Biplanar Crossing Numbers I:
A Survey of Results and Problems

Éva Czabarka
NCBI/NLM/NIH, Bethesda, MD 20892-6510, USA

Ondrej Sýkora*
Department of Computer Science, Loughborough University
Loughborough, Leicestershire LE11 3TU, The United Kingdom

László A. Székely†
Department of Mathematics, University of South Carolina
Columbia, SC 29208, USA

Imrich Vrťo‡
Department of Informatics, Institute of Mathematics
Slovak Academy of Sciences
Dúbravská 9, 842 35 Bratislava, Slovak Republic

This paper is dedicated to the 70th birthdays of András Hajnal and Vera T. Sós.

Abstract

We survey known results and propose open problems on the biplanar crossing number. We study biplanar crossing numbers of specific families of graphs, in particular, of complete bipartite graphs. We find a few particular exact values and give general lower and upper bounds for the biplanar crossing number. We find the exact biplanar crossing number of $K_{5,q}$ for every q.

1 Introduction

During WWII in a forced work camp, Paul Turán [27] introduced the crossing number problem, in particular the Brick Factory Problem, which asks for the crossing number of complete bipartite graphs. The present paper surveys the few known results and proposes open problems on a variant of the crossing number, the *biplanar crossing number*, and solves the biplanar version of the Brick Factory Problem for $K_{5,q}$ exactly.

Recall that a graph G is *biplanar* [5], if one can write $G = G_1 \cup G_2$, where G_1 and G_2 are planar graphs. Let $\mathrm{cr}(G)$ denote the standard crossing number of the graph G, i.e. the minimum number of crossings of its edges over all possible drawings of G in the plane, under the usual rules for drawings for crossing numbers [20, 26]. Motivated by printed

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circuit boards, Owens [15] introduced the *biplanar crossing number* of a graph G, that we denote by $\mathrm{cr}_2(G)$. By definition $\mathrm{cr}_2(G) = \min\{\mathrm{cr}(G_1) + \mathrm{cr}(G_2)\}$, where the minimum is taken over all unions $G = G_1 \cup G_2$. A *biplanar drawing* of a graph G means drawings of two subgraphs, G_1 and G_2, of G, on two disjoint planes under the usual rules for drawings for crossing numbers, such that $G_1 \cup G_2 = G$. Owens described a biplanar drawing of the complete graph K_n with $\mathrm{cr}_2(K_n) \le 7n^4/1536 + O(n^3)$. One can define $\mathrm{cr}_k(G)$ similarly for any $k \ge 2$, making G a union of k subgraphs. Determining $\mathrm{cr}_k(G)$ would have application to the design of multilayer VLSI circuits [1]; but perhaps the case $k = 2$ is the most interesting, and even this simplest case is little explored so far. Note that one always can realize $\mathrm{cr}_2(G)$ by drawing the edges of G_1 and G_2 on two different sides of the same plane, while identical vertices of G_1 and G_2 are placed to identical locations on the plane on the two sides.

The biplanar crossing number problem is related to the *thickness* and *book crossing number problems*. The *thickness* $\Theta(G)$ of G is the minimum number of planar graphs whose union is G. By definition, $\mathrm{cr}_2(G) = 0$ if and only if $\Theta(G) \le 2$, i.e. G is *biplanar*. The nature of the crossing number and the biplanar crossing number problems seems different, since testing whether $\mathrm{cr}(G) = 0$ can be done in linear time, while testing biplanarity is an NP-complete problem [12]. Asano's result [3] implies that if a graph is toroidal, then $\mathrm{cr}_2(G) = 0$. Surveys on biplanar graphs and the thickness problem can be found in [5, 13].

A k-book embedding of a graph G consists of placing vertices of G on the spine of a book and drawing each edge on one of the k pages. The *book crossing number* of G, denoted by $\mu_k(G)$, is the minimum total number of crossings on all pages among all k-page book embedding of G [21]. One can easily observe that $\mathrm{cr}_2(G) \le \mu_4(G)$.

We denote by n the order and by m the size of a graph, and we deviate from this rule only for complete bipartite graphs.

We are indebted to an anonymous referee for his comments.

2 General Results

2.1 Variants of Euler's Formula

Little is known about the biplanar crossing number in general. Some of the lower bounds for crossing numbers, *mutatis mutandis* apply to biplanar crossing numbers. For example, the lower bound resulting from Euler's formula, $\mathrm{cr}(G) \ge m - 3n + 6$ for $n \ge 3$, generalizes to

$$\mathrm{cr}_2(G) \ge m - 6n + 12. \tag{1}$$

There is a strengthening of the lower bound resulting from Euler's formula for graphs G with girth $\ge g$, $\mathrm{cr}(G) \ge m - g(n-2)/(g-2)$ for $n \ge g$; and we get

$$\mathrm{cr}_2(G) \ge m - 2\frac{g}{g-2}(n-2) \tag{2}$$

for $n \ge g$ (it follows from combining Theorem 2.1 in [5] with the arguments in [20]). Pach and Tóth showed ([18] and personal communication from G. Tóth) that with $n \ge 3$

$$\mathrm{cr}(G) \ge 6m - 33n + 66, \tag{3}$$

and for triangle-free graphs with $n \ge 4$

$$\mathrm{cr}(G) \ge 6m - 27n + 54. \tag{4}$$

These results immediately imply their counterparts for the biplanar crossing number:

$$\mathrm{cr}_2(G) \geq 6m - 66n + 132, \tag{5}$$

for $n \geq 3$; and for triangle-free graphs with $n \geq 4$

$$\mathrm{cr}_2(G) \geq 6m - 54n + 108. \tag{6}$$

2.2 Other Lower Bounds

Using our (1) instead of formula (1) from [20] in the second proof of Theorem 3.2 in [20], one obtains the following biplanar counterpart of the Leighton [10] and Ajtai et al. [2] bound: for all $c > 6$, if $m \geq cn$, then

$$\mathrm{cr}_2(G) \geq \frac{c-6}{c^3} \cdot \frac{m^3}{n^2}. \tag{7}$$

For somewhat denser graphs one can improve (7) using the Pach-Tóth results cited above.

