A New Lower Bound for the Bipartite Crossing Number with Applications *

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Abstract

Let G be a connected bipartite graph.We give a short proof, using a variation of Menger's Theorem, for a new lower bound which relates the *bipartite crossing number* of G, denoted by $bcr(G)$, to the edge connectivity properties of G. The general lower bound implies a weaker version of a very recent result, establishing a bisection based lower bound on $bcr(G)$ which has algorithmic consequences. Moreover, we show further applications of our general method to estimate $bcr(G)$ for "well structured" families of graphs, for which tight isoperimetric inequalities are available. For hypercubes and 2-dimensional meshes, the upper bounds (asymptotically) are within multiplicative factors of 4 and 2, from the lower bounds, respectively. The general lower bound also implies a lower bound involving eigenvalues of G.

1 Introduction and summary

The *planar crossing number* problem is the problem of drawing a graph with minimum number of edge crossings in the plane. This is a difficult, and important problem which is studied in

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graph theory and also in the theory of VLSI [11, 21, 27]. Computing the value of the planar crossing number is NP-hard [14], and exact values are known only for very restricted classes of graphs. The exact value of the planar crossing number is not known, even for the complete graph K_n, for arbitrary values of n. Indeed, there has been numerous results regarding the approximate values of the crossing number for very specific graphs [21, 32]. In this paper we study a variation of the planar crossing number. Let $G = (V_0, V_1, E)$ be a connected bipartite graph, where V_0, V_1 is the bipartition of vertices into independent sets, and E is the edge set. A *bipartite drawing* of G consists of placing the vertices of V_0 and V_1 into distinct points on two horizontal lines $y = 0, y = 1$ in the xy-plane, respectively, and then drawing each edge with one straight line segment which connects the points of $y = 0$ and $y = 1$ where the endvertices of the edge were placed. Hence, placing the vertices will determine the whole drawing. The *bipartite crossing number* of G, denoted by $bcr(G)$ is the minimum number of crossings of edges over all bipartite drawings of G.

A motivation behind studying $bcr(G)$ is the routing of VLSI (see for example [21, 29]). Desirable features of a VLSI chip include small area and small delay. A crucial step in the VLSI design is the *routing* stage in which the modules are interconnected, see [29] for details. The modules are usually placed on the rows of a grid (grids); certain modules on consecutive rows must be connected using wires. The wires are splitted into horizontal and vertical segments, where horizontal and vertical segments are assigned to different layers. Although no two wires are allowed to cross each other physically, wires can *cross over* each other, that is, one horizontal segment may run on the top of a vertical segment so that the projection of the two segments cross. Cross overs are undesirable since they create delay. We can think of modules and wires connecting them on two consecutive rows of grid as vertices and edges of a bipartite graph. Hence, by relaxing the requirements, and allowing the routes to be straight lines between the modules, the routing problem can be modeled as a bipartite drawing problem. A good solution to the bipartite crossing number problem will allow the designer at an early stage of the design, to approximate the location of modules, minimizing the number of cross overs, assuming that the modules will be connected using the straight line segments. Later, the designer can refine and change the shape of wires at a final stage of routing. It should be emphasized that, minimizing the number of the crossings in the initial design, also will help to reduce the grid sizes, and hence reducing the area [21].

Another motivation behind studying the bipartite crossing number comes from graph drawing. It is well known that $bcr(G)$ is one of the parameters which strongly influence the understanding and the aesthetics of drawings of graphs drawn in a hierarchical fashion. For a survey on drawing graphs see [10].

The notion of $bcr(G)$ was first introduced in [15], [16] and [37], where in [16] exact values for $bcr(G)$ of complete and complete bipartite graphs and even cycles were obtained. Some basic observations on $bcr(G)$ were made in [26]. The bipartite crossing number problem is known to be NP-complete[1] [14] but can be solved in polynomial time for bipartite permutation graphs [33], and trees [30]. A great deal of research has been devoted to the design of algorithms and heuristics for solving this problem (see for example [6, 7, 12, 19, 24, 34, 36]). Mutzel, and Jünger and Mutzel [19, 28] had reported experiments indicating the success of their algorithms in computing near-optimal values of $bcr(G)$ in certain cases. Despite their success in a practical sense over the range of the applied data, these algorithms did not have a performance guarantee, and thus one could not expect that they always generate a solution in polynomial time which is

[1]Technically speaking, the NP-hardness of the problem was proved for multigraphs, but it is widely assumed that it is also NP-hard for simple graphs.

provably close to the optimal solution. Thus, these algorithms would not fit the notion of the theoretically efficient approximation algorithm [14]. For a more restricted problem when the positions of the vertices of V_0 are fixed Eades and Wormald [12] designed a polynomial time algorithm which approximates the bipartite crossing number within a multiplicative factor of 3 in this restricted problem. See also [7, 36], and the survey [10].

