Counting rooted spanning forests in complete multipartite graphs

László A. Székely[1]
Department of Mathematics
University of South Carolina
Columbia, SC 29208

Abstract

Jin and Liu discovered an elegant formula for the number of rooted spanning forests in the complete bipartite graph K_{a_1,a_2}, with b_1 roots in the first vertex class and b_2 roots in the second vertex class. We give a simple proof to their formula, and a generalization for complete m-partite graphs, using the multivariate Lagrange inverse.

Y. Jin and C. Liu [3] give a formula for $f(m,l;n,k)$, the number of spanning forests of the labelled complete bipartite graph $K_{n,m}$, where in the forest every tree is rooted, there are k roots in the first vertex class (among the n vertices) and l roots in the second vertex class (among the m vertices), and the trees in the forest are not ordered. They discovered the elegant formula

$$f(m,l;n,k) = \binom{m}{l}\binom{n}{k} n^{m-l-1} m^{n-k-1}(km + ln - lk). \tag{1}$$

The goal of the present note is generalization of (1) from complete bipartite to complete multipartite graphs, through a simple proof using the multivariate Lagrange inverse.

Let $f(a_1,b_1;...;a_m,b_m)$ denote the number of spanning forests of the labelled complete multipartite graph $K_{a_1,a_2,...,a_m}$, where in the forest every tree is rooted, there are b_i roots in the i^{th} vertex class for $i=1,2,...,m$, and the trees in the forest are not ordered. Let $w_i(t_1,...,t_m)$ denote the multivariate exponential generating function (EGF) of the numbers $f(a_1,0;...;a_i,1;...;a_m,0)$ (the number of rooted spanning trees of the complete multipartite graph $K_{a_1,a_2,...,a_m}$, if the root has to be in the i^{th} class), i.e.

$$w_i(t_1,...,t_m) = \sum_{a_1=0}^{\infty} ... \sum_{a_i=1}^{\infty} ... \sum_{a_m=0}^{\infty} f(a_1,0;...;a_i,1;...;a_m,0) \prod_{k=1}^{m} \frac{t_k^{a_k}}{a_k!}. \tag{2}$$

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The key identity for our argument is

$$t_i e^{(w_1+w_2+\ldots+w_m)-w_i} = w_i \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{3}$$

The proof of formula (3) is based on the following combinatorial decomposition. Given a rooted spanning tree of the complete multipartite graph $K_{a_1,a_2,\ldots,a_m}$, where the root is in the i^{th} class, remove the root vertex from the tree to obtain a spanning forest of $K_{a_1,a_2,\ldots,a_i-1,\ldots,a_m}$, and mark the former neighbors of the eliminated root vertex as roots in the forest. This decomposition establishes a bijection between the following two sets:
the set of rooted spanning trees of the complete multipartite graph $K_{a_1,a_2,\ldots,a_m}$, *where the root is in the* i^{th} *vertex class,*
and
the set of some ordered pairs, where the first entry of the ordered pair is one of the vertices of the i^{th} *vertex class, the second element of the ordered pair is a rooted spanning forest of* $K_{a_1,a_2,\ldots,a_i-1,\ldots,a_m}$, *where the vertex from the first entry is removed from the* i^{th} *vertex class, and the trees of the forest are not ordered.*
Now $t_i e^{(w_1+w_2+\ldots+w_m)-w_i}$ is the EGF of the set of ordered pairs in question, according to the Exponential Formula; and w_i is the same EGF by the bijection. Set $\Phi_i(w_1, w_2, \ldots, w_m) = e^{(w_1+w_2+\ldots+w_m)-w_i}$.

