Biplanar crossing numbers II: comparing crossing numbers and biplanar crossing numbers using the probabilistic method

Éva Czabarka
Department of Mathematics, College of William & Mary
Williamsburg, VA 23187, USA

$\boxed{\text{Ondrej Sýkora}}$[*]
Department of Computer Science, Loughborough University.
Loughborough, Leicestershire LE11 3TU, The United Kingdom

László A. Székely[†]
Department of Mathematics, University of South Carolina
Columbia, SC 29208, USA

Imrich Vrťo[‡]
Department of Informatics, Institute of Mathematics
Slovak Academy of Sciences
Dúbravská 9, 841 04 Bratislava, Slovak Republic

[*]This researcher was supported in part by the EPSRC grants GR/R37395/01 and GR/S76694/01. These grants supported visits of all other authors at Loughborough University

[†]This researcher was supported in part by the NSF contracts 007 2187 and 030 2307.

[‡]This researcher was supported in part by VEGA grant 2/6089/26.

Abstract

The biplanar crossing number $cr_2(G)$ of a graph G is $\min\{cr(G_1)+cr(G_2)\}$, where cr is the planar crossing number and $G_1 \cup G_2 = G$. We show that $cr_2(G) \leq (3/8)cr(G)$. Using this result recursively, we bound the thickness by $\Theta(G) - 2 \leq K cr_2(G)^{.4057} \log_2 n$ with some constant K. A partition realizing this bound for the thickness can be obtained by a polynomial time randomized algorithm. We show that for any size exceeding a certain threshold, there exists a graph G of this size, which simultaneously has the following properties: $cr(G)$ is roughly as large as it can be for any graph of that size, and $cr_2(G)$ is as small as it can be for any graph of that size. The existence is shown using the probabilistic method.

We dedicate this paper to our late colleague and friend, Ondrej Sýkora.

1 Introduction

This paper is a sequel to our earlier research on biplanar drawings [20] and biplanar crossing numbers [5]. Motivation for this research came from Beineke's study of biplanar drawings of graphs [4], Owens's study of the biplanar crossing number of K_n [12], from the theory of thickness (see the survey [15]) and the theory of crossing numbers (see the surveys [17, 22]).

Recall that a graph G is *biplanar*, if one can write $G = G_1 \cup G_2$, where G_1 and G_2 are planar graphs with the same vertex set as G, i.e. for the *thickness* of G, $\Theta(G)$, we have $\Theta(G) \leq 2$. Although planarity can be tested in polynomial time, testing biplanarity is NP-complete [14].

Owens [12] introcuced the *biplanar crossing number* of a graph G, that we denote by $cr_2(G)$. By definition $cr_2(G) = \min_{G_1 \cup G_2 = G}\{cr(G_1) + cr(G_2)\}$, where cr is the planar crossing number. One can define $cr_k(G) = \min_{G_1 \cup G_2 \cup \ldots \cup G_k = G}\{cr(G_1) + cr(G_2) + \ldots + cr(G_k)\}$, [18] similarly for any $k \geq 2$, making G a union of k subgraphs; but perhaps $k = 2$ is more relevant for VLSI for the following reason: one always can realize $cr_2(G)$ by drawing the edges of G_1 and G_2 on two different sides of the same plane, while identical vertices of G_1 and G_2 are placed to identical locations on the two sides of the plane.

Little is known about the biplanar crossing number in general. Some of the lower bounds for crossing numbers, *mutatis mutandis* apply to biplanar crossing numbers. Here and later $n = n(G)$ is the order and $m = m(G)$ is the size of the graph G. The lower bounds resulting from Euler's formula are

$$cr_2(G) \geq m - 6n + 12 \tag{1}$$

(for $n \geq 3$, however a slightly weaker version of (1), $cr_2(G) \geq m - 6n$ holds for all n); and a stronger version of (1) for graphs G with girth g

$$cr_2(G) \geq m - 2 \cdot \frac{g}{g-2} \cdot (n-2). \tag{2}$$

Formulae (1) and (2) follow easily by combining Theorem 2.1 in [4] with the arguments in [17]. Similarly, using (1) instead of (1) from [17] in the second proof of Theorem 3.2 in [17], one obtains the following biplanar counterpart of the Leighton [10] and Ajtai et al. [1] bound: for all $c > 6$, if $m \geq cn$, then

$$cr_2(G) \geq \frac{c-6}{c^3} \cdot \frac{m^3}{n^2}. \tag{3}$$

Lower bounds for the crossing number based on the *counting method* [17] generalize to similar arguments setting lower bounds for the biplanar crossing number.