Pach, Spencer and Tóth [17] proved a conjecture of Simonovits, improving the bound of (7). If G has girth $> 2r$ and $m \geq 4n$, then

$$\mathrm{cr}(G) = \Omega\left(\frac{m^{r+2}}{n^{r+1}}\right). \tag{8}$$

It is easy to see that (8) also holds for cr_2 instead of cr, if $m \geq 8n$.

Lower bounds for the crossing number based on the *counting method* [20] generalize to similar arguments setting lower bounds for the biplanar crossing number. Since we are going to use it, we review the counting method. Assume that we have a sample graph H. Take a graph G together with a biplanar drawing which realizes its biplanar crossing number. Without loss of generality we may assume that no adjacent edges cross and any two edges cross at most once in the drawing [26]. If we find A copies of H in G, and no crossing of the drawing belongs to more than B copies of H, then

$$\mathrm{cr}_2(G) \geq \mathrm{cr}_2(H)\frac{A}{B}.$$

However, important techniques as the *embedding method* [10] or the *bisection width method* [16], [24] (see also the survey [20]) do not seem to generalize to biplanar crossing numbers. Even worse, as Tutte noted [5], the biplanar crossing number is not an invariant for homeomorphic graphs; in fact, the edges of every graph can be subdivided such that the subdivided graph is biplanar! Furthermore, Beineke [5] shows that the minimum number of subdivisions needed to make a graph biplanar equals the minimum number of edges whose deletion leaves a biplanar graph.

Open Problem 1. *Find lower bound arguments for the biplanar crossing number based on structural properties of graphs, not merely on the density of graphs.*

J. Spencer [25] was the first to find such a lower bound. Say that a graph of order n and size m has property (*), if for every vertex set A with $n/6 \leq |A| \leq 5n/6$, the number of edges between A and $\bar{A}$ is at least $m/10000$. Spencer showed that if $m > cn$ for a certain c, $\sum d_i^2 = o(m^2)$, and the graph has the (*) property, then $\mathrm{cr}_2(G) = \Omega(m^2)$. Since random graphs have the (*) property, the biplanar crossing number of the random graph is $\Omega(p^2\binom{n}{2}^2)$ for $p \geq c'/n$. Bounded degree expander graphs also have property (*).

2.3 Drawings, Upper Bounds

We showed [23] using a randomized algorithm, that for all graphs G,

$$\mathrm{cr}_2(G) \leq \frac{3}{8}\mathrm{cr}(G). \tag{9}$$

However, one cannot give an upper bound for $\mathrm{cr}(G)$ in terms of $\mathrm{cr}_2(G)$, since there are graphs G of order n and size m, with crossing number $\mathrm{cr}(G) = \Theta(m^2)$ (i.e. as large as possible) and biplanar crossing number $\mathrm{cr}_2(G) = \Theta(m^3/n^2)$ (i.e. as small as possible), for any $m = m(n)$, where m/n exceeds a certain absolute constant. As [23] shows, such graphs G can be obtained from a certain graph H with $\mathrm{cr}(H) = \Theta(m^3/n^2)$, such that vertices of H are identified with identically named vertices of H^π, where H^π is obtained from H by permuting the vertices randomly.

Open Problem 2. *What is the smallest number c^* (in place of 3/8), with which (9) is true?*

Owens [15] came up with a conjectured cr_2-optimal drawing of K_n which has about 7/24 of the crossings of a conjectured cr-optimal drawing of K_n. This might give some basis to conjecture that $c^* \leq 7/24$. On the other hand, we will show in (19)y that $cr_2(K_n) \geq n^4/952$ for large n, and comparison with $cr(K_n) \leq n^4/64$ [29] proves $c^* \geq 64/952$. We used (9) to prove that for any graph G, $\Theta(G) - 2 = O(\mathrm{cr}(G)^{.4057})$ [23]. It is likely that .4057 can be replaced by smaller constants, perhaps with .25. The example of a complete graph shows that the constant cannot be smaller than .25.

We see a curious phenomenon. Call a biplanar drawing realizing the biplanar crossing number of a graph G *self-complementary*, if the subgraphs G_1 and G_2 are isomorphic in the graph theoretic sense. K_8 is biplanar, and a self-complementary drawing shows it [5], and the same can be told about $K_{5,12}$. Self-complementary biplanar drawings are very convenient to draw. As G_1 and G_2 are isomorphic we only need to label the vertices by symbols like $(a : b)$, which means that the vertex in question is vertex a in the drawing on the first plane, and is vertex b in the drawing on the second plane. (See Figs. 1, 3, 4.) Our drawing in Theorem 6 for the hypercube Q_k with even k—although clearly not optimal, but probably near-optimal—is also self-complementary.

Open Problem 3. *Find conditions, which imply that a graph G has some self-complementary optimal biplanar drawing, i.e. where G_1 is isomorphic to G_2. In particular, does $K_{6,q}$ have such a drawing?*

Concerning upper bounds for $\mathrm{cr}_2(G)$, in terms of m, we proved in a joint paper with Shahrokhi [21] a general upper bound for the k-page book crossing numbers of graphs:

$$\mu_k(G) \leq \frac{1}{3k^2}\left(1 - \frac{1}{2k}\right)m^2 + O\left(\frac{m^2}{kn}\right), \tag{10}$$

which together with $\mathrm{cr}_2(G) \leq \mu_4(G)$ gives a general upper bound on $\mathrm{cr}_2(G)$

$$\mathrm{cr}_2(G) \leq \frac{7}{384}m^2 + O\left(\frac{m^2}{n}\right). \tag{11}$$

3 Results and Problems on Complete Bipartite Graphs

The famous Zarankiewicz's Crossing Number Conjecture or Turán's Brick Factory Problem is as follows:

$$\mathrm{cr}(K_{p,q}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor. \tag{12}$$

D. J. Kleitman showed that (12) holds for $q \leq 6$ [9] and also proved that the smallest counterexample to the Zarankiewicz's conjecture must occur for odd p and q. D. R. Woodall used elaborate computer search to show that (12) holds for $K_{7,7}$ and $K_{7,9}$. Thus, the smallest unsettled instances of Zarankiewicz's conjecture are $K_{7,11}$ and $K_{9,9}$. The following remarkable construction suggests Zarankiewicz's conjecture: place $\lfloor p/2 \rfloor$ vertices to negative positions on the x-axis, $\lceil p/2 \rceil$ vertices to positive positions on the x-axis, $\lfloor q/2 \rfloor$ vertices to negative positions on the y-axis, $\lceil q/2 \rceil$ vertices to positive positions on the y-axis, and draw pq edges by straight line segments to obtain a drawing of $K_{p,q}$.

