On Bipartite Drawings and the Linear Arrangement Problem*

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Abstract

The bipartite crossing number problem is studied, and a connection between this problem and the linear arrangement problem is established. It is shown that when the arboricity is close to the minimum degree and the graph is not too sparse, then the optimal number of crossings has the same order of magnitude as the optimal arrangement value times the arboricity. The application of the results to a tree provides for a closed formula which expresses exactly, the optimal number of crossings in terms of the optimal value of the linear arrangement and the degree sequence, resulting in an $O(n^{1.6})$ time algorithm for computing the bipartite crossing number. Two polynomial time approximation algorithms for computing the bipartite crossing number are derived, with approximation factors, $O(\log^2 n)$, and $O(\log n \log\log n)$, from the optimal, respectively, for approximating the number of crossings, and at the same time, total edge lengths, for a large class of graphs on n vertices. No approximation algorithm which could generate a provably good solution was previously known.

The problem of computing a largest weighted biplanar subgraph of an acyclic graph is also studied and a linear time algorithm for it is derived. This problem was known to be NP-hard when the graph is planar and very sparse, and all weights are 1.

1 Introduction

The planar crossing number problem calls for placing the vertices of a graph in the plane and drawing the edges with Jordan curves, so that the number of edge crossings is minimized. This problem has been extensively studied in graph theory [32], combinatorial geometry [22], and theory of VLSI [16]. In this paper we study the *bipartite crossing number* problem which is an important variation of the planar crossing number. Throughout this paper $G = (V_0, V_1, E)$ denotes a connected bipartite graph, where V_0, V_1 are the two classes of independent vertices, and E is the edge set. We will assume that

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$|V_0 \cup V_1| = n$ and $|E| = m$. A *bipartite drawing* [13], or *2-layer drawing* of G consists of placing the vertices of V_0 and V_1 into distinct points on two parallel lines and then drawing each edge using a straight line segment connecting the points representing the endvertices of the edge. Let $bcr(G)$ denote the *bipartite crossing number* of G, that is, $bcr(G)$ is the minimum number of edge crossings over all bipartite drawings of G.

Computing $bcr(G)$ is NP-hard [11][1] but can be solved in polynomial time for bipartite permutation graphs [29]. The problem of obtaining nice multiple layer drawings of graphs (i.e. drawings with small number of crossings), has been extensively studied by the graph drawing, VLSI, and CAD communities [6, 7, 19, 30, 31]. In particular one of the most important aesthetic objectives in graph drawing is reducing the number of crossings [23]. Very recently Jünger and Mutzel, [14] and Mutzel [20] succeeded to employ integer programming methods in order to compute $bcr(G)$ exactly, or to estimate it, nevertheless, these methods do not guarantee polynomial time convergence. In fact, although a $O(\log^4 n)$ times optimal polynomial time algorithm for approximating the planar crossing number of degree bounded graphs has been known [17], no polynomial time approximation algorithm whose performance is guaranteed has been previously known for approximating $bcr(G)$. A nice result in this area is a fast polynomial time algorithm of Eades and Wormald [7] which approximates the bipartite crossing number by a factor of 3, when the positions of vertices in V_0 are fixed.

In this paper we explore an important relationship between the bipartite drawings and the linear arrangement problem, which is another well-known problem in the theory of VLSI [4, 5, 15, 18, 28]. In particular, it is shown that for many graphs the order of magnitude for the optimal number of crossings is bounded from below, and above, respectively, by minimum degree times the optimal arrangement value, and by arboricity times the optimal arrangement value, where the arboricity of G is the minimum number of acyclic graphs that G can be decomposed to. Hence for a large class of graphs, it is possible to estimate $bcr(G)$ in terms of the optimal arrangement value. Our general method for constructing the upper bound is shown to provide for an optimal solution and an exact formula, resulting to an $O(n^{1.6})$ time algorithm for computing $bcr(G)$ when G is a tree. The presence of arboricity in our upper bound allows us to relate some important topological properties such as genus and page number, to $bcr(G)$. In particular, our results easily imply that when G is "nearly planar", i.e. it either has bounded genus, or bounded page number, then, the asymptotic values of $bcr(G)$, and the optimal arrangement are the same, provided that G is not too sparse.

A direct consequence of our results is that for many graphs, the bipratite drawings with small sum of edge lenghts also have small bipartite crossings, and vis versa, and therefore, a suboptimal solution to the bipartite crossing number problem can be extracted from a suboptimal solution to the linear arrangement problem. Hence, we have derived here, the first polynomial time approximation algorithms for $bcr(G)$, which perform within a multiplicative factor of $O(\log n \log\log n)$ from the optimal, for a large class of graphs. Moreover, we show here that the traditional divide and conquer paradigm in which the divide phase approximately bisects the graph, also obtains a provably good approximation, in polynomial time, for $bcr(G)$ within a multiplicative factor of $O(\log^2 n)$ from the optimal, for a variety of graphs. Crucial to verifying the performance guarantee of the divide and conquer algorithm, is a lower bound of $\Omega(\delta_G n b_\beta(G))$, derived here, for $bcr(G)$, where $b_\beta(G)$, $\beta < 1/2$, and δ_G are the size of the β-bisection and minimum degree of G, respectively. This significantly improves Leighton's well-known lower bound of $\Omega(b^2_{\frac{1}{3}}(G))$ [16] which was derived for the planar crossing number of degree bounded graphs. The class of graphs for which the performance of our approximation algorithms is guaranteed is very large, and in particular contains those regular graphs, degree bounded graphs, and genus bounded graphs, which are not too sparse. Another notable aspect of relating $bcr(G)$ to the linear arrangement problem is that, both algorithms produce drawings with near optimal number of crossings in which the coordinates of all vertices are integers, so that the total edge length is also

[1]Technically speaking, the NP-hardness of the problem was proved for multigraphs, but it is widely assumed that it is also NP-hard for simple graphs.

near optimal, with the same performance guarantee as for the number of crossings.

We also study biplanar graphs. A bipartite graph $G = (V_0, V_1, E)$ is called a *biplanar*, if it has a bipartite drawing in which no two edges cross each other. Eades and Whitesides [8] have shown that the problem of determining largest biplanar subgraph is NP-hard even when G is planar, and the vertices in V_0 and V_1 have degrees at most 3 and 2, respectively. This raised the question of whether or not computing a largest biplanar subgraph can be done in polynomial time when G is acyclic [20]. In this paper we present a linear time dynamic programming algorithm for the weighted version of this problem in an acyclic graph. (The weighted version was first introduced by Mutzel [20].)

Section 2 contains our general results regarding the relation between $bcr(G)$ and the linear arrangement problem. Section 3 contains the applications, and includes several important observations, the bisection based lower bound for $bcr(G)$, and the approximation algorithms. Finally, Section 4 contains our linear time algorithm for computing a largest biplanar subgraph of a tree.

2 Linear arrangement and bipartite crossings

Let $G = (V_0, V_1, E)$, $V = V_0 \cup V_1$, $|V| = n$, and $v \in V$. We denote by d_v the degree of v, and by d^*_v denote the number vertices adjacent to v of degree 1. We denote by δ_G the minimum degree of G.

A *bipartite drawing* of G is obtained by: (i) placing the vertices of V_0 and V_1 into distinct points on two horizontal lines y_0, y_1, respectively, (ii) drawing each edge with one straight line segment which connects the points of y_0 and y_1 where the endvertices of the edge were placed. Hence, the order in which the vertices are placed on y_0 and y_1 will determine the drawing.

Let D_G be a bipartite drawing of G; when the context is clear, we omit the subscript G and write D. For any $e \in E$, let $bcr_D(e)$ denote the number of crossings of the edge e with other edges. Edges sharing an endvertex do not count as crossing edges. Let $bcr(D)$ denote the total number of crossings in D, i.e. $bcr(D) = \frac{1}{2}\sum_e bcr_D(e)$.

