Diameter of 4-Colourable Graphs

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Abstract

We prove that for every connected 4-colourable graph G of order n and minimum degree $\delta \geq 1$, $\mathrm{diam}(G) \leq \frac{5n}{2\delta} - 1$. This is a first step toward proving a conjecture of Erdős, Pach, Pollack and Tuza [4] from 1989.

1 Introduction

Let $G = (V, E)$ be a simple, finite, connected graph on n vertices, with minimum degree $\delta \geq 2$ and diameter $\mathrm{diam}(G)$. The natural problem of bounding the diameter of a graph in terms of its order and minimum degree was solved by several authors [5, 4, 6, 7], who independently proved that, for fixed $\delta \geq 2$ and large n,

$$\mathrm{diam}(G) \leq \frac{3n}{\delta + 1} + O(1). \tag{1}$$

In 1989, Erdős, Pach, Pollack, and Tuza [4] showed that this upper bound on the diameter can be improved if G is triangle-free, or if G does not contain a 4-cycle. Their results were extended in [1] to graphs not containing a subgraph isomorphic to the complete bipartite graph $K_{2,s}$, for $s \geq 2$, and in [2] to graphs not containing a complete subgraph $K_{3,3}$.

In the same paper [4], Erdős, Pach, Pollack, and Tuza also conjectured that the upper bound (1) can be improved further if G does not contain a large complete subgraph K_k:

Conjecture 1 *Let $r, \delta \geq 2$ be fixed integers and let G be a connected graph with n vertices and minimum degree δ.*
(i) If G is K_{2r}-free and δ is a multiple of $(r-1)(3r+2)$ then, for large n,

$$\mathrm{diam}(G) \leq \frac{2(r-1)(3r+2)}{(2r^2-1)\delta} n + O(1).$$

(ii) If G is K_{2r+1}-free and δ is a multiple of $3r-1$, then, for large n,

$$\mathrm{diam}(G) \leq \frac{3r-1}{r\delta} n + O(1).$$

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They also constructed graphs showing that, if the above bounds hold, then they are sharp, apart from an additive constant. For $r = 2$, which is relevant for our paper, the graph construction is the following. Let X_i and Y_i be disjoint sets of vertices, such that $|X_0| = |Y_0| = 3\delta/5 = |X_d| = |Y_d|$ and for $0 < i < d$, $|X_i| = |Y_i| = \delta/5$; and join vertices of X_i to the vertices of Y_i, and vertices of $X_i \cup Y_i$ to vertices of $X_{i-1} \cup Y_{i-1}$ and $X_{i+1} \cup Y_{i+1}$

So far, no progress on the above conjecture, even for specific values of r, has been reported. In this paper, we consider a slight weakening of the above conjecture for K_5-free graphs. We show that the conjecture holds for all $\delta \geq 1$ under the somewhat stronger assumption that G is 4-colourable.

2 Proof of theorem

Custom-taylored Bonferroni-type inequalities have a large literature, see [3]. The following variant will be central to our proof.

Lemma 1 *Let $\{A_i \mid i = 1, 2, \ldots, d\}$ be a finite set system. If no element of $\bigcup_{i\in I} A_i$ is contained in more than 4 sets among the A_i, then*

$$3|\bigcup_{i=1}^{d} A_i| \geq 2\sum_{1\leq i\leq d} |A_i| - \sum_{1\leq i,j\leq d} |A_i \cap A_j| + \sum_{1\leq i<j<k<l\leq d} |A_i \cap A_j \cap A_k \cap A_l|.$$

Proof. Let $x \in \bigcup_{i\in I} A_i$. Then x contributes exactly 3 to the left hand side of the above inequality. If x is in p sets A_i then x contributes $2p - \binom{p}{2} + \binom{p}{4}$ to the right hand side, which for $0 \leq p \leq 4$ is at most 3. Summing this over all x yields the lemma. $\Box$

We use standard notation. Specifically, we denote the vertex set and the edge set of a graph by V and E, respectively. The neighbourhood of a vertex v is denoted by $N_G(v)$. If $P = v_1v_2, \ldots v_k$ is a sequence of vertices, and v_0, v_{k+1} are two further vertices, then we denote the extended sequence $v_0v_1 \ldots v_kv_{k+1}$ by v_0Pv_{k+1}.