In this section we work towards a biplanar analogue of the Zarankiewicz's Conjecture and make conjectures for the cases $q = 6$ and 8.

3.1 Lower Bounds for Complete Bipartite Graphs

The girth formula (2) yields

$$\mathrm{cr}_2(K_{p,q}) \geq pq - 4(p + q - 2). \tag{13}$$

One can use the counting argument with $H = K_{10,10}$, $G = K_{p,q}$, and the fact that $\mathrm{cr}_2(K_{10,10}) \geq 28$ from (13), to obtain:

Theorem 1. *For $10 \leq p \leq q$, we have*

$$\mathrm{cr}_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{290}. \tag{14}$$

For $p \leq 9$ we make a finer analysis of $\mathrm{cr}_2(K_{p,q})$.

3.2 Exact Results for Complete Bipartite Graphs

It is easy to see that $K_{4,q}$ is always biplanar. The result on the thickness of complete bipartite graphs of Harary et al. [4] implies that for $q \leq 12$, $\Theta(K_{5,q}) \leq 2$ and $\Theta(K_{5,13}) = 3$. Hence $\mathrm{cr}_2(K_{5,13}) \geq 1$. Paterson [19] observed that $\mathrm{cr}_2(K_{5,13}) = 1$. Determining the biplanar crossing number of $K_{5,q}$ for $q \geq 14$ is the main result of this paper.

Theorem 2. *For any $q \geq 1$, we have*

$$\mathrm{cr}_2(K_{5,q}) = \left\lfloor \frac{q}{12} \right\rfloor \left(q - 6 \left\lfloor \frac{q}{12} \right\rfloor - 6 \right),$$

and for even q there is an optimal drawing, which is self-complementary.

Proof. We provide a drawing first. Assume that $q = 12a + b$, $0 \leq b < 12$. Partition the q vertices into 12 consecutive arcs, which are as equal as possible. Let these arcs be $S_1, S_2, ..., S_{12}$. Clearly b arcs contain $a + 1$ vertices and $12 - b$ arcs contain a vertices. Consider the regular 12-gon inscribed into the unit circle centered at (0,0), with one vertex