The *bipartite crossing number* of G, denoted by $bcr(G)$ is the minimum number of crossings of edges over all bipartite drawings of G. Clearly, $bcr(G) = \min_D bcr(D)$.

We assume throughout this paper that the vertices of V_0 are placed on the line y_0 which is taken to be the x-axis, and vertices of V_1 are placed on the line y_1 which is taken to be the line $y = 1$. For a vertex $v \in V_0 \cup V_1$ let $x_D(v)$ denote v's x-coordinate in the drawing D. We call the function $x_D : V \to \mathbb{R}$ the coordinate function of D. Throughout this paper, we often omit the y coordinates. Note that x_D is not necessarily an injection, since for $a \in V_0$, and $b \in V_1$, we may have $x_D(a) = x_D(b)$.

Given an arbitrary graph $G = (V, E)$, and a real function $f : V \to \mathbb{R}$, define the *length* of f, as

$$L_f = \sum_{uv \in E} |f(u) - f(v)|.$$

The *linear arrangement problem* is to find a bijection $f : V \to \{1, 2, 3, ..., |V|\}$, of minimum length. This minimum value is denoted by $\hat{L}(G)$.

Let $G = (V_0, V_1, E)$ and D be a bipartite drawing of G. Define the *length* of D to be

$$L_{x_D} = \sum_{uv \in E} |x_D(u) - x_D(v)|.$$

In this section we derive a relation between the bipartite crossing number and the linear arrangement problem.

Let D be a bipartite drawing of $G = (V_0, V_1, E)$ such that the vertices of V_0 are placed into the points

$$(1, 0), (2, 0), ..., (|V_0|, 0).$$

For $v \in V_1$, let $u_1, u_2, ..., u_{d_v}$ be its neighbors satisfying $x_D(u_1) < x_D(u_2) < ... < x_D(u_{d_v})$. Define the *median vertex* of v, $med(v) = u_{\lfloor \frac{d_v}{2} \rfloor}$, if $d_v \geq 2$, and $med(v) = u_1$, if $d_v = 1$ [7]. We say that D has the *median property* if the vertices of G have distinct x-coordinates and the x-coordinate of any vertex v in V_1 is larger than, but arbitrarily close to, $x_D(med(v))$, with the restriction that if a vertex of odd degree and a vertex of even degree have the same median vertex, then the odd degree vertex has a smaller x-coordinate. Note that if D has the median property, then x_D is an injection.

When the bipartite drawing D does not have the median property, one can always convert it to a drawing which has the property, by first placing the vertices of V_0 in the same order in which they appear in D into the locations $(1,0),(2,0),....,(|V_0|,0)$, and then placing each $v \in V_1$ on a proper position so that the median property holds. Such a construction is called the *median construction* and was utilized by Eades and Wormald [7] to obtain the following remarkable result.

Theorem 2.1 [7] *Let $G=(V_0,V_1,E)$, and D be a bipartite drawing of G. If D' is obtained using the median construction from D, then*

$$bcr(D') \leq 3bcr(D).$$

□

2.1 Lower bounds

Let $G=(V_0,V_1,E)$ and D be a bipartite drawing of G. Consider an edge $e=ab \in E$, and let u be a vertex in $V_0 \cup V_1$ so that $u \notin \{a,b\}$. We say e *covers* u in D, if the line parallel to the y axis passing through u has a point in common with the edge e. Thus for $e=ab$, $a \in V_0, b \in V_1$, neither a nor b are covered by e. However, a vertex $c \in V_1$ with $x_D(c)=x_D(a)$ is covered by e. Let $N_D(e)$ denote the number of those vertices in V_1 which are covered by e in D. We will use the following two lemmas later.

Lemma 2.1 *For $G=(V_0,V_1,E)$, let D be a bipartite drawing of G. Recall that x_D is the coordinate function of D. Then, the following hold.*

(i) Assume that $x_D(v)$ is an integer for all $x \in V_0$. Then, there is a bijection $f^*: V_0 \cup V_1 \rightarrow \{1,2,...,n\}$ so that for any $e=ab \in E$, it holds

$$|f^*(a)-f^*(b)| \leq N_D(e)+|x_D(a)-x_D(b)|+1.$$

(ii) Assume that D has the median property. Then for the bijection f^* in (i), it holds

$$L_{f^*} \leq \frac{8bcr(D)}{\delta_G}+L_{x_D}+\sum_{a \in V_0} d_a d_a^* + m.$$

Proof. To prove (i), we construct f^* by moving all vertices in V to integer locations. Formally, let $w_1, w_2, ..., w_n$ be the order of vertices of $V_0 \cup V_1$ such that $x_D(w_1) \leq x_D(w_2) \leq ... \leq x_D(w_n)$. (Note that we may have $x_D(w_i)=x_D(w_{i+1})$, for some i, $w_i \in V_0$, $w_{i+1} \in V_1$, since x_D may not be an injection.) Define $f^*(w_i)=i$, $1 \leq i \leq n$, then the proof of (i) easily follows. (In particular note that the factor $+1$ appears in the upper bound, since the end point of e which belongs to V_1 may not have an integer coordinate.) For (ii), let $e=ab \in E$, $a \in V_0, b \in V_1$. Assume $x(a)>x(b)$, and let v be any vertex in V_1 covered by e in D. Since D has the median property, at least $\lfloor d_v/2 \rfloor$ of vertices adjacent to v are separated from v in D by the straight line segment e. This means, in this case, that vertex v generates at least $\lfloor \delta_G/2 \rfloor \geq (\delta_G-1)/2$ crossings on e. Moreover, vertex v, even if it has degree 1, generates one crossing on e, since v and $med(v)$ are separated by the line segment e in D. Thus $bcr(e) \geq \frac{1}{2}N_D(e)(1+\frac{\delta_G-1}{2})=N_D(e)\frac{\delta_G+1}{4}$. Now assume $x_D(a)<x_D(b)$, and let v be a vertex covered by e. Then, v generates at least $d_v-\lfloor \frac{d_v}{2} \rfloor \geq d_v/2$ crossings on e provided that v is not a vertex of degree 1 which is adjacent only to a. Consequently, in this case, $bcr_D(e) \geq (N_D(e)-d_a^*)\delta_G/2$. We conclude that in either case, $bcr_D(e) \geq \frac{1}{4}(N_D(e)-d_a^*)\delta_G$, and hence $N_D(e) \leq \frac{4bcr(e)}{\delta_G}+d_a^*$, and consequently, using (i),

$$|f^*(a)-f^*(b)| \leq \frac{4bcr(e)}{\delta_G}+d_a^*+|x_D(a)-x_D(b)|+1.$$

To finish the proof of (ii) take the sum over all $e = ab \in E$. □

Lemma 2.2 *Let $G = (V_0, V_1, E)$, and let D be a bipartite drawing of G which has the median property, then*

$$L_{x_D} \le \epsilon + \sum_{\substack{uv\in E, u\in V_0, v\in V_1 \\ d_v \ge 2}} |x_D(u) - x_D(v)|.$$

with an arbitrary small $\epsilon > 0$.

Proof. To prove the claim, let $uv \in E$ with $v \in V_1$ so that $d_v = 1$. Since D has the median property, $med(v) = u$, and thus v is placed arbitrary close to u. So we may assume that $|x_D(v) - x_D(u)| \le \frac{\epsilon}{V_1}$. This way the total sum of the contributions of all edges which are incident to a vertex of degree one in V_1 to L_{x_D} is at most $|V_1| \frac{\epsilon}{|V_1|} = \epsilon$ and the claim follows. □

We now prove the main result of this section.