The latest progress in this area was made recently [30] in which we fully explored the structure of bipartite drawing by relating them to the linear arrangement problem. In particular, we showed (Theorem 2.2 in [30]) that when the maximum and minimum degrees are close to each other, then the asymptotic values of $bcr(G)$ and the optimal linear arrangement of G have the same order of magnitude. Hence, we derived a provably good approximation algorithm, with performance guarantees of $O(\log n \log\log n)$ from the optimal, for computing $bcr(G)$. Moreover, we verified in [30], using the connection between the linear arrangement problem and bipartite drawings, that $bcr(G)$ is large compared to the bisection of G. Consequently, we showed that a standard divide and conquer algorithm also approximates $bcr(G)$ within a factor of $O(\log^2 n)$ from the optimal, in polynomial time, when the maximum and minimum degrees in G are close to each other.

In this paper we develop a new lower bound argument using Menger's Theorem which relates the bipartite crossing number of a graph to the edge connectivity properties of G (Theorem 2.1). The result easily implies good lower bounds involving the bisection, the edge isoperimetric properties and the eigenvalues of the graph. In particular we give a short proof, establishing a large lower bound involving the bisection of G (Corollary 2.1), on $bcr(G)$. The bisection based lower bound presented here is weaker than the one in [30]. Nonetheless, its proof is short, and in fact the lower bound is strong enough to show that for sparse graphs arising in the VLSI applications, the standard divide and conquer algorithm can approximate $bcr(G)$ within a factor of $O(\log^2 n)$ from the optimal value, in polynomial time. Moreover, the approach taken here allows to derive lower bounds on the values of $bcr(G)$ which are within small multiplicative constants from the upper bounds, for well structured graphs in which tight isoperimetric inequalities are available. Results of this nature are significant in graph theory, much in the spirit of similar results regarding estimating the approximate values of the planar crossing number for certain graphs. For instance for the 2-dimensional mesh (or grid) $\mathcal{M}(M, N)$ we get $\frac{3}{4}M^2N - O(M^3 + MN) \leq bcr(\mathcal{M}(M, N)) \leq \frac{3}{2}M^2N - O(MN)$ and for the $N-$dimensional hypercube graph $\mathcal{Q}_N$ we get $N4^{N-2} - O(4^N) \leq bcr(\mathcal{Q}_N) \leq N4^{N-1}$. Finally, we provide a general lower bound for $bcr(G)$ in terms of the smallest positive Laplacian eigenvalue of the graph.

This paper is the extended version of our conference paper [31].

For $G = (V_0, V_1, E)$, we will assume throughout this paper that $V = V_0 \cup V_1$ and $n = |V|$. We will denote the degree of vertex v by $d(v)$, and denote by δ_0 the minimum degree among the vertices in V_0. For a bipartite drawing $D(G)$ of a graph G, let $bcr(D(G))$ denote the number of the crossings in $D(G)$ (i.e. the number of unordered pairs of crossing edges). When the context is clear we write D and $bcr(D)$. Note that $bcr(G) = \min_D bcr(D)$.

2 A general lower bound method

For $X \subseteq V$ define

$$\partial(X) = \{uv \in E : u \in X, v \in V - X\}.$$

The problem of finding good lower bounds for $|\partial(X)|$, for all $X \subseteq V$, is an important problem in graph theory and computer science and is studied under the heading *edge isoperimetric* *inequalities* [3].

For $X, Y \subset V$, $X \cap Y = \emptyset$ define $sep(X,Y)$ to be a set of edges in G of smallest cardinality which separates X from Y in G. Note that $|sep(X,Y)| \leq |\partial(X)|$.

For $0 < \gamma < 1/2$, the $\gamma-$bisection of G, denoted by $b_\gamma(G)$, is the smallest $|\partial(X)|$, over all X, with $\gamma|V| \leq |X| \leq (1-\gamma)|V|$.

For a bipartite drawing D of G, let v_k be the kth vertex on $y=0$ from left, and let A_k denote the set of the first k vertices on $y=0$ from the left, $1 \leq k \leq |V_0|$.

Theorem 2.1 *Let D be a bipartite drawing of $G = (V_0, V_1, E)$, then the following holds:*

$$2bcr(D) \geq \sum_{k=1}^{|V_0|-1} \left(d(v_{k+1}) - 1\right)\left(|sep(A_k, V_0 - A_k)| - d(v_{k+1})\right).$$

Proof. Let P_k be a set of edge disjoint paths of largest cardinality with one end point in A_k and the other in $V_0 - A_k$. A variant of Menger's theorem [35] says that $|P_k| = |sep(A_k, V_0 - A_k)|$. Observe that each path in P_k, except for those including the $(k+1)$-st vertex v_{k+1} on $y=0$ from the left, must cross all but one edges incident to the $(k+1)$-st vertex v_{k+1}. This observation can be verified by considering two cases. In the first case, P_k does not go through any neighbor of v_{k+1}, then it crosses all edges incident to v_{k+1}. In the second case, if P_k does go through any neighbor t of v_{k+1}, then P_k has to cross all edges incident to v_{k+1} but the edge $v_{k+1}t$.