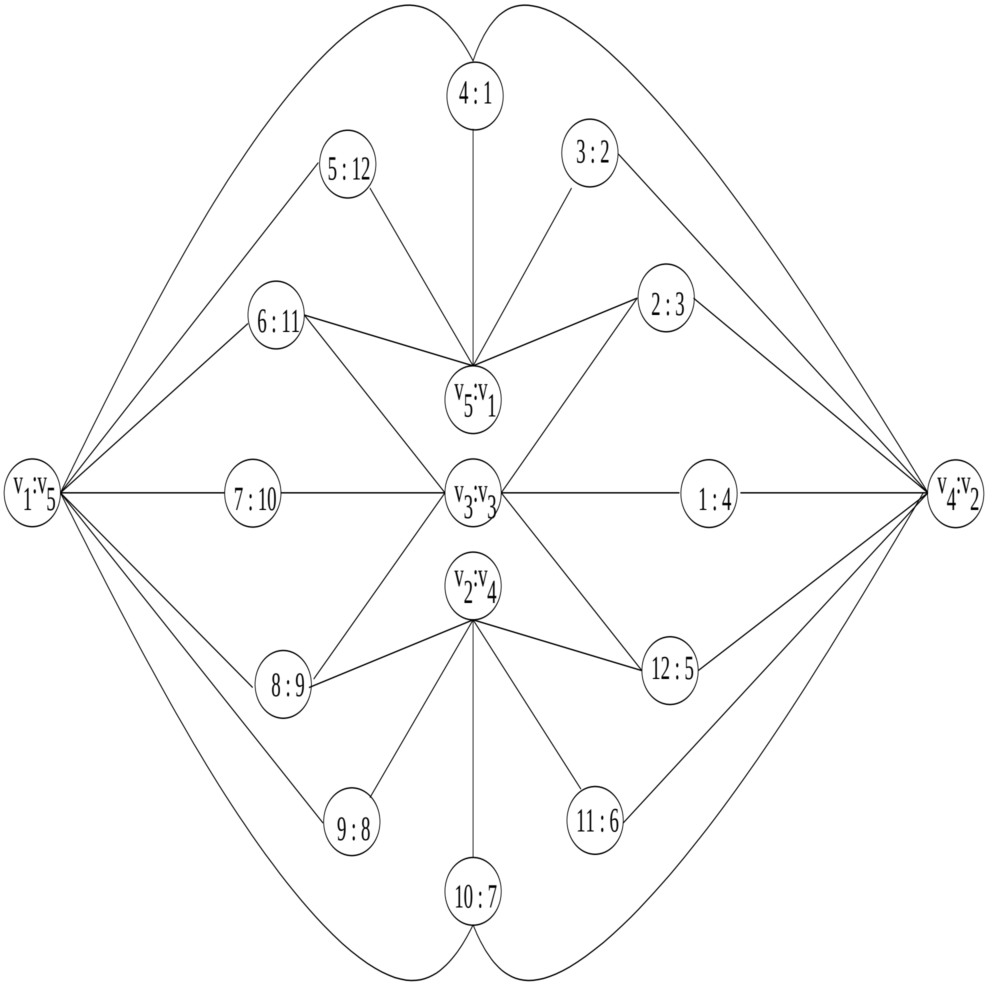

Figure 1: Self-complementary drawing of $K_{5,12}$

placed in (1,0). Fig. 1 shows a self-complementary biplanar drawing of $K_{5,12}$, where the 12 vertices are placed into the vertices of the regular 12-gon. To draw $K_{5,q}$, we place the 5 vertices into the locations as they take in Fig. 1.We use small neighborhoods of the vertices of this regular 12-gon for the placement of the 12 arcs on the circumscribed circle of the 12-gon, starting with S_1 at $(1,0)$, and going counterclockwise, i.e. put S_i where the vertex is $(i : 5-i)$ on the figure. Now we describe a drawing of $K_{5,q}$ on the first plane.
Place v_1 at $(-2,0)$ and join it to $S_4, S_5, S_6, S_7, S_8, S_9, S_{10}$.
Place v_2 at $(0,-\frac{1}{2})$ and join it to $S_8, S_9, S_{10}, S_{11}, S_{12}$.
Place v_3 at $(0,0)$ and join it to S_{12}, S_1, S_2 and S_6, S_7, S_8.
Place v_4 at $(2,0)$ and join it to $S_{10}, S_{11}, S_{12}, S_1, S_2, S_3, S_4$.
Place v_5 at $(0,\frac{1}{2})$ and join it to S_2, S_3, S_4, S_5, S_6.

On the second plane, place v_1 at $(0,\frac{1}{2})$, v_2 at $(2,0)$, v_3 at $(0,0)$, v_4 at $(0,-\frac{1}{2})$, and v_5 at $(-2,0)$. Put S_{5-i} (counting mod 12) where S_i was in the first plane and and draw the remaining edges exactly with the same curves that we used in the first plane.

In general, vertex $(i : 5-i)$ represents an arc with S_i in the first plane and an arc with S_{5-i} in the second plane. Clearly the number of crossings—as we made the necessary crossings only— is exactly

$$\sum_{i=1}^{12} \binom{|S_i|}{2} = b\binom{a+1}{2} + (12-b)\binom{a}{2}.$$

Substituting $a = \lfloor q/12 \rfloor$ and $b = q - 12\lfloor q/12 \rfloor$ into the previous formula we get the required upper bound.

We obtained above a self-complementary drawing of $K_{5,12q}$. To make this drawing self-complementary for every even q, the question is, where we put the extra $b = 2b'$ vertices. Whenever we have to add two new vertices, they must be added to arcs S_i and S_{5-i} for some i. Note that the twelve arcs make exactly 6 such pairs.

The lower bound is proved by induction on q. The claim is true for $12 \le q \le 24$, as formula (2) gives a lower bound of $q - 12$. Assume that it is true for some $q \ge 24$. Using the counting argument with $H = K_{5,q}$, $G = K_{5,q+1}$, we argue that

$$\begin{aligned}
\mathrm{cr}_2(K_{5,q+1}) &- \left\lfloor \frac{q+1}{12} \right\rfloor \left(q - 6\left\lfloor \frac{q+1}{12} \right\rfloor - 5 \right) \\
&\ge \left\lceil \frac{\binom{q+1}{q}}{\binom{q-1}{q-2}} \mathrm{cr}_2(K_{5,q}) \right\rceil - \left\lfloor \frac{q+1}{12} \right\rfloor \left(q - 6\left\lfloor \frac{q+1}{12} \right\rfloor - 5 \right) \\
&\ge \left\lceil \frac{q+1}{q-1} \left\lfloor \frac{q}{12} \right\rfloor \left(q - 6\left\lfloor \frac{q}{12} \right\rfloor - 6 \right) - \left\lfloor \frac{q+1}{12} \right\rfloor \left(q - 6\left\lfloor \frac{q+1}{12} \right\rfloor - 5 \right) \right\rceil.
\end{aligned}$$

To conclude the proof, one has to show that the expression inside the big brackets of the last line is greater than -1. This can be done by distinguishing two cases: whether $q = 11 \pmod{12}$, or not, and doing some algebra. ♠

Other exact results that we know about $\mathrm{cr}_2(K_{p,q})$ are summarized in the following table. In some interesting cases we also included lower and upper bounds.

p vs. q	7	8	9	10	11	12	13	14	15	16
6	0	0	2	4	6	8	10	12	14	16
7	1	4	7	10	13	16	19,21			
8	4	8	12	16	20	24	29,32			
9	7	12	17,19	22, 24						
10	10	16	22,24	28,32						

All the lower bounds in the table follow from the lower bound (13). Exactness for $p = 6$ follows from Theorem 3 in Subsection 3.3. Exactness for $p = 7$ follows from the drawing Fig. 2 of $K_{7,12}$ for $q = 12$; and optimal drawings for $K_{7,q}$ for $8 \le q \le 11$ can be obtained from Fig. 2 by successively erasing vertices $12, 11, 10, 9$, in this order. Note that the drawings obtained for $K_{7,8}$, $K_{7,10}$, and $K_{7,12}$ are also self-complementary. Unfortunately, we do not have a biplanar drawing of $K_{7,q}$ that we would dare to think optimal.

Exactness for $p = 8$ follows from the self-complementary drawing Fig. 3 of $K_{8,12}$; optimal drawings for $K_{8,q}$ for $6 \le q \le 11$ can be obtained from that drawing by e.g. successively erasing vertices 12,1,7,6,10,3, in this order.

One can get drawings for $K_{9,q}$ and $K_{10,q}$ from the general drawing described in Subsection 3.4. We know that as early as for $K_{11,11}$ or $K_{10,13}$, the estimation (13) is no longer the best lower bound. This follows from the arguments that lead to (6).