Theorem 2.2 *Let $G = (V_0, V_1, E)$, then*

$$bcr(G) + \frac{1}{12} \sum_{v\in V} d_v^2 \ge \frac{1}{36} \delta_G \hat{L}(G).$$

Proof. Let D be a bipartite drawing of G. We will construct an appropriate bijection $f^* : V_0 \cup V_1 \to \{1, 2, ..., n\}$. Let D' be a drawing which is obtained by applying the median construction to D. Let $v \in V_1$ with $d_v \ge 2$, and let $u_1, u_2, ..., u_{d_v}$ be its neighbors with $x_{D'}(u_1) < x_{D'}(u_2) < ... < x_{D'}(u_{d_v})$. Let i be an integer, $1 \le i \le \lfloor d_v/2 \rfloor$, and let u be a vertex in V_0 so that $x_{D'}(u_i) < x_{D'}(u) < x_{D'}(u_{d_v-i+1})$. Observe that u generates d_u crossings on the edges $u_i v$ and $u_{d_v-i+1}v$, if it is not adjacent to v. Similarly, u generates $d_u - 1$ crossings on the edges $u_i v$ and $u_{d_v-i+1}v$, if it is adjacent to v. Thus

$$bcr_{D'}(u_i v) + bcr_{D'}(u_{d_v-i+1}v) \ge (x_{D'}(u_{d_v-i+1}) - x_{D'}(u_i) - 1)\delta_G - d_v$$

$$= (x_{D'}(u_{d_v-i+1}) - x_{D'}(v) + x_{D'}(v) - x_{D'}(u_i) - 1)\delta_G - d_v. \tag{1}$$

Note that D' has the median property, thus for $i = 1, 2, ..., \lfloor d_v/2 \rfloor$,

$$x_{D'}(u_i) < x_{D'}(v) < x_{D'}(u_{d_v-i+1})$$

and hence (1) implies

$$\begin{aligned} bcr_{D'}(u_i v) + bcr_{D'}(u_{d_v-i+1}v) \;&\ge\; (|x_{D'}(v) - x_{D'}(u_{d_v-i+1})| + |x_{D'}(v) \\ &-\; x_{D'}(u_i)| - 1)\delta_G - d_v. \end{aligned} \tag{2}$$

Using (2) observe that, for $v \in V_1$ with $d_v \ge 2$,

$$\sum_{i=1}^{\lfloor \frac{d_v}{2} \rfloor} (bcr_{D'}(u_i v) + bcr_{D'}(u_{d_v-i+1}v))$$

$$\ge \delta_G \sum_{i=1}^{\lfloor \frac{d_v}{2} \rfloor} (|x_{D'}(v) - x_{D'}(u_i)| + |x_{D'}(v) - x_{D'}(u_{d_v-i+1})|) - \delta_G \left\lfloor \frac{d_v}{2} \right\rfloor - \left\lfloor \frac{d_v}{2} \right\rfloor d_v. \tag{3}$$

Thus, using (3), when $d_v \geq 2$ is even, we have

$$\sum_{i=1}^{d_v} bcr_{D'}(u_i v) = \sum_{i=1}^{\lfloor \frac{d_v}{2} \rfloor} (bcr_{D'}(u_i v) + bcr_{D'}(u_{d_v-i+1} v))$$
$$\geq \delta_G \sum_{i=1}^{\lfloor \frac{d_v}{2} \rfloor} (|x_{D'}(v) - x_{D'}(u_i)| + |x_{D'}(v) - x_{D'}(u_{d_v-i+1})|) - \delta_G \left\lfloor \frac{d_v}{2} \right\rfloor - \left\lfloor \frac{d_v}{2} \right\rfloor d_v$$
$$= \delta_G \sum_{i=1}^{d_v} |x_{D'}(v) - x_{D'}(u_i)| - \delta_G \left\lfloor \frac{d_v}{2} \right\rfloor - \left\lfloor \frac{d_v}{2} \right\rfloor d_v. \tag{4}$$

Moreover, when $d_v \geq 2$ is odd, we have,

$$\begin{aligned} \sum_{i=1}^{d_v} bcr_{D'}(u_i v) &\geq bcr_{D'}(u_{\lfloor \frac{d_v}{2} \rfloor} v) + bcr_{D'}(u_{\lceil \frac{d_v}{2} \rceil} v) \\ &\geq (x_{D'}(u_{\lceil \frac{d_v}{2} \rceil}) - x_{D'}(u_{\lfloor \frac{d_v}{2} \rfloor}) - 1)\delta_G, \end{aligned}$$

where the upper bound is obvious, and the lower bound holds since no vertex adjacent to v is between $u_{\lceil \frac{d_v}{2} \rceil}$ and $u_{\lfloor \frac{d_v}{2} \rfloor}$. Consequently, when $d_v \geq 2$ is odd, we have,

$$\begin{aligned} \sum_{i=1}^{d_v} bcr_{D'}(u_i v) &\geq bcr_{D'}(u_{\lfloor \frac{d_v}{2} \rfloor} v) + bcr_{D'}(u_{\lceil \frac{d_v}{2} \rceil} v) \\ &\geq (x_{D'}(u_{\lceil \frac{d_v}{2} \rceil}) - x_{D'}(v) + x_{D'}(v) - x_{D'}(u_{\lfloor \frac{d_v}{2} \rfloor}) - 1)\delta_G \\ &\geq \delta_G |x_{D'}(v) - x_{D'}(u_{\lceil \frac{d_v}{2} \rceil})| - \delta_G, \end{aligned}$$

where the last line is obtained by observing that $x_{D'}(u_{\lceil \frac{d_v}{2} \rceil}) > x_{D'}(v) > x_{D'}(med(v)) = x_{D'}(u_{\lfloor \frac{d_v}{2} \rfloor})$. Combining this with (3), for odd d_v, we obtain

$$2 \sum_{i=1}^{d_v} bcr_{D'}(u_i v) \geq \delta_G \sum_{i=1}^{d_v} |x_{D'}(v) - x_{D'}(u_i)| - \delta_G - \delta_G \left\lfloor \frac{d_v}{2} \right\rfloor - \left\lfloor \frac{d_v}{2} \right\rfloor d_v. \tag{5}$$

We note that since (5) is weaker than (4), it must also hold when d_v is even, and conclude by summing (5) over all $v \in V_1$ with $d_v \geq 2$, that

$$\begin{aligned} 4bcr(D') &\geq \delta_G \sum_{\substack{uv \in E, v \in V_1 \\ d_v \geq 2}} |x_{D'}(v) - x_{D'}(u)| \\ &- \delta_G |V_1| - \delta_G \sum_{v \in V_1} \left\lfloor \frac{d_v}{2} \right\rfloor - \sum_{v \in V_1} \left\lfloor \frac{d_v}{2} \right\rfloor d_v \\ &\geq \delta_G \sum_{\substack{uv \in E, v \in V_1 \\ d_v \geq 2}} |x_{D'}(v) - x_{D'}(u)| - 2 \sum_{v \in V_1} d_v^2. \end{aligned}$$

Using Lemma 2.2, we get

$$4bcr(D') \geq \delta_G L_{x_{D'}} - \epsilon - 2 \sum_{v \in V_1} d_v^2. \tag{6}$$

Consider the bijection f^* in Part (ii) of Lemma 2.1. Then

$$\delta_G L_{x_{D'}} \geq \delta_G L_{f^*} - 8bcr(D') - \delta_G m - \delta_G \sum_{v \in V_0} d_v d_v^*.$$

Observe that $\delta_G \geq 2$ implies $\sum_{v\in V_0} d_v d_v^* = 0$, and hence

$$\delta_G L_{x_{D'}} \geq \delta_G L_{f^*} - 8bcr(D') - \delta_G m - \sum_{v\in V_0} d_v d_v^*.$$

Hence (6) implies

$$12bcr(D') \geq \delta_G L_{f^*} - \delta_G m - \epsilon - \sum_{v\in V_0} d_v d_v^* - 2\sum_{v\in V_1} d_v^2. \tag{7}$$

Observing that $L_{f^*} \geq \hat{L}(G)$, $bcr(D') \leq 3bcr(D)$, $\delta_G m + \epsilon = \epsilon + \sum_{v\in V_0} d_v \delta_G \leq \sum_{v\in V} d_v^2$, and $\sum_{v\in V_0} d_v d_v^* + 2\sum_{v\in V_1} d_v^2 \leq 2\sum_{v\in V} d_v^2$, we obtain

$$36bcr(D) + 3\sum_{v\in V} d_v^2 \geq \delta_G \hat{L}(G),$$

which finishes the proof. □

Next, we investigate the cases for which the error term $\sum_{v\in V} d_v^2$ can be eliminated from Theorem 2.2.