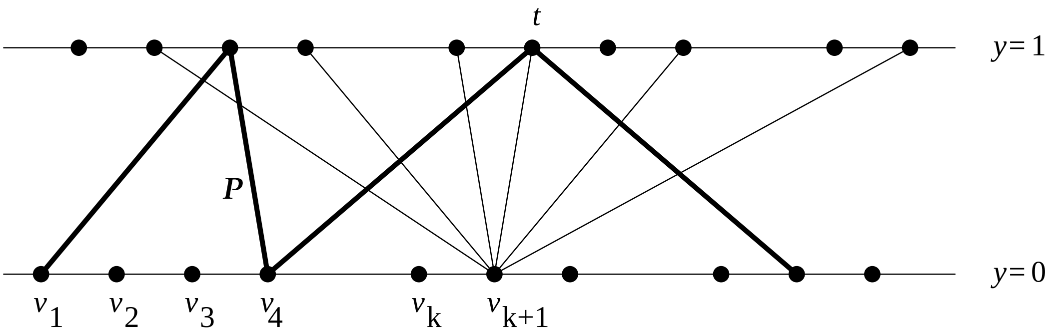

Fig. 1 : A path $P \in P_k$ crosses all but one edges incident with v_{k+1}.

Thus the paths in P_k generate a total of at least $[d(v_{k+1}) - 1][|P_k| - d(v_{k+1})]$ crossings on the edges incident to v_{k+1} and the theorem follows by taking the sum over all k. (Note that a factor of 2 is needed on the left hand side, since a crossing will be counted twice.) $\square$

Leighton [21] proved that $\Omega(b_{\frac{1}{3}}(G)^2 - n)$ is a lower bound on the planar crossing number of any graph G of bounded degree. In [30], we developed a general theory for studying the bipartite drawings by relating them to the linear arrangement problem which is another well known problem in the theory of VLSI [20, 29]. In particular, using an elaborated proof, we verified that $bcr(G) + \sum_{x \in V} d^2(x) = \Omega(\delta L(G))$, where δ is the min degree, and $L(G)$ is the optimal arrangement value. A consequence was that $bcr(G) + \sum_{x\in V} d^2(x) = \Omega(\delta n b_{\gamma(G)})$. A nice application of Theorem 2.1, is to provide a weaker version of the bisection related result using a short and direct proof. This weaker lower bound, however, is strong enough to show that the standard divide and conquer algorithm has a good performance guarantee for approximating $bcr(G)$, when G is sparse.

Corollary 2.1 *Let $G = (V_0, V_1, E)$. Assume that $|V_0| \geq |V_1|$ and the number of vertices of degree 1 in $|V_0|$ is at most $\epsilon|V_0|$, where $0 \leq \epsilon < 1$ is a constant. Let ϵ' be any positive constant so that $\epsilon' < 1 - \epsilon$, and define $\gamma_\epsilon = \frac{1-\epsilon-\epsilon'}{4}$. Then for any $\gamma \leq \gamma_\epsilon$ it holds,*

$$2bcr(G) \geq \frac{\epsilon' \delta_0}{2} |V_0| b_\gamma(G) - \sum_{v \in V_0} d^2(v).$$

In particular, for $\epsilon' = \frac{1-\epsilon}{5}$, we have $\gamma_\epsilon = \frac{1-\epsilon}{5}$, and it holds

$$2bcr(G) = \Omega\left(n \delta_0 b_{\frac{1-\epsilon}{5}}(G)\right) - \sum_{v \in V_0} d^2(v).$$

Proof. Consider the sum in Theorem 2.1 for those values of k which are at least $2\gamma_\epsilon|V_0|$ (and hence $\geq \gamma_\epsilon n$), and are at most $|V_0|(1-2\gamma_\epsilon)$. Next note that there are $|V_0|(1-4\gamma_\epsilon) = |V_0|(\epsilon+\epsilon')$ such values of the index k, and also that for at least $|V_0|\epsilon'$ values, the corresponding term has $d(v_{k+1}) \geq 2$, and hence $d(v_{k+1}) - 1 \geq \frac{d(v_{k+1})}{2} \geq \frac{\delta_0}{2}$. To finish the proof, since $\gamma_\epsilon \geq \gamma$ and hence $b_{\gamma_\epsilon} \geq b_\gamma$, we will show that for the prescribed values of k, $|sep(A_k, V_0 - A_k)| \geq b_{\gamma_\epsilon}(G)$. The set $sep(A_k, V_0 - A_k)$ partitions V into X_k and $V - X_k$ such that $A_k \subseteq X_k$ and $V_0 - A_k \subseteq V - X_k$. Clearly $|sep(A_k, V_0 - A_k)| = |\partial(X_k)|$, and it suffices to show that $\gamma_\epsilon n \leq |X_k| \leq (1-\gamma_\epsilon)n$. Note that $|A_k| \leq |X_k| \leq n - |V_0 - A_k|$, for any k. Now observe that for $\gamma_\epsilon n \leq k \leq |V_0|(1-2\gamma_\epsilon)$, it holds that $|X_k| \geq \gamma_\epsilon n$, and $|V_0 - A_k| \geq 2|V_0|\gamma_\epsilon \geq n\gamma_\epsilon$, and hence proving the claim . □