According to the multiplication rule of EGF's, $\prod_{k=1}^m w_k^{b_k}$ is the multivariate exponential generating function of the number of rooted spanning forests of complete m-partite graphs, with b_k roots in the k^{th} vertex class, where the trees rooted in the same part *are ordered*; hence

$$f(a_1, b_1; \ldots; a_m, b_m) = \frac{a_1! a_2! \cdots a_m!}{b_1! b_2! \cdots b_m!} [t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}] \prod_{k=1}^m w_k^{b_k}. \tag{4}$$

According to Part 1 of Theorem 1.2.9 (Multivariate Lagrange Formula) from [2], (3) implies

$$\begin{aligned}
[t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}] \prod_{k=1}^m w_k^{b_k} &= [\lambda_1^{a_1} \cdots \lambda_m^{a_m}] \Bigg\{ \det \left| \delta_{ij} - \frac{\lambda_j}{\Phi_i} \cdot \frac{\partial \Phi_i}{\partial \lambda_j} \right| & (5) \\
&\times \lambda_1^{b_1} \cdots \lambda_m^{b_m} \prod_{k=1}^m e^{a_i(w_1+\ldots+w_m)-a_i w_i} \Bigg\}, & (6)
\end{aligned}$$

where Φ_i is a short-hand notation for $\Phi_i(\lambda_1, \ldots, \lambda_m)$. Observe that $\frac{\lambda_j}{\Phi_i} \cdot \frac{\partial \Phi_i}{\partial \lambda_j} = (1-\delta_{ij})\lambda_j$, and the for the determinant in (5) we have the well-known evaluation

$$\det \left| \delta_{ij} - (1-\delta_{ij})\lambda_j \right| = (\lambda_1 + 1) \cdots (\lambda_m + 1) \left(1 - \frac{\lambda_1}{\lambda_1 + 1} - \ldots - \frac{\lambda_m}{\lambda_m + 1} \right) \tag{7}$$

(see for example Exercise 225 in [1]). Now (7) is easily rewritten as

$$1 - \sum_{j=2}^m (j-1) \sum_{1 \le i_1 < i_2 < \cdots < i_j \le m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_j}, \tag{8}$$

and (8) is rewritten as

$$\sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\cdots\sum_{l_m=0}^{1}(1-l_1-l_2-\ldots-l_m)(\lambda_1)_{l_1}(\lambda_2)_{l_2}\cdots(\lambda_m)_{l_m}, \tag{9}$$

where $(x)_t$ stands for the falling factorial, $(x)_0 = 1$ and $(x)_1 = x$. Introducing the notation $A = a_1 + a_2 + \ldots + a_m$ and using (9), we find that (5) and (6) are equal to

$$[\lambda_1^{a_1-b_1}\lambda_2^{a_2-b_2}\cdots\lambda_m^{a_m-b_m}]\sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\cdots\sum_{l_m=0}^{1}(1-l_1-l_2-\ldots-l_m)$$

$$\times(\lambda_1)_{l_1}(\lambda_2)_{l_2}\cdots(\lambda_m)_{l_m}e^{(A-a_1)\lambda_1}e^{(A-a_2)\lambda_2}\cdots e^{(A-a_m)\lambda_m}$$

$$=\left(\prod_{k=1}^{m}\frac{(A-a_k)^{a_k-b_k-1}}{(a_k-b_k)!}\right)\sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\cdots\sum_{l_m=0}^{1}(1-l_1-l_2-\ldots-l_m) \tag{10}$$

$$\times\left(\prod_{j=1}^{m}(A-a_j)^{1-l_j}(a_j-b_j)_{l_j}\right). \tag{11}$$

Combining (10), (11), and (4), we obtain the main result:

Theorem 1

$$f(a_1,b_1;\ldots;a_m,b_m) = \left(\prod_{k=1}^{m}\binom{a_k}{b_k}(A-a_k)^{a_k-b_k-1}\right) \tag{12}$$

$$\times\sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\cdots\sum_{l_m=0}^{1}(1-l_1-l_2-\ldots-l_m)\left(\prod_{j=1}^{m}(A-a_j)^{1-l_j}(a_j-b_j)_{l_j}\right). \tag{13}$$

For the case $m = 2$, formula (12), (13) specializes to the formula of Jin and Liu (1), and formula (12), (13) yields a closed formula for every fixed m. Note that for the case $m = 2$ we do not even have to evaluate the determinant in general, since for $m = 2$ simply

$$\det\left|\delta_{ij}-(1-\delta_{ij})\lambda_j\right| = 1-\lambda_1\lambda_2.$$

References

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[3] Y. Jin and C. Liu, Enumeration for spanning forests of complete bipartite graphs, to appear in Ars Combinatoria.