However, important techniques as the *embedding method* [10] or the *bisection width method* [10], [16], [21] (see also the survey [17]) do not seem to generalize to biplanar crossing numbers. Even worse, as Tutte noted [4], the biplanar crossing number is not an invariant for homeomorphic graphs; in fact, the edges of every graph can be subdivided such that the subdivided graph is biplanar!

The only other study on biplanar crossing numbers that we are aware of is Spencer's result that proved our conjecture: $cr_2(G)$ for a random graph G with edge-probability $p > c_0/n$ is at least $c_1(n^2p)^2$ [19].

The present paper is the first attempt to establish general and non-trivial bounds on the biplanar crossing number.

2 The Main Results

The first natural problem is that of comparing $cr(G)$ and $cr_2(G)$.

Theorem 1. *For all finite simple graphs G, $cr_2(G) \leq \frac{3}{8} cr(G)$.*

Unfortunately, not any kind of converse of Theorem 1 can be true, as the following theorem shows:

Theorem 2. *There are numbers $c_1, c_2 > 0$, k_1 and n_1, such that for all positive integer $n \geq n_1$ and $m \geq k_1 n$, there exists a graph G of order n and size m, with crossing number*

$$cr(G) \geq c_1 m^2, \tag{4}$$

and with biplanar crossing number

$$cr_2(G) \leq c_2 m^3/n^2. \tag{5}$$

Since for any graph G we have $cr(G) = O(m^2)$, and by (3) $cr_2(G) = \Omega(m^3/n^2)$, whenever $m/n > 6$, the theorem above shows the existence of graphs with prescribed size, with roughly as large crossing number as it can be for any graph, and with roughly as small biplanar crossing number as it can be for any graph of this size.

Open Problem 1. *What is the smallest c^* of those constants c, for which $cr_2(G) \leq c \cdot cr(G)$ holds for every graph G?*

Owens [12] came up with a conjectured cr_2-optimal drawing of K_n which has about 7/24 of the crossings of a conjectured cr-optimal drawing of K_n. This might give some basis to conjecture that $c^* \leq 7/24$. On the other hand, $cr_2(K_9) = 1$ ([3] or [11] p. 34) and $cr(K_9) = 36$ [11] proves $c^* \geq 1/36$.

With a refined argument, we showed in (19) in [5] that $cr_2(K_n) \geq n^4/952$ for large n, and comparison with $cr(K_n) \leq n^4/64$ [24] proves $c^* \geq 64/952$.

We give here the proof of Theorem 1, since we already need it for the exposition of the next result. We provide a randomized algorithm which proves that $cr_2(G) \leq (3/8)cr(G)$ for any finite simple graph G. Without loss of generality we may assume that the input drawing is *nice* [22, 24], i.e. any two edges of G cross at most once, edges do not "touch", and edges sharing an endvertex do not cross; since all these assumptions do not change the crossing number. In the proofs of Theorems 1 and 3 the computational complexity is estimated for this kind of restricted input, using a table look-up which tells if two edges do or do not cross.

`Splitting Algorithm.` `INPUT` any nice drawing D of G in the plane. Let $cr(D)$ denote the number of crossings in this drawing.