Corollary 2.1 *Let $G = (V_0, V_1, E)$ so that $m \geq (1+\gamma)n$, and $\sum_{v\in V} (d_v - d_v^*)^2 \geq \alpha \sum_{v\in V} d_v^2$, where γ and α are positive constants. Then*

$$bcr(G) \geq C_{\alpha,\gamma}\delta_G \hat{L}(G), \quad \textit{where} \quad C_{\alpha,\gamma} = \frac{1}{36}\cdot\frac{1}{1+\frac{8+4\gamma}{3\alpha}}.$$

Proof. To prove the result we will first show that for any bipartite drawing D of G it holds,

$$bcr(D) \geq \frac{\sum_{v\in V}(d_v - d_v^*)^2}{16} - m. \tag{8}$$

For now assume that (8) holds. It is easy to see that $bcr(G) \geq m - n + 1$ [19], and since $n \leq \frac{\gamma}{1+\gamma}m$, we conclude that $m \leq (\gamma+1)bcr(G)$. Combining this inequality with (8), we obtain $(2+\gamma)bcr(G) \geq \frac{1}{16}\sum_{v\in V}(d_v - d_v^*)^2 \geq \frac{\alpha}{16}\sum_{v\in V} d_v^2$, and thus

$$\frac{16(2+\gamma)}{\alpha} bcr(G) \geq \sum_{v\in V} d_v^2,$$

and the claim follows from Theorem 2.2.

To prove (8), let D be any bipartite drawing of G, and let $v \in V_0$ so that $d_v - d_v^* \geq 2$. Let $u_1, u_2, ..., u_{d_v - d_v^*}$ be the set of vertices of degree at least 2 which are adjacent to v, and assume with no loss of generality that $x_D(u_1) < x_D(u_2) < ... < x_D(u_{d_v - d_v^*})$. Let i be an integer, $1 \leq i \leq \lfloor \frac{d_v - d_v^*}{2} \rfloor$, and note that any vertex u_j, $d_v - d_v^* - i + 1 > j > i$ generates at least one crossing on the edges $u_i v$ and $u_{d_v - i+1}v$. Thus $bcr(vu_i) + bcr(vu_{d_v - d_v^* - i+1}) \geq d_v - d_v^* - 2i$, $1 \leq i \leq \lfloor \frac{d_v - d_v^*}{2} \rfloor$, and therefore

$$\begin{aligned}\sum_{i=1}^{\lfloor \frac{d_v - d_v^*}{2} \rfloor} [bcr_D(u_i v) + bcr_D(u_{d_v - i - d_v^* + 1} v)] &\geq \sum_{i=1}^{\lfloor \frac{d_v - d_v^*}{2} \rfloor} d_v - d_v^* - 2i \\ &\geq (d_v - d_v^*)\frac{d_v - d_v^* - 1}{2} - \frac{d_v - d_v^*}{2}\cdot\frac{d_v - d_v^* + 2}{2} \\ &\geq \frac{1}{4}(d_v - d_v^*)^2 - d_v. \end{aligned} \tag{9}$$

We conclude that by summing (9) over all $v \in V_1$ that,

$$2bcr(D) \geq \frac{\sum_{v\in V_1}(d_v - d_v^*)^2}{4} - 2m.$$

Similarly we can show that $2bcr(D) \geq (\sum_{v\in V_0}(d_v - d_v^*)^2/4) - 2m$, and hence the claim follows. □

Remarks. The conditions of Corollary 2.1, involving α and γ are not restrictive at all. For instance, any bipartite graph of minimum degree at least 3, satisfies the conditions. We identify more additional graphs which satisfy these conditions in Section 3.

2.2 An upper bound

We now derive an upper bound on $bcr(G)$. We need the following obvious lemma.

Lemma 2.3 *Let D be a bipartite drawing of $G = (V_0, V_1, E)$. Let $e = uv$ and $\bar{e} = ab$, $u, a \in V_0, v, b \in V_1$ be two edges which cross in D. Assume that $|x_D(v) - x_D(u)| \geq |x_D(a) - x_D(b)|$, then either a or b is covered by e in D. Moreover, if a is covered by e, then*

$$|x_D(b) - x_D(u)| \leq |x_D(v) - x_D(u)|,$$

whereas, if b is covered by e, then

$$|x_D(a) - x_D(v)| \leq |x_D(v) - x_D(u)|.$$

□

Let V_H and E_H, denote the vertex set and the edge set of a subgraph H, of G. The *arboricity* of G, denoted by a_G, is $\max_H \lceil \frac{|E_H|}{|V_H|-1} \rceil$, where the maximum is taken over all subgraphs H, with $|V_H| \geq 2$. Note that $\delta_G/2 \leq a_G \leq \Delta_G$, where Δ_G denotes the maximum degree of G. A well-known theorem of Nash-Williams [21] asserts that a_G is the minimum number of edge disjoint acyclic subgraphs that edges of G can be decomposed to.

Theorem 2.3 *Let $G = (V_0, V_1, E)$, then*

$$bcr(G) \leq 5a_G \hat{L}(G).$$

Proof. Consider a solution (not necessarily optimal) of the linear arrangement of G, realized by a bijection $f^* : V_0 \cup V_1 \to \{1, 2, ..., n\}$. The mapping f^* induces an ordering of vertices of $V_0 \cup V_1$ in y_0. Lift up the vertices of V_1 into y_1 and draw the edges with respect to the new locations of these vertices to obtain a bipartite drawing D. Note that

$$L_{x_D} = \sum_{uv \in E} |x_D(u) - x_D(v)| = L_{f^*} \tag{10}$$

for this drawing D. Let $e = uv \in E$, $u \in V_0$, $v \in V_1$, and define I_e to be the set all edges crossing e in D so that for any $ab \in I_e$,

$$|x_D(a) - x_D(b)| \leq |x_D(v) - x_D(u)|.$$

Observe that if any edge $e' \notin I_e$ crosses e, then $e \in I_{e'}$. Hence, in this case the crossing of e and e' contributes one to $|I_{e'}|$. We conclude that

$$bcr(D) \leq \sum_{e \in E} |I_e|,$$

and will show that $|I_e| \leq a_G(4|x_D(u) - x_D(v)| + 1)$. For $e = uv \in E$, with $u \in V_0, v \in V_1$, let V_0^e be the set of all those vertices y of V_0 so that $|x_D(y) - x_D(v)| \leq |x_D(u) - x_D(v)|$. Similarly, let V_1^e be the set of all those vertices y of V_1 so that $|x_D(y) - x_D(u)| \leq |x_D(u) - x_D(v)|$. Note that, $|V_i^e| \leq 2|x_D(u) - x_D(v)| + 1$, $i = 0, 1$, since the coordinates of all vertices are integers. Therefore, we have $|V_0^e \cup V_1^e| \leq 4|x_D(u) - x_D(v)| + 2$. Let $\bar{e} = ab \in I_e$, $a \in V_0, b \in V_1$, and observe that by Lemma 2.3, $a \in V_0^e$ and $b \in V_1^e$. Consequently, $|I_e| \leq |E_H|$, where E_H is the edge set of the induced subgraph of G on the vertex set $V_0^e \cup V_1^e$. Clearly,

$$|I_e| \leq |E_H| \leq a_G(4|x_D(u) - x_D(v)| + 2 - 1) = a_G(4|x_D(u) - x_D(v)| + 1)$$

by the definition of a_G, and thus

$$bcr(D) \leq \sum_{e \in E} I_e \leq a_G(4L_{x_D} + m).$$

To complete the proof we take f^* to be the optimal solution to the linear arrangement problem, that is, $L_{f^*} = \hat{L}(G) \geq m$. □

2.3 Bipartite crossings in trees

We note that if a_G is small, then, the gap between the upper bound and the lower bound in Theorems 2.2 and 2.3 is small, and hence, it is natural to investigate the case $a_G = 1$, that is, when G is acyclic. In fact, in this case the method in the proof of Theorem 2.3 provides for an optimal bipartite drawing.