Lower bounds that involve the bisection of a graph are known to be useful in showing the performance guarantee of the approximation algorithms [2, 18, 30]. Hence, a simple algorithmic application of Corollary 2.1 is that the traditional divide and conquer algorithm can also be used to approximate $bcr(G)$ within a factor of $O(\log^2 n)$ from the optimal. The divide stage of the algorithm uses an approximation algorithm for bisecting a graph such as those in [13, 23]. These bisecting algorithms have a performance guarantee of $O(\log n)$ from the optimal. The details of the next result are standard, and similar (but not identical) to [2, 18, 30]. For completeness we have included a proof in the appendix.

Theorem 2.2 *Let $G = (V_0, V_1, E)$, with $|E| = m$, $|V_0| \geq |V_1|$ be a degree bounded graph. Assume that the number of vertices of degree 1 in $|V_0|$ is at most $\epsilon|V_0|$, where $\epsilon < 1$ is a constant. Let A be a polynomial time algorithm to approximate the $\frac{1-\epsilon}{5}$-bisection of a graph with a performance guarantee $O(\log n)$. Consider a divide and conquer algorithm which recursively bisects the graph G, using A, obtains the two drawings, and then inserts the edges of the bisection between these two drawings. This divide and conquer algorithm generates, in polynomial time, a bipartite drawing D so that $bcr(D)$ is within a factor of $O(\log^2 n)$ from the optimal, provided that $m \geq n(1+\mu)$, where $\mu > 0$ is any positive constant.*

Remarks. One may think that the above result is not too strong, since it is only valid for degree bounded graphs. First, it should be noted that for problems arising in the applications such as VLSI design, the underlying graphs are always degree bounded, and hence fit the framework described above. Second, the strength of the above result is justified by noting that the best existing approximation algorithm for the planar crossing number has the performance guarantee of $O(\log^4 n)$ [23], only when the graph is degree bounded and has degree at least 4. Hence, we have obtained a factor of $O(\log^2 n)$ improvement in the performance guarantee compared to the case of the planar crossing number. Finally, it should be noted that working with the lower bound of Corollary 2.1 is essential and the previous lower bound of $\Omega(b_{\frac{1}{3}}(G)^2 - n)$ can not be used to show the suboptimality of the solution, since it is too small compared to the error terms appearing in the right hand side of the recurrence relation in Theorem 2.2.

3 Bipartite crossing numbers of meshes and hypercubes

For $M \leq N$, let $\mathcal{M}(M, N)$ denote the 2-dimensional mesh i.e. the graph defined by the Cartesian product of an $M-$vertex path with an $N-$vertex path. Let $\mathcal{Q}_N$ denote the $N-$dimensional hypercube graph, i.e. the Cartesian product of N 2-vertex paths.

Using the results in [30] we can obtain bounds on $bcr(G)$ for hypercubes and meshes which are tight within large constant multiplicative factors. In particular the ratio of upper to lower bound will be about 60 for the mesh, and about 30 for the cube. In this section we improve these constants. In the case of $\mathcal{M}(M, N)$ we provide the exact values for the small values $M = 2, 3$. We emphasize that our main contribution is improving the constants involving the lower bounds, and that the constructions for the upper bounds are not difficult to see.

Theorem 3.1 *For a mesh $\mathcal{M}(M, N), 4 \leq M \leq N$ it holds:*

$$\frac{3}{4}M^2N - \frac{1}{4}M^3 - \frac{9}{2}MN - \frac{3}{2}M^2 - 3M \leq bcr(\mathcal{M}(M,N)) \leq \frac{3}{2}M^2N - \frac{3}{2}MN.$$

Proof. Upper bound. View $\mathcal{M}(M, N)$ as M rows and N columns. Note that each row or column is a path. First we place all vertices in $V_0 \cup V_1$ on the line $y = 0$ in a column after column manner. Then we project the vertices of V_1 on $y = 1$. Note that edges in the same row or same column do not cross each other. Moreover, edges in a column do not cross edges in another column.