Theorem 1 *For every connected 4-colourable graph G of order n and minimum degree $\delta \geq 1$,*

$$\operatorname{diam}(G) \leq \frac{5n}{2\delta} - 1.$$

Proof. Let $d := \operatorname{diam}(G)$. We can assume that G is edge-maximal, i.e., addition of any edge decreases the diameter or increases the chromatic number. It suffices to show that there exists a sequence of vertices $P = \alpha_0\alpha_1 \ldots \alpha_d$ of G such that, with $A_i := N_G(a_i)$, $i = 0, 1, \ldots, d$, we have

$$\sum_{0\leq i<j\leq d} |A_i \cap A_j| - \sum_{0\leq i<j<k<l\leq d} |A_i \cap A_j \cap A_k \cap A_l| \quad \leq \quad 2n. \tag{2}$$

since then, by $|A_i| \geq \delta$ and Lemma 1,

$$\begin{aligned}
3n &\geq 3|\bigcup_{i=0}^{d} A_i| \\
&\geq 2\sum_{i=0}^{d} |A_i| - \sum_{0\leq i<j\leq d} |A_i \cap A_j| + \sum_{0\leq i<j<k<l\leq d} |A_i \cap A_j \cap A_k \cap A_l| \\
&\geq 2(d+1)\delta - 2n,
\end{aligned}$$

which implies $d \le \frac{5n}{2\delta} - 1$, as desired.

For a subset V' of V we define $g(P,V')$ to be the contribution of V' to the right hand side of (2), i.e.,

$$g(P,V') = \sum_{0\le i<j\le d} |A_i \cap A_j \cap V'| - \sum_{0\le i<j<k<l\le d} |A_i \cap A_j \cap A_k \cap A_l \cap V'|.$$

So equation (2) becomes $g(P,V) \le 2n$. Often we need only the first sum of the right hand side above, so we also let

$$f(P,V') = \sum_{0\le i<j\le d} |A_i \cap A_j \cap V'|.$$

Note that $g(P,V') \le f(P,V')$. Let u and v be two vertices at distance d, let V_i be the set of all vertices at distance i from u, and for $i \le j$ let $V_{i,j} := V_i \cup V_{i+1} \cup \ldots \cup V_j$. Denote by χ_i the number of colours that occur in V_i. Note that $\chi_i = 1$ implies $\chi_{i+1} \le 3$ since no vertex of V_{i+1} can have the colour of the vertices in V_i. Note that all vertices in V_i of the same colour have the same neighbourhood by the assumption on edge-maximality.

Consider the sequence $C = \chi_0\chi_1\ldots\chi_d$. We will provide an algorithm that shows that there exist integers $0 = c_1 < c_2 < \ldots < c_t = d+1$ such that, if we let $r = c_i$ and $s = c_{i+1} - 1$, each of the $t-1$ segments $S_i = \chi_r\chi_{r+1}\chi_{r+2}\ldots\chi_s$ is of one of the 4 types described below. (For shortness, we sometimes also say that $V_{r,s}$ is the corresponding type as well.)

Type 1: $\chi_r = \chi_{r+1} = \ldots = \chi_s = 1$, $s \ge r$;

Type 2: $\chi_r \ge 1$ and $\chi_{r+1}, \chi_{r+2}, \ldots, \chi_s \ge 2$, $s \ge r+1$. If $s < d$ then $\chi_{s+1} = 1$. If $\chi_r > 1$, then $r \ge 1$ and $\chi_{r-1} = 1$. If $s = r+1$ then $(\chi_r, \chi_{r+1}) \ne (1,3)$;

Type 3: $s - r$ is even and positive; $\chi_r = \chi_{r+2} = \chi_{r+4} = \ldots = \chi_s = 1$ and $\chi_{r+1} = 3$ and $\chi_{r+3}, \chi_{r+5}, \ldots, \chi_{s-1} \ge 2$;

Type 4: $s = r = d$, $\chi_r \ge 2$ and $\chi_{r-1} = 1$.

During the algorithm we will consider the sequence $\chi_a\chi_{a+1}\ldots\chi_b$ that still needs to be processed with $a \le b$; initially $a = 0$ and $b = d$. The preliminary step decides whether a sequence of type 4 will be used at the end, and the final step will take care of processing the c_i's for the type 4 sequence. After the preliminary step $V_{a,b}$ will have the property that $\chi_a = 1$, $b = d$ or $b = d-1$ depending on the existence of a type 4 sequence, and if $\chi_b \ne 1$ then $\chi_{b-1} \ne 1$. This property will be maintained during the processing step, where there only the value of a is changed. By contraposition, in the processing step the set $V_{a,b}$ must satisfy the conditions that if $\chi_{b-1} = 1$ then $\chi_b = 1$. Some remarks that may be necessary to see the correctness of the algorithm are included between // dividers and set in italic.