Theorem 2.4 *Let T be a tree on the vertex set $V = V_0 \cup V_1$, where V_0 and V_1 are the partite sets, and $|V| = n$. Let f^* be a bijection utilizing the optimal solution to the linear arrangement problem. Let D^* be a bipartite drawing constructed by the method of Theorem 2.3, that is, by lifting the vertices in V_1 into the line $y = 1$. Then*

$$bcr(D^*) = bcr(T) = \hat{L}(T) - n + 1 - \sum_{v \in T} \left\lfloor \frac{d_v}{2} \right\rfloor \left\lceil \frac{d_v - 2}{2} \right\rceil . \tag{11}$$

Proof. We prove the Theorem by induction on n. The result is true for $n = 1, 2$. Let $n \geq 3$. Assume that the Theorem is true for all l-vertex trees, $l < n$, and let T be a tree on n vertices. We first show that the RHS of (11) is a lower bound on $bcr(T)$. We then show that $bcr(D^*)$ equals to RHS of (11). Consider an optimal bipartite drawing D of T. It is not difficult to see that one of the leftmost (rightmost) vertices is a leaf. Denote the left leaf by v_0, the right leaf by v_k, and let $P = v_0 v_1 ... v_k$ be the path between v_0 and v_k. Note that P will cross any edge in T which is not incident to v_i, $0 \leq i \leq k$, it follows that path P will generate at least

$$c_P = n - 1 - k - \sum_{i=1}^{k-1} (d_{v_i} - 2) \tag{12}$$

crossings, where c_P counts exactly the number of edges in T (in D) which are not incident to any vertex on P. Deleting the edges of P we get trees T_i, on the vertex set $V^i = V_0^i \cup V_1^i$, rooted in $v_i, i = 1, 2, ..., k-1$. Consider the optimal bipartite drawings of T_i, $i = 1, 2, ..., k-1$, and place them consecutively such that T_i does not cross T_j, for $i \neq j$. Then draw the path P without self crossings such that v_0 (v_k) is placed to the left (right) of the drawing of $T_1 (T_{k-1})$. Then clearly the number of crossings in this new drawings is $\sum_{i=1}^{k-1} bcr(T_i) + c_P$, so we conclude that

$$bcr(D) = \sum_{i=1}^{k-1} bcr(T_i) + c_P = \left(\sum_{i=1}^{k-1} bcr(T_i) \right) + n - 1 - k - \sum_{i=1}^{k-1} (d_{v_i} - 2),$$

for otherwise D is not an optimal drawing. For any $v \in V$, let d_v^i denote the degree of v in T_i; applying the inductive hypothesis to T_i, $i = 1, 2, ..., k-1$, we obtain

$$\begin{aligned} bcr(T) &= \sum_{i=1}^{k-1} \left(\hat{L}(T_i) - |V^i| + 1 - \sum_{v \in V^i} \left\lfloor \frac{d_v^i}{2} \right\rfloor \left\lceil \frac{d_v^i - 2}{2} \right\rceil \right) \\ &\quad + n - 1 - k - \sum_{i=1}^{k-1} (d_{v_i} - 2) \\ &= \sum_{i=1}^{k-1} \left(\hat{L}(T_i) - \sum_{v \in V^i} (\left\lfloor \frac{d_{v_i}}{2} \right\rfloor \left\lceil \frac{d_{v_i} - 2}{2} \right\rceil + d_{v_i} - 2) \right). \end{aligned} \tag{13}$$

Now observe that for $v \in V^i$, $d_v^i = d_v$, if $v \neq v_i$; otherwise $d_v^i = d_v - 2$, $i = 1, 2, ..., k-1$. Consequently,

$$\begin{aligned} \sum_{v \in V^i} \left\lfloor \frac{d_v^i}{2} \right\rfloor \left\lceil \frac{d_v^i - 2}{2} \right\rceil + d_{v_i} - 2 &= \left\lfloor \frac{d_{v_i} - 2}{2} \right\rfloor \left\lceil \frac{d_{v_i} - 4}{2} \right\rceil + d_{v_i} - 2 + \sum_{v \in V^i - v_i} \left\lfloor \frac{d_v}{2} \right\rfloor \left\lceil \frac{d_v - 2}{2} \right\rceil \\ &= \sum_{v \in V^i} \left\lfloor \frac{d_v}{2} \right\rfloor \left\lceil \frac{d_v - 2}{2} \right\rceil , \end{aligned} \tag{14}$$

where the last line is obtained by observing that $\left\lfloor\frac{d_{v_i}-2}{2}\right\rfloor\left\lceil\frac{d_{v_i}-4}{2}\right\rceil + d_{v_i} - 2 = \left\lfloor\frac{d_{v_i}}{2}\right\rfloor\left\lceil\frac{d_{v_i}-2}{2}\right\rceil$. Thus it follows using (13) that

$$bcr(D) = \sum_{i=1}^{k-1} \hat{L}(T_i) - \sum_{v\in V} \left\lfloor\frac{d_v}{2}\right\rfloor\left\lceil\frac{d_v-2}{2}\right\rceil. \tag{15}$$

Now consider the optimal linear arrangements of the trees T_i, for $i = 0, 1, 2, ..., k$ and place them consecutively in that order on a line, and the path P. Let g denote the bijection associated with this arrangement, then $L_g = \sum_{i=1}^{k-1} \hat{L}(T_i) + n - 1$. Using this fact (15) implies

$$bcr(T) \geq \hat{L}(T) - n + 1 - \sum_{v\in T} \left\lfloor\frac{d_v}{2}\right\rfloor\left\lceil\frac{d_v-2}{2}\right\rceil,$$

since $L_g \geq \hat{L}(T)$.

To finish the proof we will show that $bcr(D^*)$ equals to the RHS of (11). Consider an optimal linear arrangement f^* of the tree T. It is not difficult to see that, $f^{*-1}(1)$ and $f^{*-1}(n)$ are leaves, [25, 4]. Let $P = v_0v_1...v_k$ be the path between $v_0 = f^{*-1}(1)$ and $v_k = f^{*-1}(n)$ in T, and let T_i be trees defined in the first part of the proof. Note that for the bijection g, described earlier, it holds $L_g = \sum_{i=1}^{k-1} \hat{L}(T_i) + n - 1$, and thus we conclude that,

$$L_{f^*} = \hat{L}(T) = \sum_{i=1}^{k-1} \hat{L}(T_i) + n - 1, \tag{16}$$

and note that the above equation implies that P does not cross itself, in the arrangement associated with f^*. It follows that P does not cross itself in the bipartite drawing D^*. Let f_i^* be the restriction of f^* to V^i, and D_i^* be the subdrawing in D^* which is associated with T_i, $i = 1, 2, ..., k-1$. Note that $bcr(D^*) = \sum_{i=1}^{k-1} bcr(D_i^*) + c_P$. However, it is easy to see that D_i^* is obtained from f_i^* by lifting the vertex set V_1^i to the line $y = 1$, and hence we can apply the induction hypothesis to D_i^*, $i = 1, 2, ...k-1$, to obtain

$$bcr(D^*) = \sum_{i=1}^{k-1}\left(\hat{L}(T_i) - |V_i| + 1 - \sum_{v\in V_i} \left\lfloor\frac{d_v}{2}\right\rfloor\left\lceil\frac{d_v-2}{2}\right\rceil\right) + c_P. \tag{17}$$

Substituting c_P its value from (12), and repeating the same steps used in deriving (15), we obtain

$$bcr(D^*) = \sum_{i=1}^{k-1} \hat{L}(T_i) - \sum_{v\in V} \left\lfloor\frac{d_v}{2}\right\rfloor\left\lceil\frac{d_v-2}{2}\right\rceil. \tag{18}$$

To complete the proof use (16) in (18) and obtain,

$$bcr(D^*) = \hat{L}(T) - n + 1 - \sum_{v\in T} \left\lfloor\frac{d_v}{2}\right\rfloor\left\lceil\frac{d_v-2}{2}\right\rceil.$$

□

Since the optimal linear arrangement of an n-vertex tree can be found in $O(n^{1.6})$ time [4], computing D^* can also be done in $O(n^{1.6})$ time.