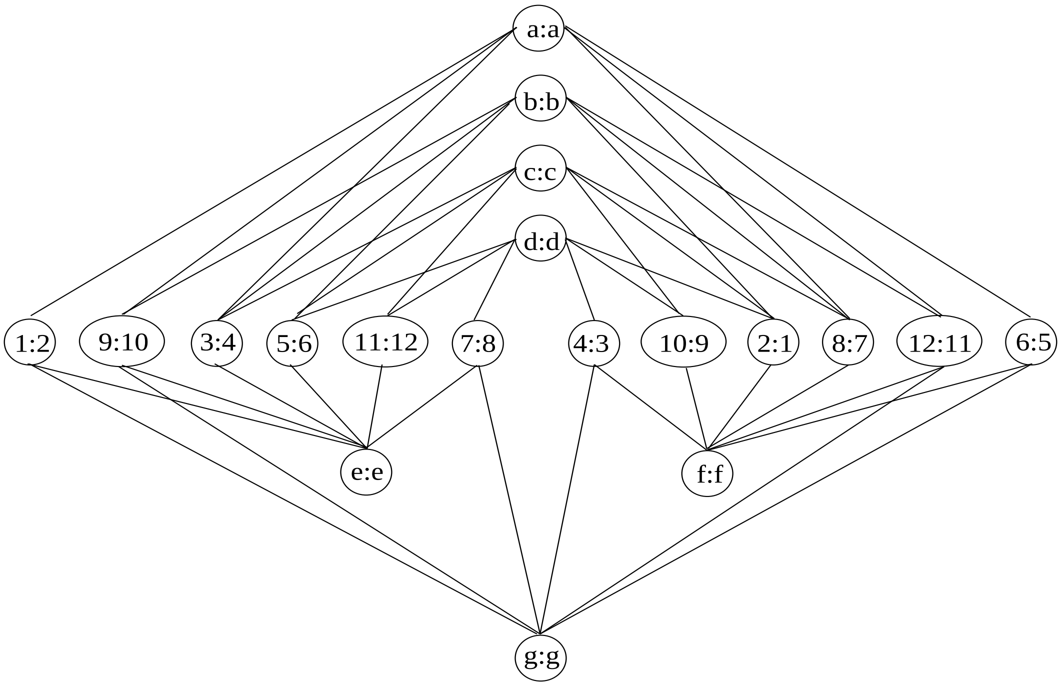

Figure 2: Self-comlementary drawing of $K_{7,12}$.

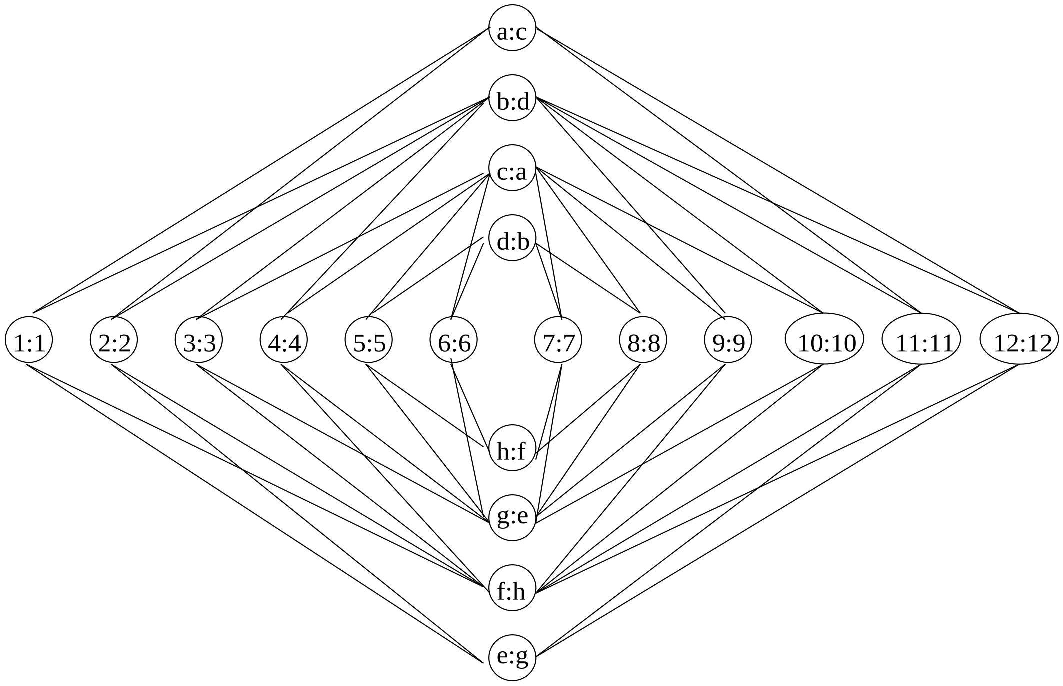

Figure 3: Self-complementary drawing of $K_{8,12}$.

3.3 Conjectured Exact Results for Complete Bipartite Graphs

Theorem 3. *For any $q \geq 1$, we have*

$$\mathrm{cr}_2(K_{6,q}) \leq 2\left\lfloor \frac{q}{8} \right\rfloor \left(q - 4\left\lfloor \frac{q}{8} \right\rfloor - 4\right).$$

This bound is optimal for any $q \leq 16$.

Proof. We provide two different drawings. *First drawing.* On both planes we draw a "thinned out" copy of the drawing from the Zarankiewicz conjecture. Place the vertices v_1, v_2 and v_3 (resp. u_1, u_2 and u_3) on the positive (resp. negative) part of the x axis, in this order from the origin. Partition the q vertices into 8 almost equal sets, S_1, S_2, S_3, S_4 and T_1, T_2, T_3, T_4. Place $S_i(T_i)$, $i = 1, 2, 3, 4$ consecutively from the origin toward infinity (minus infinity) on the y axis. On both planes we connect any v_i, u_j to all or no vertices of any S_k or T_l, and all connections are straight line segments. For the drawing on the first plane join v_1 and u_1 with S_1, S_2, T_1, T_2; v_2 and u_2 with S_2, S_3, T_2, T_3; v_3 and u_3 with S_3, S_4, T_3, T_4. For the drawing on the second plane the locations of v_i's and u_i's are the same. But place the S_i's vertices in the order S_3, S_4, S_1, S_2, from the origin toward infinity; and place the T_i's vertices in the order T_3, T_4, T_1, T_2, from the origin toward minus infinity. Draw the remaining edges with straight line segments. The number of crossings is precisely

$$2\left(\binom{|S_1|}{2} + \binom{|S_2|}{2} + \binom{|S_3|}{2} + \binom{|S_4|}{2} + \binom{|T_1|}{2} + \binom{|T_2|}{2} + \binom{|T_3|}{2} + \binom{|T_4|}{2}\right). \quad (15)$$