Consider a row, and the corresponding path in the drawing. This path produces at most a total of $M - 2$ crossings with all the edges in a fixed column, since this path can intersect all edges in any column with the exception of at least 1 edge which is incident to a vertex on the path. We conclude that the total number of crossings between rows and columns is at most $(M - 2)MN$. Now consider any two rows, and the two paths p_1, and p_2 associated with them. Observe that if an edge e in p_1 crosses an edge e' in p_2, then either e and e' must both have endpoints in two consecutive columns i, $i+1$, or e and e' must have endpoints in 3 consecutive columns i, $i+1$, $i+2$. (Note that in this case column $i+1$ contains one end point of e and one end point of e'.) In the former case we refer to the crossing associated with e and e' as type one, and in the latter we refer to it as type two. Assume with no loss of generality that both end points of p_1 are in V_0. If p_2 has endpoints in V_0, then crossing of any edge in p_1 with any edge in p_2 must be a type two crossing. In this case the total number of crossings between p_1 and p_2 is exactly $N - 2$. On the other hand, if p_2 has both ends in V_1, then crossing of any edge in p_1 and any edge in p_2 must be a type one crossing. In this case the total number of crossings between p_1 and p_2 is $N - 1$. Thus the the total number of crossings between all rows is at most $\binom{M}{2}(N-1) = \frac{M(M-1)(N-1)}{2}$.

We conclude that the total number of crossings in our drawing is at most $M(M-1)\frac{3N-1}{2} \leq \frac{3}{2}M^2N - \frac{3}{2}MN$.

Lower bound. For the sake of simplicity assume that both M and N are even. Consider a bipartite drawing of $\mathcal{M}(M, N)$. Then $|V_0| = MN/2$. Let A_k denote for $k = 1, 2, ..., MN/2$ the set of the first k vertices on $y = 0$ from the left. We use a variant of the proof of Theorem 2.1. Define a function

$$f(x) = \begin{cases} 2\sqrt{x}, \text{ if } 0 \leq x \leq M^2/4, \\ M, \text{ if } M^2/4 \leq x \leq MN - M^2/4, \\ 2\sqrt{MN - x}, \text{ if } MN - M^2/4 \leq x \leq MN. \end{cases}$$

Now we use an edge isoperimetric inequality for meshes. It is known [1, 4] that for any $X \subset V$, $|\partial(X)| \geq f(|X|)$ holds. The set $sep(A_k, V_0 - A_k)$ partitions V into X_k and $V - X_k$ such that $A_k \subseteq X_k$ and $V_0 - A_k \subseteq V - X_k$. Clearly $|sep(A_k, V_0 - A_k)| = |\partial(X_k)|$. As $A_k \subseteq X_k \subseteq V - (V_0 - A_k)$, the concavity of f gives

$$|sep(A_k, V_0 - A_k)| \geq \min\{f(|A_k|), f(|V - (V_0 - A_k)|)\} = \min\{f(k), f(\frac{MN}{2} + k)\}.$$

There are at most $M + N$ vertices in V_0 whose degree is less than 4. Let I denote the set of these vertices. We are going to give a lower bound for the number of crossings for the edges incident to v_{k+1}. It is convenient not to countthe contribution of vertices v_{k+1} whose degree is less than 4. Hence if k runs from 1 to $MN/2 - 1$, using only vertices v_{k+1} whose degree is 4, Theorem 2.1 yields that all $(A_k, V_0 - A_k)$ paths but 4 intersect at least 3 of the edges adjacent to v_{k+1}. We obtain

$$bcr(\mathcal{M}(M, N)) \geq \frac{3}{2} \sum_{\substack{k=1 \\ k+1 \notin I}}^{\frac{MN}{2}-1} (|sep(A_k, V_0 - A_k)| - 4).$$

The denominator 2 occurs before the sum since each crossing is counted at twice. Further,

$$\begin{aligned}
bcr(\mathcal{M}(M, N)) &\geq \frac{3}{2}\left(\sum_{\substack{k=1 \\ k+1 \notin I}}^{\frac{MN}{2}-1} \min\{f(k), f(\frac{MN}{2} + k)\} \right) - 3MN \\
&= \frac{3}{2} \sum_{\substack{k=1 \\ k+1 \notin I}}^{\frac{M^2}{4}} \min\{f(k), f(\frac{MN}{2} + k)\} \\
&+ \frac{3}{2} \sum_{\substack{k=\frac{M^2}{4}+1 \\ k+1 \notin I}}^{\frac{MN}{2}-\frac{M^2}{4}-1} \min\{f(k), f(\frac{MN}{2} + k)\} \\
&+ \frac{3}{2} \sum_{\substack{k=\frac{MN}{2}-\frac{M^2}{4} \\ k+1 \notin I}}^{\frac{MN}{2}-1} \min\{f(k), f(\frac{MN}{2} + k)\} - 3MN \\
&\geq 6 \sum_{k=1}^{\frac{M^2}{4}} \sqrt{k} + \frac{3}{4}(M^2N - M^3 - 4M) - 3MN - (M + N)\frac{3M}{2} \\
&\geq 6 \int_0^{\frac{M^2}{4}} \sqrt{x}dx + \frac{3}{4}(M^2N - M^3 - 4M) - 3MN - (M + N)\frac{3M}{2} \\
&\geq \frac{3}{4}M^2N - \frac{1}{4}M^3 - \frac{9}{2}MN - \frac{3}{2}M^2 - 3M.
\end{aligned}$$

□

Theorem 3.2 *For $N \geq 3$ it holds:*

$$bcr(\mathcal{M}(3, N)) = 5N - 6.$$

Proof. Upper bound. Use the same "column after column" principle as in Theorem 3.1. It is easy to see by induction on N that the resulting drawing has $5N - 6$ crossings.