Consider a random bipartition (U, W) of $V(G)$: for every vertex, independently toss a fair coin, and if Head is obtained, add it to U, otherwise to V. Now any crossing in D occurs in 6 possible forms, according to which classes the endpoints of the crossing edges belong to:

it is a crossings of UU,UU edges with probability 1/16

it is a crossings of WW,WW edges with probability 1/16

it is a crossings of UW,UW edges with probability 1/4

it is a crossings of UU,WW edges with probability 1/8

it is a crossings of UU,UW edges with probability 1/4

it is a crossings of WW,UW edges with probability 1/4

Draw in the first plane the subdrawings spanned by U and spanned by W, draw in the second plane the subdrawing of edges connecting U to W. In the second plane we have the UW,UW type crossings, in expectation $\frac{1}{4}cr(D)$. In the first plane, we have the UU,UU and WW,WW type crossings, and also the UU,WW type crossings. However, we easily get rid of the latter type of crossing, by a translation of the W point set and its induced edges to sufficiently far away. Therefore, the first plane has in expectation $\frac{1}{8}cr(D)$

crossings after the translation.

The randomized algorithm above can be derandomized by routine arguments. However, if we want to keep the number of crossings on *both* planes near the respective expected value, the standard derandomization techniques fail. Therefore, it is hard to tell what happens if we try to iterate the algorithm above.

Next we make a refined analysis of the iteration of the randomized algorithm above. The goal is to show, that if a graph can be drawn with few crossings, then it has small thickness.

Theorem 3. *For all $\gamma > \frac{\ln x_0}{\ln 2} \approx .4057$, where x_0 is the real root of $x^3 = x+1$, there exists a c_γ constant, such that*

$$\Theta(G) - 1 \leq c_\gamma cr(G)^\gamma \log_2 n, \tag{6}$$

$$\Theta(G) - 2 \leq c_\gamma cr_2(G)^\gamma \log_2 n. \tag{7}$$

Furthermore, for every $\delta > 0$ and $\gamma > \frac{\ln x_0}{\ln 2}$, there exists a randomized algorithm of the following description. The input of the algorithm is a table. This table tells which edges cross in a nice drawing D of G, which has $cr(D)$ ($cr_2(D)$) crossings. The algorithm outputs with probability at least $1-\delta$ a decomposition of the graph G into $1 + c_\gamma cr(D)^\gamma \log_2 n$ ($2 + c_\gamma cr(D)^\gamma \log_2 n$) planar graphs. The running time of the algorithm is bounded by a polynomial in the variables $\ln \frac{1}{\delta}$ and $n(G)$.

Theorem 3 can be interpreted in two ways. In one way, it is the first nontrivial, structural lower bound for $cr_2(G)$, in terms of other graph parameters. Unfortunately, it rarely happens that the thickness of a graph G is known.

In the other way, the theorem is interpreted as estimating the thickness in terms of other graph parameters. There are some results in this direction: Halton [9] proved $\Theta(G) \leq \lceil \frac{\Delta}{2} \rceil$, and Dean, Hutchinson, and Scheinerman [7] proved $\Theta(G) \leq \lfloor \sqrt{\frac{m}{3}} + \frac{7}{6} \rfloor$, where Δ is the maximum degree and m is the number of edges. Malitz [13] proved $\Theta(G) = O(\sqrt{g})$, where g is the genus of the graph G. These results are not directly comparable to Theorem 3.

Improvement on Theorem 1 would yield improvement on Theorem 3, but we were unable give Theorem 3 in an "abstract" way, since for example, an improvement may involve drawings on 3 planes.

Open Problem 2. *Derandomize the algorithm in the proof of Theorem 3.*

Open Problem 3. *What is the smallest value γ^* with which (6), (7) hold for all graphs?*

It is clear from the example of the complete graph that $\gamma^* \geq 1/4$, since there is a closed form for the thickness of the complete graph [24], and it is linear in n; while $cr(K_n)$ and $cr_2(K_n)$ are at least cn^4 with some constant c.

3 Notations and Graph Definitions

In this section we introduce some notations and graphs that we will use in our proofs.

Let $G = (V, E)$ be a graph. For $X, Y \subset V$ we will denote the set of edges between X and Y by $E(X, Y)$, i.e.

$$E(X, Y) = \{\{x, y\} : x \in X, y \in Y, \{x, y\} \in E\}.$$

Note that if $Y = V - X$, $E(X, Y)$ is the edge set of a cut.