The description of the algorithm is self-explanatory:

PRELIMINARY STEP: $a \leftarrow 0$; $c_1 \leftarrow 0$; $m \leftarrow 2$; `DONE`$\leftarrow$`FALSE`;

  `IF` ($\chi_d > 1$ `and` $\chi_{d-1} = 1$) `THEN` $\{b \leftarrow d-1\}$ // *This means χ_d will be type 4.*//

PROCESSING STEP: `REPEAT UNTIL DONE=TRUE`

```
{IF (a = b or χ_{a+1} = 1) // Removal of type 1 sequence.//
    {
    LET c_m BE THE LARGEST INTEGER SUCH THAT FOR ALL i : a ≤ i ≤ c_m WE
    HAVE χ_i = 1; //Clearly c_m − 1 ≥ a or c_m = a = b.//
    IF c_m = b THEN {c_m ← b + 1; DONE←TRUE} // V_{a,b} will be type 1.//
    ELSE // V_{a,c_m−1} will be type 1, χ_{c_m} = 1.// {a ← c_m; m ← m + 1}
    }
ELSEIF (χ_{a+1} ≠ 3 OR χ_{a+2} ≠ 1) // χ_{a+1} > 1; removal of type 2 sequence.//
     IF (χ_i ≠ 1 FOR ALL i : a < i ≤ b) THEN {c_m ← b + 1; DONE←TRUE}
        // V_{a,b} will be type 2.//
     ELSE // now some χ_i is 1.//
        {LET c_m BE THE LEAST INTEGER SUCH THAT (c_m > a AND χ_{c_m} = 1);
        // V_{a,c_m−1} will be type 2, χ_{c_m} = 1.//
        a ← c_m; m ← m + 1}
ELSE // Now χ_{a+1} = 3 and χ_{a+2} = 1; removal of type 3 sequence.//
    {SET k TO THE LARGEST INTEGER SUCH THAT FOR ALL i : 1 ≤ i ≤ k WE HAVE
    (χ_{a+2i} = 1 AND χ_{a+2i−1} > 1); c_m ← a + 2k + 1; // Clearly k ≥ 1.//
     IF c_m = b + 1 THEN DONE←TRUE // V_{a,b} will be type 3.//
     ELSE // V_{a,a+2k} = V_{a,c_m−1} will be type 3; but χ_{c_m} may not be 1.//
         IF (χ_i = 1 FOR SOME i : c_m ≤ i ≤ b)
            {SET w TO THE LEAST INTEGER SUCH THAT (w ≥ c_m AND χ_w = 1);
            // Clearly w ≠ c_m + 1, as this would contradict the maximality of k.//
             IF w = c_m THEN {a ← c_m; m ← m + 1} // continue as χ_{c_m} = 1.//
             ELSE //χ_{c_m} ≠ 1, V_{c_m,w−1} will be type 2 since w > c_m + 2, χ_w = 1.//
              {c_{m+1} ← w; a ← c_{m+1}; m ← m + 2}
            }
         ELSE DONE←TRUE //In this case there are no more 1's among the χ_i's.
            From χ_{c_m−1} = 1, we get b − 1 > c_m − 1 and V_{c_m,b} is type 2.//
    } // End of case χ_{a+1} = 3 and χ_{a+2} = 1.//

} // End of repeat loop.//
```

FINAL STEP: IF $c_m = d + 1$ THEN $\{t \leftarrow m\}$ ELSE $\{t \leftarrow m + 1;\ c_t \leftarrow d + 1\}$

Consider a segment $S_i = \chi_r\chi_{r+1}\ldots\chi_s$ of C of type 1, 2, 3, or 4. If $r > 0$, let $\beta_{r-1} \in V_{0,r-1}$ be arbitrarily fixed. We will show that for this arbitrarily fixed choice of β_{r-1} (if such a choice was made) there exists a sequence of vertices $P_{r,s} = \alpha_r\alpha_{r+1}\ldots\alpha_s$ such that

Property (i): V_r (V_s) contains no vertex of $P_{r,s}$, except possibly α_r (α_s),

Property (ii):

(a) If $r > 0$ and $s < d$ then for all $\beta_{s+1} \in V_{s+1,d}$ using $P' = \beta_{r-1}P_{r,s}\beta_{s+1}$ we have

$$g(P', V_{r,s}) \leq 2|V_{r,s}|.$$

(b) If $r > 0$ and $s = d$ then using $P' = \beta_{r-1}P_{r,s}$ we have $g(P', V_{r,s}) \leq 2|V_{r,s}|$.