3 Applications

It is instructive to provide examples of graphs G for which $bcr(G) = \Theta(\delta_G \hat{L}(G))$. Consider any bipartite G with $\delta_G \geq 3$ and $\delta_G = \Theta(a_G)$, for instance, take any regular bipartite graph with $\delta_G \geq 3$. Then, conditions of Corollary 2.1 are met, and thus by Theorem 2.3, $bcr(G) = \Theta(\delta_G \hat{L}(G))$. Moreover, consider any connected bipartite G of degree at most a constant k, with $m \geq (1+\gamma)n$, where $\gamma > 0$

is fixed. Note that, $d_v - d_v^* \geq 1$ for any $v \in V$, since G is connected and is not a star, and thus, $\sum_{v\in V}(d_v - d_v^*)^2 \geq n$. (Note that the star is excluded by the density condition $m \geq (1+\gamma)n$.) Now let $\alpha = \frac{1}{k^2}$, to obtain $n \geq \frac{1}{k^2}\sum_{v\in V} d_v^2$. Hence this graph satisfies the conditions of Corollary 2.1, moreover, it is easy to see that $a_G \leq k = O(1)$, and we conclude using Theorem 2.3 that $bcr(G) = \Theta(\hat{L}(G))$.

3.1 Bipartite crossings, bisection, genus, and page number

The appearance of a_G in the upper bound of Theorem 2.3 relates $bcr(G)$ to other important topological properties of G such as genus of G, denoted by g_G [32], and page number of G [1], denoted by p_G.

Observation 3.1 *Let $G = (V_0, V_1, E)$, and assume that $\delta_G \geq 2$ and $m \geq (1+\gamma)n$, for a fixed $\gamma > 0$. Then $bcr(G) = \Theta(\hat{L}(G))$, provided that $a_G = O(1)$. Consequently, under the given conditions for G, if either $p_G = O(1)$, or $g_G = O(1)$, then $bcr(G) = \Theta(\hat{L}(G))$.*

Proof. Assume that $a_G = O(1)$, then using Corollary 2.1 and Theorem 2.3, and observing that, $a_G = O(1)$, implies $\delta_G = O(1)$, we conclude that $bcr(G) = \Theta(\hat{L}(G))$. (Note that, $\delta_G \geq 2$, gives $d_v^* = 0$, for all $v \in V$.) To finish the proof, observe that $p_G = O(1)$ ($g_G = O(1)$), implies that $a_G = O(1)$. □

Next, we provide another application of our results, by deriving nontrivial upper bounds on the bipartite crossing number.

Observation 3.2 *Let $G = (V_0, V_1, E)$, with page number p_G and genus g_G. Then*

$$bcr(G) \leq 10 p_G \hat{L}(G) \quad and \quad bcr(G) \leq (10\sqrt{g_G} + 20)\hat{L}(G).$$

Proof. Since $cr(G) \leq bcr(G) \leq 5a_G\hat{L}(G)$, by Theorem 2.3, we need to bound a_G in terms of g_G and p_G. Let H be a subgraph of G with the vertex set $V_H, |V_H| \geq 2$, and the edge set E_H. Note that $p_H \leq p_G$, and $\frac{|E_H|}{|V_H|-1} \leq 2p_H$ [1], and hence $a_G \leq 2p_G$, which verifies the upper bound involving p_G . To finish the proof observe that $\frac{|E_H|}{4} - \frac{|V_H|}{2} + 1$ is a lower bound on the genus of H, or g_H [32]. Thus,

$$\frac{g_H}{|V_H|-1} \geq \frac{1}{4}\frac{|E_H|}{|V_H|-1} - \frac{|V_H|}{2|V_H|-2} + \frac{1}{|V_H|-1}.$$

Since g_H is at most $(|V_H|-1)^2/12$ [32], it follows that for any subgraph H, $\sqrt{g_G/12} \geq \sqrt{g_H/12} \geq \frac{g_H}{|V_G|-1} \geq \frac{1}{4}\frac{|E_H|}{|V_H|-1}$, and consequently $a_G \leq 2\sqrt{g_G} + 4$. □

Let $0 < \beta \leq \frac{1}{2}$ be a constant and denote by $b_\beta(G)$ size of the minimal β-bisection of G. That is,

$$b_\beta(G) = \min_{\beta n \leq |A| \leq (1-\beta)n} |(A, \bar{A})|$$

where $(A, \bar{A})$ denotes a cut which partitions V into A and $\bar{A}$. Leighton [16] proved for any degree bounded graph G, the inequality $cr(G) + n = \Omega(b_{\frac{1}{3}}^2(G))$, where $cr(G)$ is the planar crossing number of G. Another very interesting consequence of Theorem 2.2 is providing a stronger version of Leighton's result, for $bcr(G)$.

Theorem 3.1 *Let $G = (V_0, V_1, E)$, Then, for any constant $0 < \beta < \frac{1}{2}$, it holds*

$$bcr(G) + \sum_{v\in V} d_v^2 = \Omega(\delta_G n b_\beta(G)),$$

in particular when G is regular, it holds

$$bcr(G) = \Omega(m b_\beta(G)).$$

Proof. The claim follows from the lower bound in Theorem 2.2 and the well-known observation that $\hat{L}(G) \geq (1-2\beta)n b_\beta(G)$. (See for instance [12].) □

Remarks. After proving Theorem 3.1, we discovered that a weaker version of this Theorem for degree bounded graphs can be obtained by a shorter proof which uses Menger's Theorem [27].

3.2 Approximation algorithms

Given a bipartite graph G, the *bipartite arrangement problem* is to find a bipartite drawing D of G with smallest L_{x_D}, or smallest length, so that the x coordinate of any vertex is an integer. We denote this minimum value by $\bar{L}(G)$. Note that coordinate function x_D, for a bipartite drawing need not to be an injection, since we may have $x_D(a) = x_D(b)$, for $a \in V_0$, and $b \in V_1$. Thus, in general $\bar{L}(G) \neq \hat{L}(G)$. Our approximation algorithms in this section provide a bipartite drawing in which all vertices have integer coordinates, so that the number of crossings and at the same time the length of the drawing is small. We need the following Lemma giving a relation between $\bar{L}(G)$ and $\hat{L}(G)$.