Let G_1 and G_2 be two graphs on the same vertex set of order n, i.e. $V = V(G_1) = V(G_2) = \{1, 2, \ldots, n\}$. We define the random graph $\Gamma(G_1, G_2)$ as

$$\Gamma[G_1, G_2] = G_1 \cup G_2^\pi,$$

where π is a random permutation of V, and G_2^π denotes the image of G_2 under the action of π, i.e. $\pi(i)$ and $\pi(j)$ are joined in G_2^π iff i and j are joined in G_2. Clearly, $\Gamma[G_1, G_2]$ is a union of two edge sets over the same vertex set (without multiple edges).

Also, for any graph G and integers s and a, let $c(a, s, G)$ denote the number of ordered $(a, n-a)$-cuts of G with at most s edges, i.e.

$$c(a, s, G) = |\{A \subset V(G) : |A| = a, |E(A, V(G) - A)| \le s\}|. \tag{8}$$

Let n and k be a positive integers, $k \le n$. The graph $H(n, k)$ is defined as follows: For the set of vertices of $H(n, k)$, take $\{1, 2, \ldots, n\}$. Join vertex x with vertex y if they are at most $k-1$ apart , i.e if $|x - y| < k$.

We will also define k subgraphs $H_1, H_2, \ldots, H_k$ of $H(n, k)$ in the following way (see Fig. 1): The vertex set of each graph is the same as the vertex set of $H(n, k)$. The edge set of H_i consists of the edges going 'upwards' from some special points of the form $i + zk$ for some $z \ge 0$ integer and $i + zk \le n$ (we call these points *centers* in H_i), i.e.

$$E(H_i) \;=\; \Big\{\{x, y\} \in E(G) : x < y, x = i + zk \text{ or some integer } z\Big\}.$$

The following statements are clear from the construction:

Lemma 1.
- *The subgraphs $H_1, H_2, ..., H_k$ are edge-disjoint graphs that partition the edge set of $H(n, k)$;*
- *every subgraph H_i is a vertex disjoint union of w_i stars (a star is considered present when its center is present), and*
$$\frac{n}{k} - 1 \le w_i \le \frac{n}{k} + 1, \tag{9}$$
- *the centers of the stars have degree at most $k-1$,*
- $$(n-k)(k-1) < m(H(n, k)) < n(k-1). \tag{10}$$

4 The Proof of Theorem 2

If, say, $m \ge n^2/20$, then conclusions (4) and (5) hold for *all* graphs of order n and size m. First, we prove the following lemma, which is a weaker version of Theorem 2:

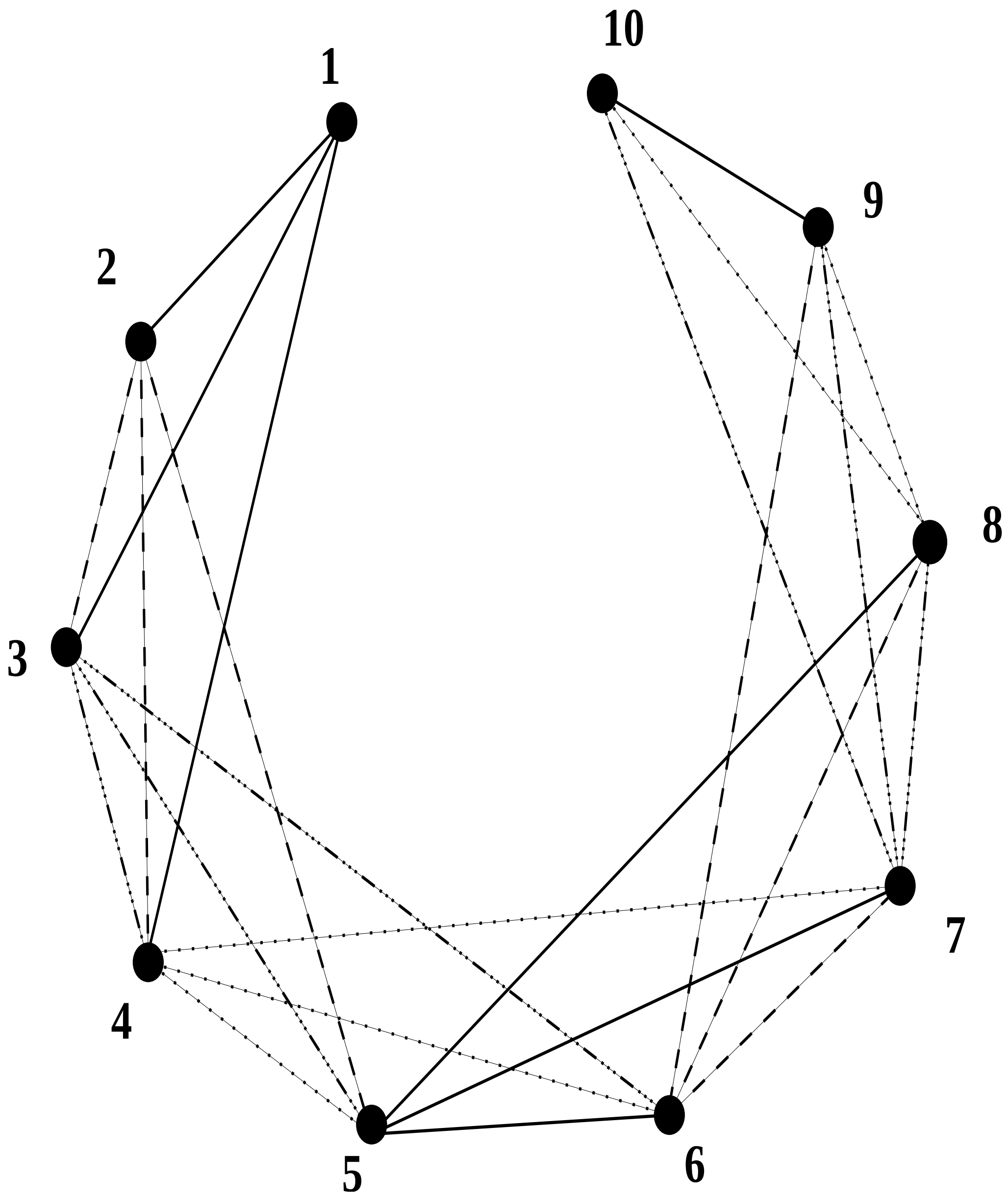

Figure 1: Circular drawing for the graph $H(10, 4)$. Edges of the subgraphs H_i ($i = 1, 2, 3, 4$) are drawn in different ways.

Lemma 2. *There exists a k_0 and an n_0, such that for all $n \geq n_0$ and all $k_0 \leq k \leq n/3$ integers, the graph $G^* = \Gamma[H(n,k), H(n,k)]$ satisfies the conclusion of the theorem with probability $1 - o(1)$, where $o(1)$is for $n \to \infty$, independent of k.*

For shortness, we denote $H(n,k)$ by H. Lemma 2 will be proved through a series of lemmas using the following obvious facts:

$$m(H) \leq m(G^*) \leq 2m(H) \tag{11}$$
$$cr_2(G^*) \leq 2cr(H). \tag{12}$$

(11) and (12) together show that $cr(H) \leq c_3 m(H)^3/n^2)$ implies $cr_2(G^*) \leq c_2(m(G^*)^3/n^2$; $cr(H) \leq c_3 m(H)^3/n^2$ will be shown in Lemma 7. $cr(G^*) \geq c_1 m(G^*)^2$ will be shown in Lemma 11, and that will finish the proof of Lemma 2.