(a) If $r = 0$ and $s < d$ then for all $\beta_{s+1} \in V_{s+1,d}$ using $P' = P_{r,s}\beta_{s+1}$ we have

$$g(P', V_{r,s}) \leq 2|V_{r,s}|.$$

(d) If $r = 0$ and $s = d$ then $g(P, V_{r,s}) \leq 2|V_{r,s}|$.

If such sequence selections can indeed be made, we will achieve our goal because of the following. Recall that C is subdivided into $t-1$ segments, with the ith segment being $S_i = \chi_{c_i}\chi_{c_i+1}\chi_{c_i+2}\cdots\chi_{c_{i+1}-1}$.

Since $c_1 = 0$, we can choose P_{c_1,c_2-1} according to properties (i)-(ii). Once the sequence P_{c_{i-1},c_i-1} has been chosen for some $i : 2 < i < t$, choose the sequence $P_{c_i,c_{i+1}-1}$ for $\beta_{c_i-1} = \alpha_{c_i-1}$ according to properties (i) and (ii).

The sequence $P = \alpha_0, \alpha_1, \ldots, \alpha_d = P_{c_1,c_2-1}P_{c_2,c_3-1}\ldots P_{c_{t-1},c_t-1}$ is constructed by concatenating the sequences $P_{c_i,c_{i+1}-1}$ in order.

Now if $c_i > 0$ and $c_{i+1} \leq d$ (i.e. $1 < i < t-1$), then by properties (i) and (ii) and the fact that the value of $g(\,\cdot\,, V_{r,s})$ is uneffected by any vertices not in $V_{r-1,s+1}$, we have that

$$g(P, V_{c_i,c_{i+1}-1}) \;=\; g(\alpha_{c_i-1}P_{c_i,c_{i+1}-1}\alpha_{c_i+1}, V_{c_i,c_{i+1}-1}) \leq 2|V_{c_i,c_{i+1}-1}|.$$

Similarly, we get that $g(P, V_{c_i,c_{i+1}-1}) \leq 2|V_{c_i,c_{i+1}-1}|$ for all $i : 1 \leq i \leq t$. Therefore

$$g(P, V) = \sum_{i=1}^{t-1} g(P, V_{c_i,c_{i+1}-1}) \leq \sum_{i=1}^{t-1} 2|V_{c_i,c_{i+1}-1}| = 2|V|,$$

as desired.

So what remains to show is that for each segment $S_i = \chi_r \ldots \chi_s$ of type 1,2,3 or 4 we can choose the appropriate sequence $P_{r,s}$ satisfying properties (i)-(ii). We have already remarked that the value of $g(\,\cdot\,, V_{r,s})$ is uneffected by any vertices not in $V_{r-1,s+1}$, and therefore it is enough to assume that $\beta_{r-1} \in V_{r-1}$ and $\beta_{s+1} \in V_{s+1}$ instead of $\beta_{r-1} \in V_{0,r-1}$ and $\beta_{s+1} \in V_{s+1,d}$ in the proof of property (ii).

If $r > 0$, fix $\beta_{r-1} \in V_{r-1}$ arbitrarily. We consider each type of segment separately. We will use $a_{r-1} = \beta_{r-1}$ and $a_{s+1} = \beta_{s+1}$ for ease of notation below.

Type 1: For $i = r, r+1, \ldots, s$ choose a vertex $a_i \in V_i$ arbitrarily and let $P_{r,s} = a_r, a_{r+1}, \ldots, a_s$ (so $\alpha_i = a_i$). Clearly, $P_{r,s}$ satisfies property (i) above. First assume that $r > 0$ and $s < d$. Choose $a_{s+1} \in V_{s+1}$ arbitrarily and let $P' = a_{r-1}, P_{r,s}, a_{s+1}$. Since for $i \in \{r, r+1, \ldots, s\}$ the distance layer V_i has only one colour class,

$$\begin{aligned} f(P', V_i) &= |N(a_{i-1}) \cap N(a_i) \cap V_i| + |N(a_i) \cap N(a_{i+1}) \cap V_i| + |N(a_{i-1}) \cap N(a_{i+1}) \cap V_i| \\ &\leq 0 + 0 + |V_i|, \text{ and} \\ f(P', V_{r,s}) &= \sum_{i=r}^{s} f(P', V_i) \leq \sum_{i=r}^{s} |V_i| \leq |V_{r,s}|. \end{aligned}$$

Hence $g(P', V_{r,s}) \leq f(P', V_{r,s}) \leq |V_{r,s}|$, independently of the choice of β_{s+1}, and so $P_{r,s}$ satisfies property (ii) as well.

