Let $\phi_1, \phi_2, \ldots, \phi_{k(k+1)+1} : \mathbb{F} \to \mathbb{F}$ be linear functions so that $\mathcal{F} = \{\phi_u(Y_0) : 1 \le u \le k(k+1)+1\}$ is a 2-intersecting family of sets. Let $\phi_u^* := \phi_1^{-1}\phi_u$ for $u = 1, \ldots, k(k+1)+1$ then $\phi_1^* = \phi_1^{-1}\phi_1 = \mathrm{Id}$ is the identity map and $\mathcal{F}^* = \{\phi_u^*(Y_0) : 1 \le u \le k(k+1)+1\}$ is also a 2-intersecting family of sets. Also $\phi_1^*(Y_0) = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}\} \subset \mathbb{P}$. As $\mathcal{F}^*$ is 2-intersecting each of the arithmetic progressions $\phi_u^*(Y_0)$ will have at least two elements in $\mathbb{P}$. Therefore by Lemma 3.2 $\phi_u^*(Y_0) \subset \mathbb{P}$. If $\phi_u^*(x) = a_u x + b_u$ then $\phi_u^*(Y_0) \subset \mathbb{P}$ implies $a_u, b_u \in \mathbb{P}$ and so $\phi_u^* : \mathbb{P} \to \mathbb{P}$. As $\Upsilon(k)$ is true in $\mathbb{P}$ this implies there are $u \ne v$ with $\phi_u^* = \phi_v^*$. But this implies $\phi_u = \phi_v$ and so $\Upsilon(k)$ is true in $\mathbb{F}$. This completes the proof. □

Theorem 3.2 *Let k be a fixed positive integer for which Conjecture 3.1 holds. For every power $n = p^l$ of any prime $p \ge p_0(k)$, condition (2) holds with $Y = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}\}$ and $t = 2$ for the affine linear group over $GF(n)$. Therefore Theorem 2.1 gives for these values of n and k a Katona type proof for the 2-intersecting Erdős-Ko-Rado theorem. This is true in particular if k is a prime or $k > e^{10200}$.*

Proof. Observe first that for $t = 2$ with the choice of the affine linear group and $Y = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}\}$, $\Upsilon(k)$ is exactly the condition (2) of Theorem 2.1. Also observe that the validity of Conjecture 3.1 for k is exactly the truth of $\Upsilon(k)$ for the field $\mathbb{Q}$.

Now we show, using a routine model theoretic argument, that for any fixed k, $\Upsilon(k)$ is true for all fields of characteristic p except for finitely many primes.

Let P denote the set of primes p with the property that $\Upsilon(k)$ is false for at least one field of characteristic p. Assume, toward a contradiction, that P is infinite and for each $p_i \in P$ let $\mathbb{F}_i$ be a field of characteristic p_i so that the statement $\Upsilon(k)$ is false in $\mathbb{F}_i$. (By Lemma 3.3 we can assume that $\mathbb{F}_i$ is $GF(p_i)$.)

Let $\mathcal{U}$ denote any non-principal ultrafilter on P (note that P is infinite) and let $\mathbb{F}$ be the ultraproduct

$$\mathbb{F} = \left(\prod_{p_i \in P} \mathbb{F}_i\right) \Big/ \mathcal{U}.$$

Then by the 'fundamental theorem' of ultraproducts (cf. [4, Thm 4.1.9]) a statement Φ in the first-order language of fields is true in $\mathbb{F}$ if and only if the set $\{p_i \in P : \Phi \text{ is true in } \mathbb{F}_i\}$ is in $\mathcal{U}$. From this it follows immediately that $\Upsilon(k)$ is false in $\mathbb{F}$. Moreover, since $\mathcal{U}$ is a non-principal ultrafilter, $\mathbb{F}$ is a field of characteristic zero. By Lemma 3.3 this implies $\Upsilon(k)$ is false in the prime field of $\mathbb{F}$ which is the rational numbers $\mathbb{Q}$. This contradicts the assumption on k and thus completes the proof. The last assertion follows from Ford's Theorem 3.1. □ □

There is also an other approach, called *direct rectification*, due to Bilu, Lev and Ruzsa [2], which yields the following quantitative version of Theorem 3.2.