We begin with recalling Markov's inequality:

Lemma 3. *For a nonnegative random variable X for every $\epsilon > 0$ we have*

$$\mathbb{P}\Big[X > (1+\epsilon)\mathbb{E}[X]\Big] < \frac{1}{1+\epsilon}, \tag{13}$$

and therefore we have

$$\mathbb{P}\Big[X \leq (1+\epsilon)\mathbb{E}[X]\Big] \geq \frac{\epsilon}{1+\epsilon}. \tag{14}$$

Lemma 4. *For arbitrary $1/3 \leq c \leq 2/3$, if n is large enough, and cn is an integer, then*

$$\frac{1}{n}\binom{n}{cn} > 1.5^n. \tag{15}$$

Proof. Recall the following well-known consequence of Stirling's formula: for any constant $c : 0 < c < 1$ one has

$$\binom{n}{cn}^{\frac{1}{n}} = H(c) + o(1) \tag{16}$$

$$H(c) = c^{-c}(1-c)^{-(1-c)}. \tag{17}$$

Using Robbins' formula instead of Stirling's, we see that the $o(1)$ term is uniform in $1/3 \le c \le 2/3$. When $1/3 \le c \le 2/3$, then the minimum value of $H(c)$ is achieved when $c = 1/3$ or $c = 2/3$, and this value is larger then 1.5; (15) follows from this. $\square$

Recall that the 1/3-2/3 *bisection width* of a graph G is the smallest possible size of an edge set of a cut $(A, V(G)-A)$, where both $|A|$ and $|V(G)-A|$ are required to be between $|V(G)|/3$ and $2|V(G)|/3$. We denote the 1/3-2/3 bisection width of G by $b(G)$.

We will need the bisection width lower bound for the $cr(G)$ that was shown in [16, 21]:

Lemma 5. *If $G = (V, E)$ and $d_i(G)$ denotes the degree sequence of the graph $(i \in V)$, then*

$$(1.58)^2\Big(16cr(G) + \sum_{i\in V} d_i^2(G)\Big) \ge b(G)^2. \tag{18}$$

The next statement provides us with a lower bound on the bisection width (see definition (8)!).

Lemma 6. *Let G_1 and G_2 be two graphs on the same vertex set of order n. If for some integer $s = s(n)$ for each $a : \frac{n}{3} \le a \le \frac{n}{2}$, we have*

$$c(a, s, G_1)c(a, s, G_2) \le \frac{g(n)}{n}\binom{n}{a}, \tag{19}$$

then $b(\Gamma[G_1, G_2]) \ge s$ with probability at least $1 - g(n)$.

Proof. For shortness, let us use $c_i(a) = c(a, s, G_i)$ and $G' = \Gamma[G_1, G_2]$. It is easy to see that (19) implies

$$\mathbb{P}[b(G') \le s] \le \sum_{a=n/3}^{n/2} c_1(a)\frac{c_2(a)a!(n-a)!}{n!} = \sum_{a=n/3}^{n/2} \frac{c_1(a)c_2(a)}{\binom{n}{a}} \le g(n), \tag{20}$$

where $\mathbb{P}$ in (20) is the probability arising from uniformly selected random permutations. $\square$

Now we are ready to estimate the crossing number of H:

Lemma 7. *For $k \geq 4$, $cr(H) \leq 48m(H)^3/n^2$.*

Proof. We are going to show that $cr(H) \leq 2nk^3$. If $k \leq n/2$, formula (10) of Lemma 1 will finish the proof; if $k \geq n/2$, then the conclusion is straightforward.

Draw the points of H on a circle in order, and draw the edges in straight lines (see Fig. 1). For each $i : 1 \leq i \leq k$ we have at most n edges of length i. Consider the $i-1$ points that a length i edge e covers in the natural way (i.e. the points between the endpoints of the edge in the cyclic order). All of them have at most $2(k-1)$ neighbors. Therefore vertices covered by e contribute at most $2(k-1)(i-1)$ to the crossings in the drawing. Hence,

$$cr(H) \leq \frac{1}{2}n\sum_{i=2}^{k} 2(k-1)(i-1) \leq 2nk^3.$$

$\square$

Lemma 8. *Fix an $0 < \epsilon < 1$, and fix an i and an a, and let $|A| = a$ be a fixed vertex set of H_i such that in H_i, $|E(A, V-A)| \leq \epsilon n$.*