If $r = 0 < s < d$ then $P' = P_{r,s}, a_{s+1}$, and the above estimate only changes when $i = 0$; and $f(P', V_0) = |N(a_i) \cap N(a_{i+1}) \cap V_i| = 0$. It is easy to see that the statement works in all other cases ($0 < r \leq s < d$ or $0 = r$ and $s = d$) as well.

Type 2: For $i = r, \ldots, s$ choose one vertex each from the largest two colour classes of V_i. (If $\chi_r = 1$ then we choose a vertex of V_r twice.) By edge maximality the graph induced by these vertices contains two geodesics $P_{r,s} = a_r, a_{r+1}, \ldots, a_s$ and $Q_{r,s} = b_r, b_{r+1}, \ldots, b_s$ from V_r to V_s that are vertex disjoint, except possibly for the first vertex. Clearly, $P_{r,s}$ and $Q_{r,s}$ satisfy property (i) above.

Assume first that $0 < r$ and $s < d$. Let $a_{s+1} \in V_{s+1}$ be arbitrary. Note that by $\chi_{s+1} = 1$ all vertices in V_{s+1} have the same neighbours. Therefore, what follows is independent of the choice of a_{s+1}. Let $P' = a_{r-1}P_{r,s}a_{s+1}$ and $Q' = a_{r-1}Q_{r,s}a_{s+1}$. We show that for each $i : r \le i \le s$

$$f(P', V_i) + f(Q', V_i) \le 4|V_i|. \tag{3}$$

For ease of notation $b_{r-1} = a_{r-1}$ and $b_{s+1} = a_{s+1}$. Consider a distance layer V_i, $r \le i \le s$. For $j = 1, 2, 3, 4$ let x_j be the number of vertices of colour j in V_i. We can assume w.l.o.g. that $x_1 \ge x_2 \ge x_3 \ge x_4$, and that a_i and b_i have colour 1 and 2, respectively. Let $a_{i-1}, b_{i-1}, a_{i+1}, b_{i+1}$ have colour j, k, l, m, respectively.

Assume first that $\chi_i > 1$. Then a_{i-1} is adjacent to a_i and b_{i-1} is adjacent to b_i (by construction if $i > r$ and by the fact that $\chi_r > 1$ implies $\chi_{i-1} = 1$ if $i = r$), so $j \ne 1$ and $k \ne 2$. If a_i (or b_i) is not adjacent to a_{i+1} (b_{i+1}), then we must have $i = s$, which implies that $a_{i+1} = b_{i+1}$ so $l = m$.

Therefore

$$\begin{aligned} f(P', V_i) &= |N(a_{i-1}) \cap N(a_i) \cap V_i| + |N(a_i) \cap N(a_{i+1}) \cap V_i| + |N(a_{i-1}) \cap N(a_{i+1}) \cap V_i| \\ &= (|V_i| - x_1 - x_j) + |N(a_i) \cap N(a_{i+1}) \cap V_i| + |N(a_{i-1}) \cap N(a_{i+1}) \cap V_i| \end{aligned}$$

If in addition $j \ne l$, then $|N(a_{i-1}) \cap N(a_{i+1}) \cap V_i| = |V_i| - x_j - x_l$.

The following 3 cases might occur: $j \ne l \ne 1$, $j \ne l = 1$ (in which case $i = s$ and $l = m = 1$) and $j = l$.

