Problem Prove that any Boolean polynomial can be written as a union of atoms in the variables of the Boolean polynomial.

Solution Recall the definitions. Boolean variables are letters \(x_1, x_2, \ldots\). Boolean polynomials are intended to describe the algebraic operations that we can carry out with sets whose names are \(x_1, x_2, \ldots\). Usual identities for set operations apply to Boolean polynomials. Boolean polynomials are defined formally as follows:

(i) Boolean variables are Boolean polynomials.

(ii) Given two Boolean polynomials already defined, say, \(u\) and \(v\), the following expressions are also Boolean polynomials: \(u \cap v, u \cup v, \bar{u}\).

(iii) Whatever you can obtain by repeating this procedure, is a Boolean polynomial.

For a Boolean variable \(x_i\) let \(x_i^f\) denote any of \(x_i\) and \(\bar{x}_i\). Atoms in the Boolean variables \(x_1, x_2, \ldots, x_n\) are defined in the following way: \(x_1^f \cap x_2^f \cap \ldots \cap x_n^f\), with any choice of \(x_i\) and \(\bar{x}_i\) for \(x_i^f\). Hence there are \(2^n\) atoms in \(n\) variables. It is easy to see that for different atoms \(a_1, a_2, a_1 \cap a_2 = \emptyset\), since \(a_1, a_2\) differ in at least one \(x_i\), and then \(a_1 = x_i \cap \ldots\) and \(a_2 = \bar{x}_i \cap \ldots\). It is easy to see by induction on \(n\), that the union of the \(2^n\) atoms is the universe. In other words, the atoms partition the universe.

Now a Boolean variable \(x_i\) is written as an atom in the single variable \(x_i\) according to the definition of an atom.

Assume now that \(u\) is a Boolean polynomial in variables \(y_1, y_2, \ldots, y_m\) and \(v\) is a Boolean polynomial in variables \(z_1, z_2, \ldots, z_n\). We assume by hypothesis that \(u\) and \(v\) can be written as union of atoms in their respective variable sets that may or may not overlap. Hence \(u = \cup_{i \in I} a_i\) \(v = \cup_{j \in J} A_j\) where \(a_i\) and \(A_j\) are atoms in the respective variables.

(1) By the distributive law
\[
 u \cap v = (\cup_{i \in I} a_i) \cap (\cup_{j \in J} A_j) = \cup_{i \in I} (a_i \cap (\cup_{j \in J} A_j)) = \cup_{i \in I} \cup_{j \in J} a_i \cap A_j.
\]

Note that \(a_i \cap A_j\) is either an atom in the union of two sets of variables or \(\emptyset\). Therefore \(u \cap v\) is written as a union of atoms.

(2) If \(u = \cup_{i \in I} a_i\) a union of atoms, then \(\bar{u} = \cup_{i \notin I} a_i\). (The union of all atoms not used to represent \(u\).) This is true since the atoms partition the universe.

(3) Finally, for \(u \cup v\), write \(u \cup v = \bar{u} \cap \bar{v}\). \(\bar{u}\) and \(\bar{v}\) can be written as a union of atoms according to (2), and so can be \(\bar{u} \cap \bar{v}\) according to (1). Now the same holds for \(\bar{u} \cap \bar{v}\) according to (2).

Definition A symmetric chain decomposition of \(P_n\) (the power set of \(\{1, 2, \ldots, n\}\)) is

(i) a partition of \(P_n\) into disjoint chains, such that

(ii) every chain has the form \(S_k \subset S_{k+1} \subset \ldots \subset S_{n-k}\), where \(|S_i| = i\), and no size between \(k\) and \(n - k\) is missing from the chain.

Problem For every \(n\), \(P_n\) has a symmetric chain decomposition.

Solution We define a symmetric chain decomposition of \(P_n\) by induction on \(n\). The base case is \(n = 1\). The single chain \(S_0 = \emptyset < S_1 = \{1\}\) satisfies the definition of the symmetric chain decomposition: every element of the power set belongs to this single chain and the chain also satisfies (ii).
Inductive step. Assume that $P_n$ has an already defined symmetric chain decomposition. Let $c = (S_k \subset S_{k+1} \subset \ldots \subset S_{n-k})$ denote generic chain in it. Now we give the generic chains in the symmetric chain decomposition of $P_{n+1}$. Every $c$ from the symmetric chain decomposition of $P_n$ gives rise to two chains in $P_{n+1}$: $c' = (S_{k+1} \subset \ldots \subset S_{n-k})$ and $c'' = (S_k \subset S_k \cup \{n+1\} \subset S_{k+1} \cup \{n+1\} \subset \ldots \subset S_{n-k} \cup \{n+1\})$. $c'$ may be empty if $k + 1 > n - k$, then we disregard it.

- It is easy to see that both $c', c''$ are chains in $P_{n+1}$.

- Every element of $P_{n+1}$ is contained by at least one chain. The reason is that every $A \in P_{n+1}$ is either $A \in P_n$, or can be written $A = B \cup \{n+1\}$ where $B \in P_n$. If $A \in P_n$, then $A$ was an $S_i$ element of a chain $c$ in the symmetric chain decomposition of $P_n$ by hypothesis. If $S_i$ was not the smallest term in $c$, then $A$ is present in $c'$, if $S_i$ was not the smallest term in $c$, then $A$ is present in $c''$. If $A = B \cup \{n+1\}$, then by hypothesis $B$ was covered by a chain $c$ in $P_n$, and now $c''$ contains $A$.

- Every element of $P_{n+1}$ is contained by at most one chain. Assume that this is false for an $A \in P_{n+1}$. Again, either $A \in P_n$, or can be written $A = B \cup \{n+1\}$ where $B \in P_n$.

CASE 1. If $A \in P_n$, then it may occur in the symmetric chain decomposition of $P_{n+1}$ in a $c'$ type chain or can be the first term in a $c''$ type chain. If $A$ occurs in two $c'$ type chains, say $c'_1$ and $c'_2$, then $A$ occurred in two chains, $c_1$ and $c_2$ in the symmetric chain decomposition of $P_n$. If $A$ occurs in a $c'_1$ and a $c''_2$, then $A$ occurred in two chains, $c_1$ and $c_2$ in the symmetric chain decomposition of $P_n$. If $A$ occurs in a $c''_1$ and a $c''_2$, then $A$ occurred in two chains, $c_1$ and $c_2$ in the symmetric chain decomposition of $P_n$.

CASE 2. $A = B \cup \{n+1\}$ where $B \in P_n$. $A$ can occur only in $c''$ type chains in the symmetric chain decomposition of $P_{n+1}$. If $A$ is in $c''_1$ and a $c''_2$, then $B$ occurred in two chains, $c_1$ and $c_2$ in the symmetric chain decomposition of $P_n$.

- It follows from the construction that $c'$ contains elements with all sizes between its smallest and largest member. Also, the first term has $k + 1$ elements, the last has $n - k = (n + 1) - (k + 1)$ as required by (ii). It follows from the construction that $c''$ contains elements with all sizes between its smallest and largest member. The smallest member has $k$ elements, the largest member has $n - k + 1 = (n + 1) - k$ elements, as required in (ii).