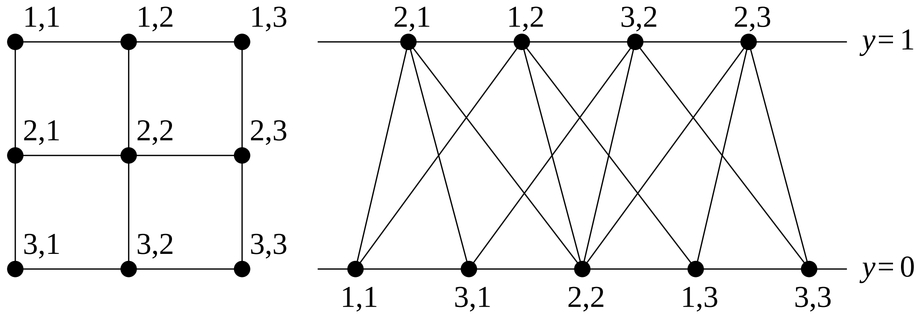

Fig. 2 : Mesh $\mathcal{M}(3,3)$ and its optimal bipartite drawing

Lower bound. Imagine that $\mathcal{M}(3,N)$ consists of 3 row and N column vertices. Let $\mathcal{M}(3,3)$ denote the submesh induced by the last 3 column vertices. We proceed by induction on N. By a case analysis we can show that $bcr(\mathcal{M}(3,3)) = 9$. Suppose that $bcr(\mathcal{M}(3,N-1)) \geq 5(N-1)-6$, for $N \geq 4$ and consider $\mathcal{M}(3,N)$. Using a case analysis again one can show that the edges incident to the last column vertices in $\mathcal{M}(3,N)$ contain at least 5 crossings. In fact this can be verified considering the submesh $\mathcal{M}(3,3)$ only. Therefore

$$bcr(\mathcal{M}(3,N)) \geq bcr(\mathcal{M}(3,N-1)) + 5 \geq 5N - 6. \qquad \square$$

The result $bcr(\mathcal{M}(2,N)) = N - 1$ can be easily deduced from the optimal bipartite drawing of the even cycle C_{2N}, [16].

Theorem 3.3 *For $N \geq 3$ it holds:*

$$N4^{N-2} - O(4^N) < bcr(\mathcal{Q}_N) \leq N4^{N-1}.$$

Proof. To prove the upper bound, we draw $\mathcal{Q}_N$ recursively and prove a stronger bound by induction:

$$bcr(D(\mathcal{Q}_{N-1})) \leq (2N-5)2^{2N-5} - ((N-1)^2 - (N-1) - 1)2^{N-3}.$$

$N = 3$ provides the base case with unique drawing of $\mathcal{Q}_2$. To construct $D(\mathcal{Q}_N)$, we consider a copy of $D(\mathcal{Q}_{N-1})$ on the usual $y = 0, y = 1$ lines, and translate it along the x axis far enough so that $D(\mathcal{Q}_{N-1})$ does not intersect the translated version denoted by $D'(\mathcal{Q}_{N-1})$. Finally, we take the mirror image of $D'(\mathcal{Q}_{N-1})$ with respect to the line $y = 1/2$ to obtain a drawing $D''(\mathcal{Q}_{N-1})$. Now connect by 2^{N-1} new edges, according to the recursive structure of the hypercube, the corresponding vertices of $D(\mathcal{Q}_{N-1})$ and $D''(\mathcal{Q}_{N-1})$ to obtain $D(\mathcal{Q}_N)$. We have $2bcr(D(\mathcal{Q}_{N-1}))$ crossings in the two subdrawings used in the recursion. Any new edge crosses exactly half of the new edges ("increasing" edges cross exactly the "decreasing" edges), so new edges make 2^{N-2} crossings each, totaling to 2^{2N-4} crossings. There are $(N-1)2^{N-2}$ old edges in $D(\mathcal{Q}_{N-1})$; each old edge has a copy in $D''(\mathcal{Q}_{N-1})$. Note that an new edge can cross either an old edge in $D(\mathcal{Q}_{N-1})$, or the copy of this edge in $D''(\mathcal{Q}_{N-1})$, but not both. Hence the number of crossings of new edges with edges of $D(\mathcal{Q}_{N-1}) \cup D''(\mathcal{Q}_{N-1})$ is at most $2^{N-1}(N-1)2^{N-2}$.