If $j \ne l \ne 1$ then $x_j + x_l \ge x_3 + x_4$ and so

$$\begin{aligned} f(P', V_i) &= (|V_i| - x_1 - x_j) + (|V_i| - x_1 - x_l) + (|V_i| - x_j - x_l) \\ &= 3|V_i| - 2(x_1 + x_j + x_l) \le 3|V_i| - 2(x_1 + x_3 + x_4) \end{aligned}$$

If $j \ne l = 1$ then

$$\begin{aligned} f(P', V_i) &= (|V_i| - x_1 - x_j) + (|V_i| - x_1) + (|V_i| - x_j - x_1) \\ &= 3|V_i| - (3x_1 + 2x_j) \le 3|V_i| - 3x_1 \end{aligned}$$

If $j = l$ then

$$f(P', V_i) \le (|V_i| - x_1 - x_j) + (|V_i| - x_1 - x_j) + (|V_i| - x_j) = 3|V_i| - (2x_1 + 3x_j)$$

In summary, we have for P' (and similarly for Q') that

$$\begin{aligned} f(P', V_i) &\le \begin{cases} 3|V_i| - 2(x_1 + x_3 + x_4), & \text{if } j \ne l \ne 1 \\ 3|V_i| - 3x_1 & \text{if } j \ne l = 1 \\ 3|V_i| - (2x_1 + 3x_j), & \text{if } j = l \end{cases} \\ f(Q', V_i) &\le \begin{cases} 3|V_i| - 2(x_2 + x_3 + x_4), & \text{if } k \ne m \ne 2 \\ 3|V_i| - 3x_2, & \text{if } k \ne m = 2 \\ 3|V_i| - (2x_2 + 3x_k), & \text{if } k = m \end{cases} \end{aligned}$$

Since we always have $f(Q', V_i) \leq 3|V_i| - 2x_2$, for $j \neq l \neq 1$ we get

$$f(P', V_i) + f(Q', V_i) \leq 3|V_i| - 2(x_1 + x_3 + x_4) + 3|V_i| - 2x_2 = 4|V_i|,$$

as claimed. This statement follows similarly if $k \neq m \neq 2$.

If $j \neq l = 1$, then $l = m = 1$. Therefore we are done when $k \neq m$, so we may assume that $k = m = 1$. Using $6x_1 \geq 2(x_1 + x_3 + x_4)$ we get

$$\begin{aligned} f(P', V_i) + f(Q', V_i) &\leq 3|V_i| - 3x_1 + 3|V_i| - (2x_2 + 3x_1) \leq 6|V_i| - (6x_1 + 2x_2) \\ &\leq 6|V_i| - 2(x_1 + x_2 + x_3 + x_4) = 4|V_i|, \end{aligned}$$

as claimed. A similar logic works when $k \neq m = 2$.

So the only case that still needs to be examined is $j = l$ and $k = m$, when

$$f(P', V_i) + f(Q', V_i) \leq 6|V_i| - 2(x_1 + x_2 + x_j + x_k)$$

If $j \neq k$ then, as before, $x_j + x_k \geq x_3 + x_4$ and we get that $f(P', V_i) + f(Q', V_i) \leq 4|V_i|$.

If $j = k$, then we must have $\chi_{i-1} = \chi_{i+1} = 1$, which implies that $i = s = r + 1$. Since the sequence is type 2, this must mean that $\chi_i = 2$, so $x_3 = x_4 = 0$ and $|V_i| = x_1 + x_2$. Therefore in this case also $f(P', V_i) + f(Q', V_i) \leq 4|V_i|$, as claimed.

So the statement is true when $\chi_i > 1$. In the case when $\chi_i = 1$ (and so $i = r$) we get $f(P', V_i) + f(Q', V_i) \leq 2|V_i|$, as before.

If $r = 0$ or $s = d$ then the corresponding estimates for $f(P', V_r)$, $f(Q', V_r)$, $f(P', V_s)$ and $f(Q', V_s)$ can only decrease.

We can assume, without loss of generality, that $f(P', V_{r,s}) \leq f(Q', V_{r,s})$ and thus $g(P', V_{r,s}) \leq f(P', V_{r,s}) \leq 2|V_{r,s}|$. Hence $P_{r,s}$ satisfies property (i) and property (ii), as desired.

Type 3: We can assume that, possibly after recolouring, the vertices in $V_r \cup V_{r+2} \cup V_{r+4} \cup \ldots \cup V_s$ all have the same colour and that this colour does not occur inbetween. We consider two cases, depending on whether $s = r + 2$ or $s \geq r + 4$. Initially, we consider $s < d$ only. Let $a_{s+1} \in V_{s+1}$ be arbitrary. Note that since $\chi_s = 1$, any member of V_s is adjacent to a_{s+1}.

Case 1: $s \geq r + 4$.