Simple algebra shows that this is equal to the expresion in the statement of the Theorem. *Second drawing.* Fig. 4 shows a crossing-free self-complementary drawing of $K_{6,8}$. We explain how to extend it into a self-complementary drawing with the same number of crossings as the first drawing. Assume first that $n = 8k$. Substitute every lettered vertex in Fig. 4 with k vertices on a very short straight line segment. We will join all three former neighbors of a lettered vertex to all k successors of the lettered vertex. Join one of the three from one side of the short straight line segment, and join the two others from the other side of the short straight line segment. Clearly the number of crossings is the same as in (15). If $q = 8k + r$ $(1 \le r \le 3)$, then use $k + 1$ successor vertices for r of the lettered vertices $(a : a)$ and $(c : c)$ and $(g : g)$. If $q = 8k + 4 + r$ $(1 \le r \le 3)$, then use $k + 1$ successor vertices for the lettered vertices $(e : b)$ and $(b : e)$ and $(d : f)$ and $(f : d)$; and also use $k + 1$ successor vertices for r of the lettered vertices $(a : a)$ and $(c : c)$ and $(g : g)$. The number of crossings is—in all cases—the same as in (15) again.

The optimality of the lower bound for $q \le 16$ follows from (2), which gives a lower bound of $2q - 16$. ♠

We would like to point out that if $\mathrm{cr}_2(K_{6,q})$ *is even* for every q, then the counting argument from the proof of Theorem 2, *mutatis mutandis*, can be repeated for Theorem 3. Note that if $K_{6,q}$ has an optimal biplanar drawing in which G_1 is isomorphic to G_2, as we conjecture, then $\mathrm{cr}_2(K_{6,q})$ is even.

Theorem 4. *For any $q \ge 1$, we have*

$$\mathrm{cr}_2(K_{8,q}) \le 4\left\lfloor \frac{q}{6} \right\rfloor \left(q - 3\left\lfloor \frac{q}{6} \right\rfloor - 3\right).$$

This bound is optimal for any $q \le 12$.

Proof. Place the vertices v_1, v_2, v_3 and v_4 (resp. u_1, u_2, u_3 and u_4) on the positive (resp. negative) part of the y axis, in this order from the origin. Partition the q vertices into 6 almost equal sets, S_1, S_2, S_3 and T_1, T_2, T_3. Place $S_i(T_i)$, $i = 1, 2, 3$, consecutively from the origin toward infinity (minus infinity) on the x axis. On both planes we connect any v_i, u_j to all or no vertices of any S_k or T_l, and all connections are straight line segments. For the drawing on the first plane join v_1 and u_1 with S_1, T_1; v_2 and u_2 with S_1, S_2, T_1, T_2; v_3 and u_3 with S_2, S_3, T_2, T_3 and v_4, u_4 to S_3, T_3. For the drawing on the second plane the

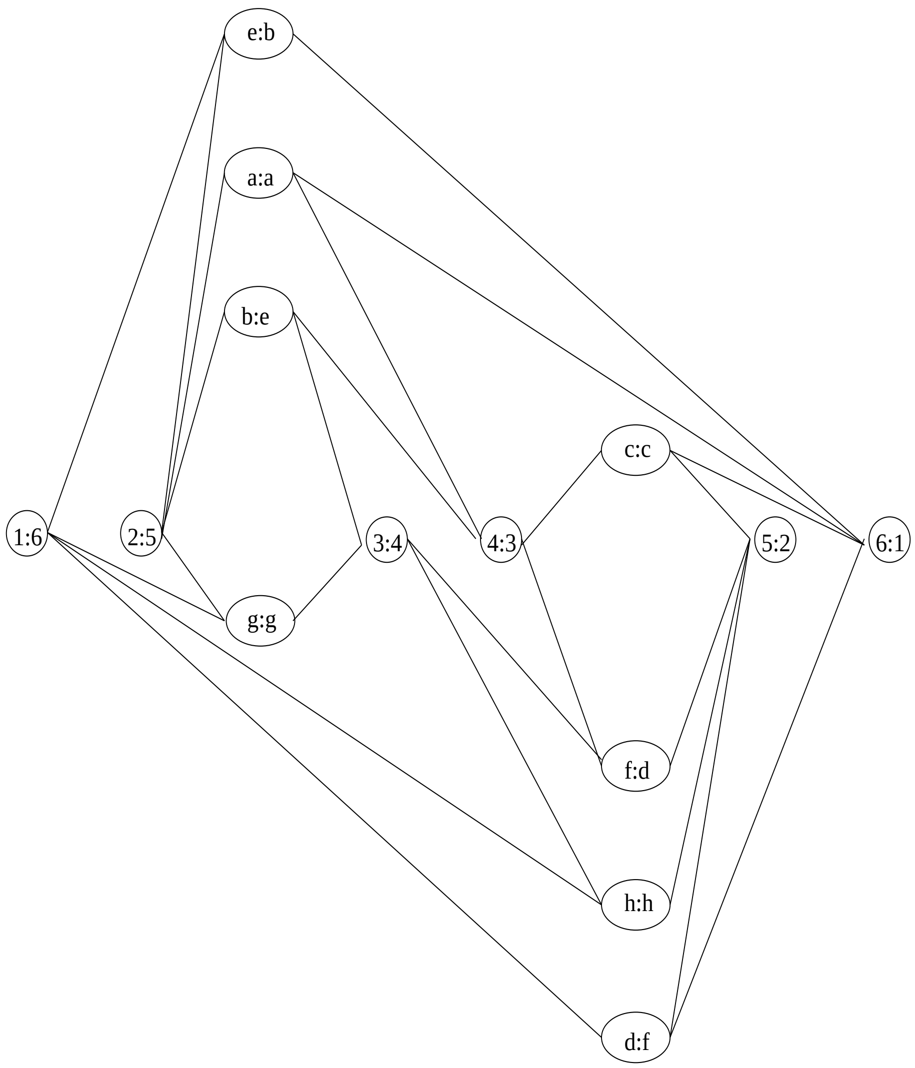

Figure 4: Self-complementary drawing of $K_{6,8}$.

locations of S_i's and T_i's are the same. But place the v_i's vertices in the order v_3, v_4, v_1, v_2, from the origin toward infinity; and place the u_i's vertices in the order u_3, u_4, u_1, u_2, from the origin toward minus infinity. Draw the remaining edges with straight line segments. The number of crossings is precisely

$$4\left(\binom{|S_1|}{2}+\binom{|S_2|}{2}+\binom{|S_3|}{2}+\binom{|T_1|}{2}+\binom{|T_2|}{2}+\binom{|T_3|}{2}\right).$$

The rest is similar as in the proof of Theorem 3. Optimality follows from (2), which gives a lower bound of $4q - 24$. ♠

Open Problem 4. *Prove that the upper bounds in Theorem 3 and in Theorem 4 are optimal.Make a first step in this direction by proving that* $\mathrm{cr}_2(K_{6,q}) = (\frac{1}{8} + o(1))q^2$.