Lemma 3.1 *For any connected bipartite graph $G = (V_0, V_1, E)$ it holds*

$$\bar{L}(G) \geq \frac{\hat{L}(G) - 1}{4}.$$

Proof. Let D be a bipartite drawing of G in which all x coordinates are integers. Let $e = ab \in E$, and note that $N_D(e) \leq |x_D(a) - x_D(b)|$, since any vertex in $V_0 \cup V_1$ has an integer x coordinate. Let f^* be the bijection in Part (i) in Lemma 2.1, then $|f^*(a) - f^*(b)| \leq 2|x_D(a) - x_D(b)| + 1$, and hence by taking the sum over all edges, we obtain $L_{f^*} \leq 2L_{x_D} + m$. To prove the lemma, we claim that there are at least $\frac{m-1}{2}$ edges $e = ab$, so that $x_D(a) \neq x_D(b)$, and consequently $L_{x_D} \geq \frac{m-1}{2}$, which implies the result. To prove our claim, note that there are at most $\frac{n}{2}$ edges ab, so that $x_D(a) = x_D(b)$, and hence at least $m - \frac{n}{2} \geq \frac{m-1}{2}$ edges ab, with $x_D(a) \neq x_D(b)$, since G is connected and therefore has at least $n - 1$ edges. □

Even et al. [9] in a breakthrough result came up with polynomial time $O(\log n \log\log n)$ times optimal approximation algorithms for several NP-hard problems, including the linear arrangement problem. Combining their result with ours, we obtain the following.

Theorem 3.2 *Let $G = (V_0, V_1, E)$, and consider the drawing D (with integer coordinates) in Theorem 2.3 obtained form an approximate solution to the linear arrangement problem provided in [9]. Then $L_{x_D} = O(\log n \log\log n \bar{L}(G))$. Moreover, if G meets the conditions in Corollary 2.1, then $bcr(D) = O(\log n \log\log n bcr(G))$, provided that $\delta_G = \Theta(a_G)$.*

Proof. Note that $L_{x_D} = O(\hat{L}(G) \log n \log\log n)$ and thus the claim regarding L_{x_D} follows from Lemma 3.1. To finish the proof note that, Theorem 2.3 gives $bcr(D) = O(a_G \log n \log\log n \hat{L}(G))$, and the claim regarding $bcr(D)$ is verified by the application of Corollary 2.1, since $\delta_G = \Theta(a_G)$.

□

The divide and conquer paradigm has been very popular in solving VLSI layout problems both in theory and also in practice. Indeed, the only known approximation algorithm for the planar crossing number is a simple divide and conquer algorithm in which the divide phase consists of approximately bisecting the graph [2]. This algorithm approximates $cr(G)+n$ to within a factor of $O(\log^4 n)$ from the optimal, when G is degree bounded [17]. A similar algorithm approximates $\hat{L}(G)$ to within a factor of $O(\log^2 n)$ from the optimal. To verify the quality of the approximate solutions, in general, one needs to show that the error term arising in the recurrence relations associated with the performance of algorithms are small compared to the value of the optimal solution. A nice algorithmic consequence of Theorem 3.1 is that the standard divide and conquer algorithm in which the divide phase consists of approximately bisecting the graph gives a good approximation for $bcr(G)$ in polynomial time. The divide stage of our algorithm uses an approximation algorithm for bisecting a graph such as those in [10, 17]. These algorithms have a performance guarantee of $O(\log n)$ from the optimal [10, 17]. It should be noted that the lower bound of $\Omega(b^2_{\frac{1}{3}}(G))$, although is sufficient to verify the the performance of the divide and conquer approximation algorithm for the planar crossing number, can not be used to show the quality of the approximation algorithm for $bcr(G)$, since (as we will see) it does not bound from above the error term in our recurrence relation. Thus our lower bound of $\Omega(n\delta_G b_{\frac{1}{3}}(G))$ is crucial to show the suboptimality of the solution.

Theorem 3.3 *Let A be a polynomial time $1/3-2/3$ bisecting algorithm to approximate the bisection of a graph with a performance guarantee $O(\log n)$. Consider a divide and conquer algorithm which (a) recursively bisects the graph G, using A, (b) obtains the two bipartite drawings, and then (c) inserts the edges of the bisection between these two drawings. This divide and conquer algorithm generates, in polynomial time, a bipartite drawing D with integer coordinates, so that $L_{x_D} = O(\log^2 n\bar{L}(G))$. Moreover, if G meets the conditions in Corollary 2.1, then $bcr(D) = O(\log^2 nbcr(G))$, provided that $\delta_G = \Theta(a_G)$.*

Proof. Assume that using A, we partition the graph G to 2 vertex disjoint subgraphs G_1 and G_2 recursively. Let $\bar{b}(G)$ denote the number of those edges having one endpoint in the vertex set of G_1, and the other in the vertex set of G_2. Let D_{G_1}, and D_{G_2} be the bipartite drawings already obtained by the algorithm for G_1 and G_2, respectively. Let D denote the drawing obtained for G. To show the claim regarding L_{x_D}, note that

$$L_{x_D} \leq L_{x_{D_{G_1}}} + L_{x_{D_{G_2}}} + \bar{b}(G)n.$$

Since, we use the approximation algorithm A for bisecting we have $\bar{b}(G) = O(\log nb_{\frac{1}{3}}(G))$, hence the error term in the recurrence relation is $O(n \log nb_{\frac{1}{3}}(G))$. Moreover, $3\hat{L}(G) \geq b_{\frac{1}{3}}(G)n$, [12], and consequently using Lemma 3.1, we obtain, $12\bar{L}(G)+3 \geq b_{\frac{1}{3}}(G)n$. Thus the error term is $O(\log n\bar{L}(G))$, and consequently,

$$L_{x_D} \leq L_{x_{D_{G_1}}} + L_{x_{D_{G_2}}} + O(\log n\bar{L}(G)),$$

which implies $L_{x_D} = O(\log^2 n\bar{L}(G))$. To verify the claim regarding $bcr(D)$, note that

$$bcr(D) \leq bcr(D_{G_1}) + bcr(D_{G_2}) + \bar{b}^2(G) + \bar{b}(G)m.$$

Now observing that $m \leq a_G n$, $\bar{b}(G) = O(\log nb_{\frac{1}{3}}(G))$, and $nb_{\frac{1}{3}}(G) \leq 3\hat{L}(G)$, we obtain,

$$bcr(D) \leq bcr(D_{G_1}) + bcr(D_{G_2}) + O(a_G\hat{L}(G)\log n)$$

which implies

$$bcr(D) = O(a_G\hat{L}(G)\log^2 n).$$

Note that by Corollary 2.1, $bcr(G) = \Omega(a_G\hat{L}(G))$, and the claim follows. □

Remarks. The method of Even *et al.* that we suggested to use in Theorem 3.2, although a theoretical breakthrough, requires the usage of specific interior point linear programming methods which may be computationally expensive or hard to code. Hence, the the divide and conquer approximation algorithm, although in theory, weaker than the method of Theorem 3.2, it may be easier to implement. Moreover, one may use very fast and simple heuristics developed by the VLSI and CAD communities [24] for graph bisection in the divide stage. Although, these heuristics do not produce provably sub-optimal solutions for bisecting a graph, they work well in practice, and are extremely fast. Therefore, one may anticipate that certain implementations of the divide and conquer algorithm are very fast and effective in practice.

Note that since a_G can be computed in polynomial time, the class of graphs with $a_G \leq c\delta_G$ is recognizable in polynomial time, when c is a given constant. Hence, those graphs which meet the required conditions in Theorems 3.2, and 3.3 can be recognized in polynomial time. Also, note that many important graphs such those introduced in Section 3 meet the conditions, and hence for these graphs the performance of both approximation algorithms is guaranteed.

4 Largest biplanar subgraphs in acyclic graphs

Let $T = (V_T, E_T)$ be a tree and w_{ij} be a weight assigned to each edge $ij \in E_T$. For any $B \subseteq E_T$, define the weight of B, denoted by $w(B)$, to be the sum of weights for all edges in B. In this section we present a linear time algorithm to compute a biplanar subgraph of T of largest weight.

A tree on at least 2 vertices is called a caterpillar if it consists of a path to which some vertices of degree 1 (leaves) are attached. We distinguish four categories of vertices in a caterpillar. First consider caterpillars which are not stars. They have a unique path connecting two internal vertices to which all leaves are attached to. We call this path the *backbone* of the caterpillar. The two endvertices of the backbone are called *endbone* vertices, internal vertices of the backbone are called *midbone* vertices. Leaves attached to endbones are called *endleaves*. Leaves attached to midbones are called *midleaves*.