For $i = r+1, r+3, r+5, \ldots, s-1$ let $a_i^1, a_i^2 \in V_i$ be vertices that belong to the largest and second largest,respectively, colour class of V_i, and for $i = r, r+2, r+4, \ldots, s$ let $a_i \in V_i$. Define the following sequences of vertices, each with $s - r + 1$ vertices:

$$\begin{aligned} P_{r,s} &= a_r a_{r+1}^1 a_{r+2} a_{r+3}^1 a_{r+4} a_{r+5} a_{r+6} \ldots a_{s-1} a_{s-1}^1 a_s, \\ Q_{r,s} &= a_r a_{r+1}^1 a_{r+1}^2 a_{r+3}^1 a_{r+3}^2 a_{r+5}^1 a_{r+5}^2 \ldots a_{s-2}^2 a_{s-1}^1 a_{s-1}^2, \\ R_{r,s} &= a_{r+1}^1 a_{r+1}^2 a_{r+3}^1 a_{r+3}^2 a_{r+5}^1 a_{r+5}^2 a_{r+7}^1 \ldots a_{s-1}^1 a_{s-1}^2 a_s. \end{aligned}$$

Clearly, $P_{r,s}, Q_{r,s}$ and $R_{r,s}$ have the required length and satisfy property (i) above.

Let $P' = a_{r-1}, P_{r,s}, a_{s+1}$, $Q' = a_{r-1}, Q_{r,s}, a_{s+1}$, and $R' = a_{r-1}, R_{r,s}, a_{s+1}$. We first consider $f(P', V_{r,s})$. Let $i \in \{r, r+1, r+2, \ldots, s\}$ with $\chi_i = 3$. So $a_{i-1} \in V_{i-1}$, $a_i^1 \in V_i$, and $a_{i+1} \in V_{i+1}$. Now $N(a_{i-1}) \cap N(a_i^1) \cap V_i$ and $N(a_i^1) \cap N(a_{i+1}) \cap V_i$ are contained in the union of the two smallest colour classes of V_i and thus have at most $\frac{2}{3}|V_i|$ vertices each, while $N(a_{i-1}) \cap N(a_{i+1}) \cap V_i$ has at most $|V_i|$ vertices each. Hence $f(P', V_i) \leq \frac{7}{3}|V_i|$ if

$\chi_i = 3$. If $\chi_i = 2$, then similar considerations show that $f(P', V_i) \le 2|V_i|$, while $\chi_i = 1$ implies that $f(P', V_i) \le |V_i|$, even when $i = r$ and perhaps $r = 0$. Hence,

$$f(P', V_{r,s}) \le (|V_r| + |V_{r+2}| + |V_{r+4}| + \ldots + |V_s|) + \frac{7}{3}(|V_{r+1}| + |V_{r+3}| + |V_{r+5}| + \ldots + |V_{s-1}|).$$

Now consider Q'. First let $i = r+1$, so $\chi_i = 3$. Then $N(a_{i-1}) \cap N(a_i^1) \cap V_i$ and $N(a_{i-1}) \cap N(a_i^2) \cap V_i$ do not contain vertices in the largest and second largest colour class, respectively, of V_i, while $N(a_i^1) \cap N(a_i^2) \cap V_i$ does not contain vertices in the two largest colour classes of V_i. Hence $f(Q', V_i) \le \frac{5}{3}|V_i|$ for $i = r+1$. (Were $\chi_i = 2$, we would have got $f(Q', V_i) \le |V_i|$—this is an estimate that we will need for R' later.) Similarly we obtain for $i = r+3, r+5, \ldots, s-1$ that $f(Q', V_i) = |N(a_i^1) \cap N(a_i^2) \cap V_i|$; therefore in this case $f(Q', V_i) \le \frac{1}{3}|V_i|$ if $\chi_i = 3$ and $f(Q', V_i) = 0$ if $\chi_i = 2$. It is easy to see that $f(Q', V_r) \le 3|V_r|$ and $f(Q', V_s) \le 3|V_s|$. For $i = r+2, r+4, r+6, \ldots, s-2$ each vertex of V_i is in the neighbourhood of exactly four vertices, $a_{i-1}^1, a_{i-1}^2, a_{i+1}^1$ and a_{i+1}^2. Hence $f(Q', V_i) = \binom{4}{2}|V_i| = 6|V_i|$ and $g(Q', V_i) = 5|V_i|$. In total

$$\begin{aligned} g(Q', V_{r,s}) \;\; &\le \;\; 3(|V_r| + |V_s|) + \frac{5}{3}|V_{r+1}| + \frac{1}{3}(|V_{r+3}| + |V_{r+5}| + |V_{r+7}| + \ldots + |V_{s-1}|) \\ &\quad +5(|V_{r+2}| + |V_{r+4}| + |V_{r+6}| + \ldots + |V_{s-2}|). \end{aligned}$$