3.4 The Best Known Drawings for other Complete Bipartite Graphs

Theorem 5. *For any $p \geq 6, q \geq 8$, we have*

$$\begin{aligned}\mathrm{cr}_2(K_{p,q}) &\leq \left\lceil \frac{p}{6} \right\rceil \left\lceil \frac{q}{8} \right\rceil \left(32 \left\lceil \frac{p}{6} \right\rceil \left\lceil \frac{q}{8} \right\rceil - 20 \left\lceil \frac{p}{6} \right\rceil - 24 \left\lceil \frac{q}{8} \right\rceil + 12 \right) \\ &\leq \frac{1}{144}(p+5)(q+7)(2pq+4p+q-7).\end{aligned}$$

Proof. We generalize the drawings for $K_{6,q}$ and $K_{8,q}$. Partition the p vertices into almost equal sets $X_1, X_2, ..., X_6$. Place X_1, X_2, X_3 (resp. X_4, X_5, X_6) on the positive (negative) part of the x axis in this order from the origin towards infinity (minus infinity). Partition the q vertices into almost equal sets $Y_1, Y_2, ..., Y_8$. Place Y_1, Y_2, Y_3, Y_4 (resp. Y_5, Y_6, Y_7, Y_8) on the positive (negative) part of the y axis in this order from the origin towards infinity (minus infinity).

On both planes we connect all vertices of any X_i to all or no vertices of any Y_j, and all connections are straight line segments. For the drawing on the first plane join X_1 and X_4 with Y_1, Y_2, Y_5, Y_6; X_2 and X_5 with Y_2, Y_3, Y_6, Y_7; X_3 and X_6 with Y_3, Y_4, Y_7, Y_8. For the drawing on the second plane the locations of X_i's are the same. Place the Y_i's vertices in the order Y_3, Y_4, Y_1, Y_2, from the origin towards infinity; and Y_7, Y_8, Y_5, Y_6, from the origin towards minus infinity on the y axis. Draw the remaining edges with straight line segments. By counting up of all kinds of crossings in the drawing and by regrouping terms we get that the number of crossings is precisely

$$\begin{aligned}&\sum_{i=1}^{6} \binom{|X_i|}{2} \sum_{j=1}^{8} \binom{|Y_j|}{2} + \\ &\left(\binom{|X_1|}{2} + \binom{|X_3|}{2} + \binom{|X_4|}{2} + \binom{|X_6|}{2} \right) (|Y_1||Y_2| + |Y_3||Y_4| + |Y_5||Y_6| + |Y_7||Y_8|) + \\ &\left(\binom{|X_2|}{2} + \binom{|X_5|}{2} \right) (|Y_1||Y_4| + |Y_2||Y_3| + |Y_5||Y_8| + |Y_6||Y_7|) + \\ &(|X_1||X_2| + |X_4||X_5|) \left(\binom{|Y_2|}{2} + \binom{|Y_4|}{2} + \binom{|Y_6|}{2} + \binom{|Y_8|}{2} \right) + \\ &(|X_2||X_3| + |X_5||X_6|) \left(\binom{|Y_1|}{2} + \binom{|Y_3|}{2} + \binom{|Y_5|}{2} + \binom{|Y_7|}{2} \right).\end{aligned}$$

First assume that p is divisible by 6 and q is divisible by 8. One can easily compute that the number of crossings is $pq(2pq - 10p - 9q + 36)/144$.

Now let p, q be arbitrary numbers. Let p' be the smallest number divisible by 6 such that $p' \geq p$ and q' be the smallest number divisible by 8 such that $q' \geq q$. Then the number of crossings is at most $p'q'(2p'q' - 10p' - 9q' + 36)/144$. Noting that $p' = 6\lceil \frac{p}{6} \rceil \leq p + 5$ and $q' = 8\lceil \frac{q}{8} \rceil \leq q + 7$ we get the claim. ♠

Open Problem 5. *Make a conjecture showing a pattern for optimal biplanar drawings of $K_{p,q}$, i.e. pose the biplanar version of the Zarankiewicz conjecture. A good conjecture for $K_{7,q}$ already seems to be hard to find.*

Open Problem 6. *Find an asymptotic formula for $\mathrm{cr}_2(K_{p,q})$ for small fixed p.*

4 Results and Problems on Other Specific Families Graphs

4.1 Complete Graphs

Note that bounding $\mathrm{cr}_2(K_n)$ is a Nordhaus-Gaddum type problem [14]. Owens gave an explicite biplanar drawing of K_n with

$$\mathrm{cr}_2(K_n) \le \frac{7}{1536}n^4 + O(n^3).$$

The same upper bound (up to the second order term), based on a different drawing follows immediately from our work with Shahrokhi [21] by setting $G = K_n$ in (11).