Hence we have:

$$\begin{aligned} bcr(D(\mathcal{Q}_N)) &\leq 2bcr(D(\mathcal{Q}_{N-1})) + 2^{2N-4} + (N-1)2^{N-1}2^{N-2} \\ &\leq 2[(2N-5)2^{2N-5} - ((N-1)^2 - (N-1) - 1)2^{N-3}] \\ &+ 2^{2N-4} + (N-1)2^{N-1}2^{N-2} \\ &\leq (N-2)2^{2N-3} + (N-1)2^{2N-3} = (N - \frac{3}{2})2^{2(N-1)} < N4^{N-1}. \end{aligned}$$

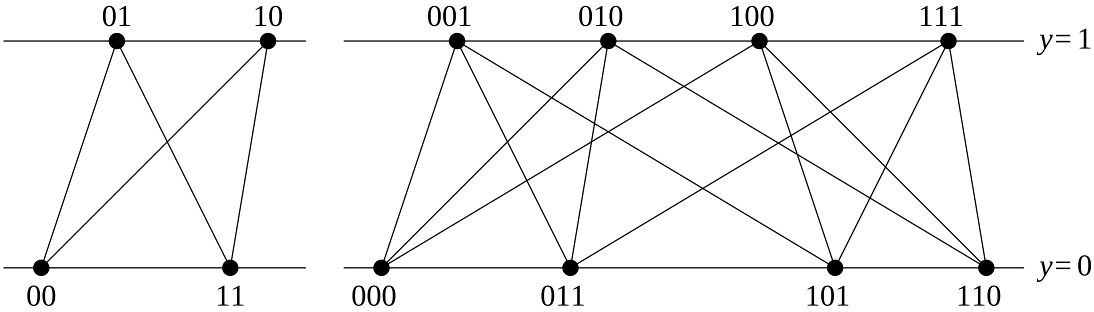

Fig. 3 : Bipartite drawings of $\mathcal{Q}_2$ and $\mathcal{Q}_3$.

Lower bound. We apply the same argument as for 2-dimensional meshes. Consider a bipartite drawing of $\mathcal{Q}_N$. Note that $|V_0| = 2^{N-1}$. For $k = 1, 2, ..., 2^{N-1} - 1$, let $A_k \subset V_0$ denote the set of the first k vertices on $y = 0$ from the left. Following Bollobás and Leader [5] define a function $f(x)$ as follows:

$$f(x) = \begin{cases} x(N - \log x), \text{ if } 1 \leq x \leq 2^{N-1}, \\ (2^N - x)(N - \log(2^N - x)), \text{ if } 2^{N-1} \leq x \leq 2^N. \end{cases}$$

(Here log denotes logarithm of base 2.) An edge isoperimetric inequality for hypercubes (see e.g [9]) says that for any $X \subset \mathcal{Q}_N$, the inequality $|\partial(X)| \geq f(|X|)$ holds. Following the reasoning applied for meshes (i.e. $|sep(A_k, V_0 - A_k)| \geq \min\ (f(k), f(2^{N-1} + k))$ for $1 \leq k \leq 2^{N-1}$) we show that

$$|sep(A_k, V_0 - A_k)| \geq \min\{k(N - \log k), (2^{N-1} - k)(N - \log(2^{N-1} - k))\}.$$

Hence if k runs from 1 to $2^{N-1} - 1$ we get

$$\begin{aligned} & bcr(\mathcal{Q}_N) \\ \geq\ & \frac{N-1}{2} \sum_{k=1}^{2^{N-1}-1} (|sep(A_k, V_0 - A_k)| - N) \\ \geq\ & \frac{N-1}{2} \sum_{k=1}^{2^{N-1}-1} \min\{k(N - \log k), (2^{N-1} - k)(N - \log(2^{N-1} - k))\} \\ -\ & N(N-1)2^{N-2} \\ \geq\ & (N-1) \sum_{k=1}^{2^{N-2}-1} k(N - \log k) + (N-1)2^{N-2} - N(N-1)2^{N-2} \\ =\ & N(N-1)2^{N-3}(2^{N-2} - 1) + (N-1)2^{N-2} - N(N-1)2^{N-2} \\ -\ & (N-1) \sum_{k=1}^{2^{N-2}-1} k \log k, \end{aligned}$$

where we used that for $k \leq 2^{N-2}$ it holds $k(N - \log k) \leq (2^{N-1} - k)(N - \log(2^{N-1} - k))$. Observe that

$$\sum_{k=1}^{2^{N-2}-1} k \log k < \int_1^{2^{N-2}} x \log x dx = (N-2)2^{2N-5} - \frac{1}{\ln 2} 2^{2N-6} + \frac{1}{4 \ln 2}.$$