Similarly we obtain

$$\begin{aligned} g(R', V_{r,s}) \;\; &\le \;\; 3(|V_r| + |V_s|) + \frac{5}{3}|V_{s-1}| + \frac{1}{3}(|V_{r+1}| + |V_{r+3}| + |V_{r+5}| + \ldots + |V_{s-3}|) \\ &\quad +5(|V_{r+2}| + |V_{r+4}| + |V_{r+6}| + \ldots + |V_{s-2}|). \end{aligned}$$

Note that each of the above three inequalities hold irrespective of the choice of a_{s+1}. moreover, they also hold when $s = d$. Now consider the weighted average of $g(P', V_{r,s})$ (counted six times) and $g(Q', V_{r,s})$ and $g(R', V_{r,s})$ (counted once each). By the above

$$\begin{aligned} &6g(P', V_{r,s}) + g(Q', V_{r,s}) + g(R', V_{r,s}) \\ &\qquad \le \;\; 12(|V_r| + |V_s|) + 16(|V_{r+2}| + |V_{r+4}| + |V_{r+6}| + \ldots + |V_{s-2}|) \\ &\qquad\quad\;\; +16(|V_{r+1}| + |V_{s-1}|) + \frac{44}{3}(|V_{r+3}| + |V_{r+5}| + |V_{r+7}| + \ldots + |V_{s-3}|) \\ &\qquad \le \;\; 16|V_{r,s}|. \end{aligned}$$

Hence at least one of $P_{r,s}$, $Q_{r,s}$ and $R_{r,s}$ satisfies also property (ii) above, as desired.

Case 2: $s = r+2$.

Then $(\chi_r, \chi_{r+1}, \chi_{r+2}) = (1, 3, 1)$. Choose vertices $a_r \in V_r$, $a_{r+2} \in V_{r+2}$ and $a_{r+1}^1, a_{r+1}^2 \in V_{r+1}$ from the largest and the second largest colour class in V_{r+1}, respectively. Let $P_{r,s} = a_r a_{r+1}^1 a_{r+2}$ and $Q_{r,s} = a_r a_{r+1}^1 a_{r+1}^2$. Clearly, $P_{r,s}$ and $Q_{r,s}$ satisfy property (i) above. Let $a_{s+1} \in V_{s+1}$ be arbitrary and let $P' = a_{r-1} P_{r,s} a_{s+1}$ and $Q' = a_{r-1} Q_{r,s} a_{s+1}$. Then

$$\begin{aligned} f(P', V_{r,r+2}) \;\; &\le \;\; |V_r| + \frac{7}{3}|V_{r+1}| + |V_{r+2}|, \\ f(Q', V_{r,r+2}) \;\; &\le \;\; 3|V_r| + \frac{5}{3}|V_{r+1}| + 3|V_{r+2}|, \end{aligned}$$

irrespective of the choice of a_{s+1}. Adding these two inequalities yields

$$f(P', V_{r,r+2}) + f(Q', V_{r,r+2}) \leq 4|V_r| + 4|V_{r+1}| + 4|V_{r+2}|,$$

and so $f(P', V_{r,r+2}) \leq 2|V_{r,r+2}|$ for all $a_{s+1} \in V_{s+1}$ or $f(Q', V_{r,r+2}) \leq 2|V_{r,r+2}|$ for all $a_{s+1} \in V_{s+1}$. Hence at least one of $P_{r,s}$ and $Q_{r,s}$ satisfies also property (ii).

TYPE 4: In this case $r = s = d > r$ Choose a vertex $\alpha_r \in V_r$ arbitrarily and set $P = \alpha_r$. Since $\chi_{r-1} = 1$, we must have β_{r-1} adjacent to α_r and therefore $g(\beta_{r-1}P, V_r) \leq f(\beta_{r-1}P, V_r) = 0$.

□

We remark that there appears to be no straightforward generalisation of the proof of Theorem 1 to $2k$-colourable graphs. However, we expect that the methods presented in this proof points a way towards a possible proof of such a generalisation.

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