Harary et al. [4] and Tutte [28] showed that for $n \le 8$, $\Theta(K_n) \le 2$ and $\Theta(K_9) = 3$. Their construction actually also shows $\mathrm{cr}_2(K_9) = 1$. Applying the counting argument for $H = K_{10,10}$, $G = K_n$, and using $\mathrm{cr}_2(K_{10,10}) \ge 28$ from (13), we obtain

$$\mathrm{cr}_2(K_n) \ge \frac{1}{1158}n^4 + O(n^3). \tag{16}$$

We can do somewhat better than (16). Consider a biplanar drawing D of K_n. Then any subset of vertices induces a biplanar subdrawing, D', of the induced complete subgraph G'. Assume that G' has order n' and size $m' = \binom{n'}{2}$. According to (5),

$$\mathrm{cr}_2(G') \ge \begin{cases} 6m' - 66n' + 132 & \text{if } n' \ge 3 \\ 6m' - 66n' + 132 - 12 & \text{if } n' = 2 \\ 6m' - 66n' + 132 - 66 & \text{if } n' = 1 \\ 6m' - 66n' + 132 - 132 & \text{if } n' = 0. \end{cases} \tag{17}$$

Pick now independently with probability p vertices of K_n to obtain a random G'. Taking expectation of the inequality of two random variables, (17), we obtain:

$$p^4\mathrm{cr}_2(K_n) \ge 6mp^2 - 66np + 132 - 12\binom{n}{2}p^2(1-p)^{n-2} - 66np(1-p)^{n-1} - 132(1-p)^n. \tag{18}$$

Setting $p = 30.073871/n$ in (18) yields that for n sufficiently large,

$$\mathrm{cr}_2(K_n) \ge \frac{n^4}{952}. \tag{19}$$

It follows from the counting argument applied to $G = K_n$ and $H = K_{n-1}$, that $\mathrm{cr}(K_n)/\binom{n}{4}$ is a non-decreasing function of n, and hence has finite limit. The same argument applies to $\mathrm{cr}_2(K_n)$ as well

Open Problem 7. *Improve the lower bound in (19). Is* $\lim_{n\to\infty} \mathrm{cr}_2(K_n)/\binom{n}{4} = \frac{7}{24} \cdot \frac{24}{64} = \frac{7}{64}$? *Find exact values for the biplanar crossing numbers of complete graphs for small values* $n = 10, 11, \ldots$.

4.2 Hypercubes

For the k-dimensional hypercube Q_k, it is known that $\Theta(Q_7) \le 2$ and the estimation (2) gives $\mathrm{cr}_2(Q_8) \ge 8$. We give a general upper bound for the biplanar crossing number of hypercubes.

Theorem 6. *For $k \geq 8$*

$$\mathrm{cr}_2(Q_k) \leq \begin{cases} \frac{165}{512} 2^{\frac{3}{2}k} + O(k^2 2^k), & \text{if } k \text{ is even,} \\ \\ \frac{176}{512} 2^{\frac{3}{2}k} + O(k^2 2^k), & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Our biplanar drawing of Q_k will be based on the best known planar drawing due to Faria and Figueiredo [6] satisfying

$$\mathrm{cr}(Q_k) \leq \frac{165}{1024} 4^k - (2k^2 - 11k + 34) 2^{k-3}. \tag{20}$$

Let $0 \leq i \leq k$. Observe that all edges belonging to the first i dimensions in Q_k induce 2^i distinct hypercubes isomorphic to Q_{k-i}. Draw these hypercubes on the first plane and the 2^{k-i} hypercubes isomorphic to Q_i, induced by the last $k-i$ dimensions on the second plane, using (20). We get a biplanar drawing with

$$\mathrm{cr}_2(Q_k) \leq \frac{165}{1024} 2^{2k-i} + \frac{165}{1024} 2^{k+i}.$$

Finally, by setting $i = \lceil k/2 \rceil$, we get the result. ♠

Unfortunately, the lower bound formula (7) gives only a weak estimation of order $\Omega(k^3 2^k)$, and even (8) improves it insignificantly to $\Omega(k^4 2^k)$. In order to use (8), we have to note that we can keep a positive percentage of edges of Q_k, while destroying all 4-cycles by throwing out edges, see [8]. We know that our drawing is not optimal: some edges between vertex disjoint copies of $Q_{\lfloor k/2 \rfloor}$ (resp. $Q_{\lceil k/2 \rceil}$) can be brought over from the other plane without making new crossings, and in this way their old crossings are eliminated.

Open Problem 8. *Is the upper bound in Theorem 6 still the best possible up to a constant multiplicative factor?*

4.3 Meshes

In the standard plane crossing number theory one of the most studied graph is the toroidal mesh, i.e. the Cartesian product of two cycles. See the recent paper [7] for the almost complete exact solution. We will concentrate on the biplanar crossing number of toroidal and ordinary meshes. It is an easy exercise to show that the graph $C_{n_1} \times C_{n_2} \times C_{n_3}$ is biplanar for any $3 \leq n_1, n_2, n_3$. On the other hand $C_{n_1} \times C_{n_2} \times C_{n_3} \times C_{n_4}$ has thickness at least 3. We can prove that $\mathrm{cr}_2(P_2 \times C_n \times C_n \times C_n) = 0$. We do not know whether

Open Problem 9. *Is it true that* $\mathrm{cr}_2(P_n \times C_n \times C_n \times C_n) = 0$*?*

If it is nonzero, it is surprisingly small, since we have a biplanar drawing showing that $\mathrm{cr}_2(P_n \times C_n \times C_n \times C_n) = O(n^4)$, which is just linear in the number of edges. (Put edges from the first two dimensions on the first plane, and edges from the second two dimensions on the second plane.)

Theorem 7. *For even k*

$$\mathrm{cr}_2\left(\prod_{i=1}^{k} C_n\right) \leq 2^{\frac{k}{2}+5} n^{k-2}.$$

Proof. Put the edges of the first $k/2$ dimensions on the first plane. They induce $2^{\frac{k}{2}}$ vertex disjoint subgraphs isomorphic to $\prod_{i=1}^{\frac{k}{2}} C_n$. Place the leftover edges on the second plane. Using the estimation

$$\mathrm{cr}\left(\prod_{i=1}^{\frac{k}{2}} C_n\right) \leq 16n^{k-2}$$

from [22] we get the result. ♠

We leave it to the Reader to prove an analogue of Theorem 7 for odd k.

Open Problem 10. *Show that the the upper bound in Theorem 7 is tight.*

5 Conclusion

Our knowledge on biplanar crossing numbers is as rudimentary as it was our knowledge on crossing numbers till Leighton's work [10] in the 70's. Bisection width and graph embedding methods cannot be used, only the counting method and density-based lower bounds are available. We hope that the development of structure-based lower bounds for the biplanar crossing numbers will shed light to some so far unknown properties of ordinary crossing numbers as well.

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