Substituting this into the previous inequality we get the result. $\square$

4 Using eigenvalues in the general lower bound

We assume familiarity with spectral graph theory and Fan Chung's recent book on the topic [8], which is our basic reference. We use Laplacian eigenvalues of a graph like [8], and define λ_G as the smallest positive Laplacian eigenvalue of the graph G. Recall that the Laplacian of a graph G is the matrix: $I(G) - A(G)$, where $A(G)$ is the adjacency matrix of the graph G ($A(G)$ is an $n \times n$ matrix with rows and columns indexed by vertices of the graph G and entries $a_{uv}, u, v \in V$ equal to 1 if there is an edge between vertices u and v and 0 if not) and $I(G)$ is the diagonal matrix with vertex degrees on the diagonal i.e. $i_{vv} = d(v)$, and $i_{uv} = 0$ if $u \neq v$.

For $X \subseteq V$, let $vol(X)$ denote $\sum_{v \in X} d(v)$.

The connection between eigenvalues and isoperimetric inequalities has been subject of study since long. We recall the following theorem from Section 2.2 of [8]: for $X \subseteq V$

$$|\partial X| \geq \frac{\lambda_G}{2} \min(vol(X), vol(V - X)). \tag{1}$$

Assume now that $G = (V_0, V_1, E)$ is a bipartite graph in an optimal bipartite drawing D. Let v_i denote the i-th vertex in V_0 and A_i denote the set of the first i vertices in V_0. Let X_i denote the side of A_i in the vertex partition defined by $sep(A_i, V_0 - A_i)$. Use (1):

$$\begin{aligned} |sep(A_i, V_0 - A_i)| &= |\partial X_i| \geq \frac{\lambda_G}{2} \min(vol(X_i), vol(V - X_i)) \\ &\geq \frac{\lambda_G}{2} \min(vol(A_i), vol(V_0 - A_i)). \end{aligned}$$

Using the previous formula for estimating $|\partial X|$ in Theorem 2.1, instead of an explicit function $f(x)$ that is rarely known, we end up with the estimate

$$2bcr(G) \geq \sum_{i=1}^{|V_0|-1} (d(v_{i+1}) - 1) \left(\frac{\lambda_G}{2} \min(vol(A_i), vol(V_0 - A_i)) - d(v_{i+1}) \right). \tag{2}$$

Formula (2) gives tighter bounds than most approaches e.g. [20] or [17] or [25] combined with [30], but is not as good as using tight isoperimetric inequalities, if they are available.

Take for example the hypercubes. In this case $\lambda_{Q_N} = 2/N$ (p. 6 in [8]) and (2) yields the lower bound of Theorem 3.3 with a slightly weaker (halved) multiplicative constant.

5 Appendix

Proof of Theorem 2.2 Assume that using A, we partition the graph G to 2 vertex disjoint subgraphs G_1 and G_2 recursively. Let m denote the number of edges in G, and $\bar{b}(G)$ denote the number of those edges having one endpoint in the vertex set of G_1, and the other in the vertex set of G_2. Let D_{G_1}, and D_{G_2} be the bipartite drawings already obtained by the algorithm for G_1 and G_2, respectively, so that the the vertices from the same part of G are on the same line. Place D_{G_1} on the left of D_{G_2} so that the drawings are disjoint. Let D_G denote the drawing obtained for G, by inserting the edges in the bisection. We have

$$bcr(D) \leq bcr(D_{G_1}) + bcr(D_{G_2}) + \bar{b}^2(G) + \bar{b}(G)(m - \bar{b}(G)) \leq bcr(D_{G_1}) + bcr(D_{G_2}) + m\bar{b}(G).$$

But since we use, the approximation algorithm A for $b(G)$, we have $\bar{b}(G) = O(b_{\gamma_\epsilon}(G)\log n)$. Now observe that $m = O(n)$, as the graph is degree bounded, and use Corollary 2.1 to obtain

$$bcr(D) \leq bcr(D_{G_1}) + bcr(D_{G_2}) + O\left(\log n(bcr(G) + \sum_{v\in V} d^2(v))\right).$$

Note that $bcr(G) + \sum_{v\in V} d^2(v) \geq bcr(G_1) + \sum_{v\in V(G_1)} d^2(v) + bcr(G_2)$ $+ \sum_{v\in V(G_2)} d^2(v)$, and hence we deduce after $O(\log n)$ iterations that

$$bcr(D) = O\left(\log^2 n(bcr(G) + \sum_{v\in V} d^2(v))\right).$$

To finish the proof, we will show that $bcr(G) = \Omega(\sum_{v\in V} d^2(v))$. Indeed we only need to show $bcr(G) = \Omega(n)$, since G is degree bounded. However, it is easy to see that $bcr(G) \geq m - n + 1$ [26], and consequently we deduce that $bcr(G) = \Omega(n)$, since $m \geq (1+\mu)n$. □

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