Theorem 3.3 *Theorem 3.2 is valid with $p_0(k) = 2^{4(k-1)^3}$.*

Proof. Let $p > k$ denote any prime. In the Abelian group $G = \mathbb{Z}/p\mathbb{Z}$, a sequence $a_1, a_2, \ldots, a_k$ is called a (nonconstant) arithmetic progression of length k and difference d ($d \ne 0$) if $a_i =$

$a_1 + (i-1)d$ for every $2 \le i \le k$. If we identify G with the Abelian group underlying $GF(p)$, then this is equivalent with the existence of a (unique) element g of the affine linear group over $GF(p)$ that satisfies $a_i = g(\mathbf{i})$ for $\mathbf{i} = \mathbf{1}$, $\mathbf{2}$,..., $\mathbf{k}$. Note that $a_1, a_2, \ldots, a_k$ is an arithmetic progression of difference d if and only if $a_k, a_{k-1}, \ldots, a_1$ is an arithmetic progression of difference $p-d$.

Let $\mathcal{A}$ be a collection of some arithmetic progressions of length k in G, any two of them having at least two elements in common. If d_1 and d_2 are differences of two progressions in $\mathcal{A}$, respectively, then it follows from the intersection property that there exist $1 \le i, j \le k-1$ such that either $id_1 = jd_2$ or $id_1 = j(p-d_2) = -jd_2$ in G. Thus, if we identify differences d and $p-d$, in $\mathcal{A}$ there can be at most $(k-1)^2$ different differences. If, moreover, $p \ge 2k$, then the standard argument yields that there exists $r \le (k-1)^2$, and $a_i, d_i \in G$, $d_i \in \{1, 2, \ldots, \lfloor p/2 \rfloor\}$ for $1 \le i \le r$ such that every progression in $\mathcal{A}$ is a (contiguous) subset of some $\{a_i, a_i + d_i, \ldots, a_i + (2k-3)d_i\}$.

Consider $K = \{a_i + jd_i \mid 1 \le i \le r, 0 \le j \le 2k-3\}$, then $|K| \le 2(k-1)^3$. Thus, if $p \ge p_0(k)$, then $|K| \le \log_4 p$, and from the rectification principle of Bilu, Lev and Ruzsa (see [2], Thm. 3.1) it follows that there exists a set of integers K' such that the canonical homomorphism $\varphi : \mathbb{Z} \to G$ induces a bijection from K' onto K with the property that for $a, b, c, d \in K'$, $a + b = c + d$ if and only if $\varphi(a) + \varphi(b) = \varphi(c) + \varphi(d)$. Assume that, for $a_1, a_2, \ldots, a_k \in K'$, $\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_k)$ is an arithmetic progression in K, then $\varphi(a_{i-1}) + \varphi(a_{i+1}) = \varphi(a_i) + \varphi(a_i)$ for every $2 \le i \le k-1$. Therefore $a_{i-1} + a_{i+1} = a_i + a_i$ also holds for $2 \le i \le k-1$ and it follows that $a_1, a_2, \ldots, a_k$ is an integer arithmetic progression. Since φ is a bijection from K' to K, we may consider the inverse image $\mathcal{A}'$ of $\mathcal{A}$ in K' which is a collection of 2-intersecting integer arithmetic progressions. If Conjecture 3.1 holds for k, then the number of increasing progressions in $\mathcal{A}'$ is at most $\binom{k}{2}$, and the same is true for the number of decreasing sequences. Thus $|\mathcal{A}| = |\mathcal{A}'| \le k(k-1)$, and the result follows for $n = p$. In the general case we only have to refer to Lemma 3.3. □

Note that if we had Theorem 3.1 (and therefore Theorem 3.2) for *every* k, then we would have a base case to prove the 2-intersecting Erdős-Ko-Rado theorem by a convenient induction for all $n \ge n_0(k)$, using the shift technique [11]. (This is interesting since for the t-intersecting Erdős-Ko-Rado theorem ($t > 1$) there is no obvious base case, unlike for the case $t = 1$. Proof techniques like the kernel method yield all n's above a threshold immediately.) In the case $k = 2$ the theorem obviously holds for all $n \ge 2$. Otherwise we will use the assumption that the theorem holds for $(k-1)$-element sets for all $n \ge n_0(k-1)$. Let $n_0(k)$ denote the smallest prime number larger than $\max\{p_0(k), n_0(k-1) - 1\}$. Then the 2-intersecting EKR theorem holds for $n = n_0(k)$ and k. When $n > n_0(k)$, shifting [11] a 2-intersecting family of k-subsets in $\{1, 2, \ldots, n\}$, we conclude that the invariant family has the following property: any two sets share at least two elements different from n. Hence the invariant family decomposes to the disjoint union of a 2-intersecting family of k-subsets in $\{1, 2, \ldots, n-1\}$, and a 2-intersecting family of $k-1$-subsets in $\{1, 2, \ldots, n-1\}$. Using induction, the number of members of this family is at most

$$\binom{n-3}{k-2} + \binom{n-3}{k-3} = \binom{n-2}{k-2}.$$

4 Open problems

It is easy to come up with a formal Bollobás inequality to generalize Theorem 2.1:

Theorem 4.1 *Let us be given a sharply t-transitive permutation group Γ acting on a set X with $|X| = n$. Assume that for a certain l there exists a sequence of subsets $Y_1 \subset \cdots \subset Y_l \subset X$ with $|Y_i| = i$ such that for any distinct group elements $\phi_1, \phi_2, \ldots, \phi_m \in \Gamma$, and any sequence $Y_{i_1}, Y_{i_2}, \ldots, Y_{i_m}$,*

$$\text{if } |\phi_r(Y_{i_r}) \cap \phi_j(Y_{i_j})| \geq t \text{ for all } r, j, \text{ then } \sum_{j=1}^{m} \frac{(|Y_{i_j}| - t)!}{|Y_{i_j}|!} \leq 1 \tag{4}$$

Then for any t-intersecting family $\mathcal{F}$ of subsets of X, so that none of them contains another as a subset, we have

$$\sum_{i=t}^{l} \frac{f_i}{\binom{n-t}{i-t}} \leq 1,$$

where f_i is the number of i-element sets in $\mathcal{F}$.

Proof. Follow the proof of Theorem 2.1 with the following changes: in formula (3), instead of $\chi_{g(Y) \in \mathcal{F}}$, sum up $\sum_{s=t}^{l} \frac{\chi_{g(Y_s) \in \mathcal{F}}}{s(s-1)\cdots(s-t+1)}$. On the one hand, the value of this summation is exactly

$$\sum_{i=t}^{m} f_i \frac{i!(n-i)!}{i(i-1)\cdots(i-t+1)}.$$

On the other hand,

$$\sum_{g\in\phi\Gamma} \sum_{s=t}^{l} \frac{\chi_{g(Y_s) \in \mathcal{F}}}{s(s-1)\cdots(s-t+1)} = \sum_{h\in\Gamma} \sum_{s=t}^{l} \frac{\chi_{h(Y_s) \in \phi^{-1}(\mathcal{F})}}{s(s-1)\cdots(s-t+1)} \leq 1$$

by (4), and the number of cosets is still $(n-t)!$. Simple algebra completes the proof. $\square$

No Bollobás inequality is known for t-intersecting families of sets, although the exact bound for the EKR theorem is known [16]. We believe that such Bollobás inequalities do hold. Our present approach, however, does not seem to yield them.

For the proof of the original Bollobás inequality (1), one needs a stronger version of the Katona Lemma, which is exactly (4) for $t = 1$ and $m = \lfloor n/2 \rfloor$. The $t = 2$ version of the Bollobás inequality would require (4) for $t = 2$. Unfortunately, the version of (4) for integer arithmetic progressions is not valid with the natural choice $Y_i = \{1, 2, \ldots, i\}$. With this choice, the sets $\{0, 6, 12\}$, $\{6, 12, 18\}$, $\{6, 9, 12\}$, and $\{6, 8, 10, 12\}$ each arises as an affine linear image of the corresponding Y_i in two ways, and (4) fails by $2 \cdot (\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12}) > 1$. Perhaps one might use another choice of Y_i.

Is the 3-intersection version of Conjecture 3.1 true? This would yield a Katona type proof for the Erdős-Ko-Rado theorem for $t = 3$.

Conjecture 4.1 *If $A_1, A_2, \ldots, A_m$ are images of the set $\{1, 2, \ldots, k\}$ under distinct non-constant fractional linear transformations with rational coefficients $x \rightarrow \frac{a_i x + b_i}{c_i x + d_i}$ ($i = 1, 2, \ldots, m$), such that $|A_i \cap A_j| \geq 3$ for all i, j, then $m \leq k(k-1)(k-2)$.*

This conjecture is the best possible, as it is easily shown by the following example: take any three distinct numbers, $x < y < z$, and for each ordered 3-set (i, j, k), $1 \leq i, j, l \leq k$, take the (unique) non-constant fractional linear transformation which maps i to x, j to y and l to z.

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