For a star with at least 3 vertices, the middle vertex is considered as endbone, the backbone path consists of this single endbone, and the leaves in the star are considered endleaves. If a star has two vertices, then we treat these vertices as endbones.

Let $T = (V_T, E_T)$ be an unrooted tree and $r \in V_T$. Then, we view r as the root of T. Then any vertex $x \in V_T,\ x \neq r$ will have a unique parent which is the first vertex on the path towards the root. For $x \in V_T$, the set of children of x, denoted by N_x, are those vertices of T whose parent is x. For any $x \in V_T,\ x \neq r$ we denote by T_x the component of T, containing x, which is obtained after removing the parent of x from T. We define T_r to be T.

We use the notation B_x for a biplanar subgraph of T_x, $x \in V_T$, and treat B_x as an edge set. We say that B_x spans a vertex a, if there is an edge $ab \in B_x$. For $x \in V_T$, we define

$$W(T_x) = \max_{B_x \subseteq E_{T_x}} w(B_x). \tag{19}$$

Our goal is to determine $W(T_r)$. To achieve this goal, we define 5 additional related optimization problems as follows:

$$\begin{aligned}
w^1(T_x) &= \max\{w(B_x) :\ x \text{ is endleaf in } B_x\}\\
w^2(T_x) &= \max\{w(B_x) :\ x \text{ is midleaf in } B_x\}\\
w^3(T_x) &= \max\{w(B_x) :\ x \text{ is endbone in } B_x\}\\
w^4(T_x) &= \max\{w(B_x) :\ x \text{ is midbone in } B_x\}\\
w^5(T_x) &= \max\{w(B_x) :\ x \text{ is not spanned by } B_x\}.
\end{aligned}$$

It is obvious that

$$W(T_x) = \max_{1\leq i\leq 5} w^i(T_x), \tag{20}$$

and therefore solving all 5 problems for T_x determines $W(T_x)$. For any leaf v set $w^1(v) = w^5(v) = 0$, $W(v) = 0$ and $w^i(v) = -\infty$ for $i = 2, 3, 4$ as initial condition. Finally, for $u \in N_x$, $x \in V_T$ define,

$$f(u) = \max\{w_{ux} + w^5(T_u), W(T_u)\}.$$

It is well-known and easy to show that a graph is biplanar iff it is a collection of vertex disjoint caterpillars. This is equivalent to saying that a graph is biplanar iff it does not contain a double claw which is a star on 3 vertices with all three edges subdivided. Therefore our problem is to find a maximum weight forest of caterpillars in an edge-weighted acyclic graph. We will use these facts in the next lemma, sometimes without explicitly referring to them.

Lemma 4.1

$$w^1(T_x) = \max_{y\in N_x}\left\{\left(\sum_{y'\in N_x\setminus\{y\}} W(T_{y'})\right)+w_{xy} + \max_{i=1,3} w^i(T_y)\right\} \tag{21}$$

$$w^2(T_x) = \max_{y\in N_x}\left\{ w_{xy} + w^4(T_y) + \sum_{y'\in N_x\setminus\{y\}} W(T_{y'}) \right\} \quad (22)$$

$$w^3(T_x) = \max\left\{ \max_{y\in N_x}\left\{ w_{xy} + \max_{i=1,3} w^i(T_y) + \sum_{y'\in N_x\setminus\{y\}} f(y') \right\}, \sum_{y\in N_x} f(y) \right\} \quad (23)$$

$$w^4(T_x) = \max_{\substack{y_1,y_2\in N_x \\ y_1\neq y_2}} \left\{ w_{xy_1} + w_{xy_2} + \max_{i=1,3} w^i(T_{y_1}) + \max_{i=1,3} w^i(T_{y_2}) + \sum_{y'\in N_x\setminus\{y_1,y_2\}} f(y') \right\} \quad (24)$$

$$w^5(T_x) = \sum_{y\in N_x} W(T_y). \quad (25)$$

Proof Sketch. The basic idea for the recurrence relations is to describe how an optimal solution for T_x decomposes in the trees rooted in N_x. Indeed, (21), (22), and (25) are obvious. For (23), note that if x is an endbone in a maximum weight biplanar B_x, then x is an endbone in a caterpillar $C \subseteq B_x$. Consider the case that C is not a star. Since, x is an endbone of C, it has at least two neighbors in C, and all but one of its neighbors are leaves in C. Then exactly one neighbor y of x is an endbone or an endleaf in $C \setminus \{x\}$. This justifies the presence of the first two terms in the inner curly bracket. To justify the presence of the sum on y', note that, in order to maximize the total weight of B_x, we must attach $y' \in N_x \setminus \{y\}$ to C as a leaf, only if $f(y') = w_{y'x} + w^5(T_{y'})$; otherwise we must include in B_x, the maximum biplanar subgraph of $T_{y'}$ which has the total weight $f(y') = W(T_{y'})$. To justify the term $\sum_{y\in N_x} f(y)$, consider the case that C is a star. Then we must attach any $y \in N_x$ to C as a leaf only if $f(y) = w_{xy} + w^5(T_y)$; otherwise we include in B_x the maximum biplanar subgraph of T_y. For (24), note that, if x is a midbone in a maximum weight B_x, then x is a midbone of $C \subseteq B_x$, and has 2 neighbors y_1 and y_2 in C. By deleting x from C, we obtain exactly two caterpillars C_1 and C_2 so that y_i is either an endbone or an endleaf for C_i, $i = 1, 2$. Now follow an argument similar to (23) to finish the proof of (24) □

Theorem 4.1 *For an edge-weighted acyclic graph $T = (V_T, E_T)$, a largest weight biplanar subgraph can be computed in $O(|V_T|)$ time.*

Proof Sketch. With no loss of generality assume that T is connected, otherwise we apply our arguments to the components of T. We select a root r for T, and then perform a post order traversal and show that we can compute $w^i(T_x)$, $1 \le i \le 5$, and $W(T_x)$ in $O(|N_x|)$ time, if all these quantities are already known for the children of x. This is obvious for (20) and (25). For (21) and (22) the expressions in curly braces are easy to evaluate in linear time, if a maximizing y is known. So the issue is to find a maximizing y in linear time. It is easy to see that for (21) we look for $y \in N_x$ which maximizes $w_{xy} + \max_{i=1,3} w^i(T_y) - W(T_y)$, and for (22) we look for $y \in N_x$ which maximizes $w_{xy} + w^4(T_y) - W(T_y)$; all these can be computed in $O(|N_x|)$ time.

For (23), it suffices to show that a $y \in N_x$ can be found in $O(|N_x|)$ time which maximizes $g(y) = w_{xy} + \max_{i=1,3} w^i(T_y) + \sum_{y'\in N_x\setminus\{y\}} f(y') = w_{xy} + \max_{i=1,3} w^i(x) - f(y) + \sum_{y'\in N_x} f(y')$. To do so find $y^* \in N_x$ which maximizes $w_{xy} + max_{i=1,3} w^i(T_y) - f(y)$. For (24), note that

$$w^4(T_x) = \left(\sum_{y\in N_x} f(y)\right) + \max_{y_1\neq y_2\in N_x}\left\{ w_{xy_1} + \max_{i=1,3} w^i(T_{y_1}) - f(y_1) + w_{xy_2} + \max_{i=1,3} w^i(T_{y_2}) - f(y_2) \right\}.$$

Thus, to maximize $w^4(T_x)$, we should find $y_1, y_2 \in N_x$, $y_1 \neq y_2$ which give the largest two values for $w_{xy} + \max_{i=1,3} w^i(T_y) - f(y)$.

It is easy to maintain for every x not just the values $w^i(T_x)$, $W(T_x)$, but also the edge-set of B_x which realizes this value, therefore, we can store the edge set of a largest biplanar subgraph as well. □

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