Short proof for a theorem of Pach, Spencer, and Tóth

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Abstract

Pach, Spencer, and Tóth showed that for a simple graph on n vertices and e edges, if $e \geq 4n$, and the girth of the graph exceeds $2r$ ($r > 0$ integer), then $cr(G) \geq c_r \frac{e^{r+2}}{n^{r+1}}$. We give a simple new proof to this theorem.

A well-known result of Ajtai et al. [ACNS] and Leighton [L] shows that for a simple graph on n vertices and e edges, if $e \geq 4n$, then $cr(G) \geq c\frac{e^3}{n^2}$, with $c = \frac{1}{64}$. For the best current constant c, see [PT]. Answering a question of Miklós Simonovits, Pach, Spencer, and Tóth [PST] showed that for a simple graph on n vertices and e edges, if $e \geq 4n$ and the girth of the graph exceeds $2r$ ($r > 0$ is a fixed integer), then $cr(G) \geq c_r \frac{e^{r+2}}{n^{r+1}}$. The aim of this note to give a very simple proof for this theorem. We also obtain an explicit constant for c_r. We prove that

$$\mathrm{cr}(G) \geq \frac{(1-o(1))c}{r^2 2^{2r+3}} \cdot \frac{e^{r+2}}{n^{r+1}}, \tag{1}$$

where $o(1)$ is for $e/n \to \infty$. At the end of the paper we comment on how to make the lower bound even more explicit.

The proof reduces the theorem to the original result of Ajtai et al. [ACNS] and Leighton [L] through the embedding method. For comparison, the original proof of Pach, Spencer, and Tóth [PST] used the bisection width method.

Assume we have a simple graph G on n vertices and e edges, with crossing number $cr(G)$ and girth $> 2r$. Consider a drawing of G realizing the crossing number, in which any two edges share at most one interior point, and no two edges with a common endpoint cross. (It is well-known that these assumptions can be made, see e.g. [S].) Let $\bar{d}$ denote the average degree of G, i.e. $\bar{d} = 2e/n$.

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Next, following Pach, Spencer, and Tóth [PST], we define a graph G' with a drawing D' as follows. We split every vertex of G whose degree exceeds $\bar{d}$ into vertices of degree at most $\bar{d}$, as we describe. Let v be a vertex of G with degree $d(v) = d > \bar{d}$, and let vw_1, vw_2, ..., vw_d be the edges incident to v, listed in clockwise order. Replace v by $\lceil d/\bar{d} \rceil$ new vertices, v_1, v_2, ..., $v_{\lceil d/\bar{d} \rceil}$, placed in clockwise order on a very small circle around v. Without introducing any new crossings, connect w_j to v_i if and only if $\bar{d}(i-1) < j \le \bar{d}i$ ($1 \le j \le d$, $1 \le i \le \lceil d/\bar{d} \rceil$). Repeat this procedure for every vertex whose degree exceeds $\bar{d}$, and denote the resulting graph by G' and the resulting drawing by D'. Observe that the number of crossings is the same in D and D', $\mathrm{cr}(G') \le \mathrm{cr}(G)$, and the girth of G' still exceeds $2r$. For the number of vertices, $|V(G')| \le \sum_{v \in V(G)} \lceil d(v)/\bar{d} \rceil \le n + \sum_{v \in V(G)} d(v)/\bar{d} = 2n$. Every degree in G' is at most $\Delta = \lfloor \bar{d} \rfloor$.

Define a graph G'' with $V(G'') = V(G')$, and $E(G'') =$ those pairs of vertices from $V(G')$ whose distance in G is exactly r. By the girth assumption on G, G'' is a simple graph with maximum degree at most $\Delta(\Delta-1)^{r-1}$. (Such graph construction for crossing number purposes was first used by Pach and Sharir [PS].)

Next, following the of the embedding method (see [L], [SSSV]), we make a drawing D'' of the graph G'', closely following the drawing D' of G'. Note that vertices of G'' are the same as the vertices of G', and keep them at the same location. Every edge f of G'' is represented by a (unique) r-path in G'. Draw the edge f by a curve "infinitesimally close" to this unique path. Do this for all edges to obtain the drawing D''.

Crossings of D'' fall into two categories. A crossing of the *first category* arises from two crossing edges a, b of D', which are parts of paths of lengths r. Notice that the number of paths of lengths r containing a fixed edge a is at most $r(\Delta-1)^{r-1}$. Therefore, every crossing in D' corresponds to at most $r^2(\Delta-1)^{2r-2}$ crossings of D'' of the first category.

A crossing of the *second category* arises at *vertices*, which have the property that in their infinitesimally small neighborhoods paths of D' representing edges of D'' cross. Fix a vertex $v \in V(G')$. Easy calculation shows that v is a vertex of at most $\frac{1}{2}(r+1)\Delta(\Delta-1)^{r-1}$ paths of length r in G'. Therefore,

$$cr(G'') \le r^2(\Delta-1)^{2r-2} cr(G) + 2n \binom{\frac{1}{2}(r+1)\Delta(\Delta-1)^{r-1}}{2}. \tag{2}$$

Combining formula (13) and Theorem 5 from the paper of Erdős and Simonovits [ES], we obtain that $|E(G'')| \ge (\frac{1}{2} - o(1)) \cdot \frac{e^r}{n^{r-1}}$.

Applying to G'' the result of Ajtai et al. [ACNS] and Leighton [L] quoted in the first sentence of the paper, we obtain $|E(G'')| \le (8 + o(1))^{1/r} n$, or,

$$cr(G'') \ge \frac{c}{(2n)^2} \left(\left(\frac{1}{2} - o(1) \right) \cdot \frac{e^r}{n^{r-1}} \right)^3. \tag{3}$$

Comparing formulae (2) and (3), the claimed lower bound for $cr(G)$ follows:

$$\begin{aligned}\mathrm{cr}(G) &\geq \frac{\mathrm{cr}(G'')}{r^2(\Delta-1)^{2r-2}} - \frac{n}{4}(1+\frac{1}{r})^2\Delta^2 \\ &\geq \frac{(1-o(1))c}{r^2 2^{2r+3}} \cdot \frac{e^{r+2}}{n^{r+1}} - (1+\frac{1}{r})^2\frac{e^2}{n},\end{aligned}$$

where the error term with negative sign is little-oh of the main term.

If preferred, more explicit lower bounds can be obtained for $|E(G'')|$, at the expense of having a smaller c_r. Also, this modified proof covers some linear size graphs. Fix any $\epsilon > 0$, and construct a large subgraph H' of G', and also H'' of G'', as follows. Throw out vertices of G' with degree $< \frac{\bar{d}}{4+\epsilon}$, and iterate this for the resulting graphs. Throwing out at most $2n$ vertices, we lost at most $\frac{4e}{4+\epsilon}$ edges, and therefore at least $\frac{\epsilon e}{4+\epsilon}$ edges are left in some subgraph H'. Since $\frac{1}{2}|V(H'')|\bar{d} \geq \frac{\epsilon e}{4+\epsilon}$, we have $|V(H'')| \geq \frac{\epsilon n}{4+\epsilon}$. H'' is a large graph with large minimum degree, and therefore has many r-paths. Complete the argument as above.

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