1. *The average number of A, $V-A$ cut edges, computed over the w_i stars of H_i, is at most $1.5\epsilon k$.*
2. *Fix any $K > 3/2$, and call a star of H_i* rich, *if it has more than $K\epsilon k$ edges in the $E(A, V-A)$ cut. The number of rich stars is at most $\frac{3w_i}{2K}$.*

Proof. For part 1, average is less or equal $\epsilon n/w_i \leq \frac{\epsilon n}{n/k-1}$ by (9), which is $\frac{\epsilon k}{1-k/n} < 1.5\epsilon k$, as $k \leq n/3$.
For part 2, apply (13), such that the probability is uniform on the w_i stars, and X counts the $E(A, V-A)$ cut edges in the stars. We have seen in part 1 that $\mathbb{E}[X] \leq 1.5\epsilon k$; and (13) immediately implies part 2. $\square$

The following rather technical lemma will help us to apply of Lemma 6 for H.

Lemma 9. *There exists an $\epsilon > 0$, such that for all n large enough, for all a, i such that $1 \le i \le k$ and $n/3 \le a \le n/2$, we have that $c(a, \epsilon n, H_i) < 1.1^n$.*

Proof. Fix an arbitrary $\epsilon > 0$ and $K > 3/2$. We estimate in their terms the number of cuts in H_i, where one side has a vertices, and the cut has at most ϵn edges. The main tool for the estimate is Lemma 8. Finally, we will assign a value to ϵ. We are going to use the following facts:

- there are 2^{w_i} placements of the midpoints of the stars of H_i to the two sides of the partition, A and $V \setminus A$;
- there are at most $\sum_{j=0}^{\frac{3w_i}{2K}} \binom{w_i}{j}$ ways to select the rich stars of H_i, since the number of such rich stars is at most $\frac{3w_i}{2K}$;
- there are at most $\sum_{t=0}^{\lfloor K\epsilon k \rfloor} \binom{k-1}{t}$ ways for cutting a star that is not a rich star (we need to decide which of the at most $\lfloor K\epsilon k \rfloor$ cut edges belong to the side of the center in the cut); and there is a total of w_i possible stars;
- in the rich stars the cut can go at most 2^{k-1} ways, and as mentioned, there are at most $\frac{3w_i}{2K}$ such stars.

By all the above, we have that

$$c(a, \epsilon n, H_i) \le 2^{w_i} \left[\sum_{j=0}^{\frac{3w_i}{2K}} \binom{w_i}{j} \right] \times \left[\sum_{t=0}^{\lfloor K\epsilon k \rfloor} \binom{k-1}{t} \right]^{w_i} \times (2^{k-1})^{\frac{3w_i}{2K}}. \tag{21}$$

We will bound the right side of (21) term-by term by $1.1^{n/3}$, as we make the appropriate choices for K and ϵ.

The first factor is at most $2^{w_i} 2^{w_i} \le 2^{2(n/k+1)} = 2^{2+2n/k}$ using (9), and this is $< 1.1^{n/3}$ if and only if $2^{\frac{6}{n}+\frac{6}{k}} < 1.1$. This is simply achieved by selecting a large enough k_0 and n.

To estimate the third factor, use again (9): $(2^{k-1})^{\frac{3w_i}{2K}} \leq 2^{(k-1)\frac{3}{2K}(n/k+1)}$, and the last term $< 1.1^{n/3}$ if and only if $2^{(k-1)\frac{9}{2K}(1/k+1/n)} \leq 1.1$. This can be achieved by selecting K sufficiently large, say $K = 100$.

For the second factor, we recall a well-known inequality: For $4b \leq N$

$$\sum_{l=0}^{b} \binom{N}{l} \leq 2\binom{N}{b}. \tag{22}$$

Given our choice of K, we want to select ϵ so small that

$$\left[\sum_{t=0}^{\lfloor K\epsilon k\rfloor} \binom{k-1}{t}\right]^{\frac{w_i}{n}} < 1.1^{1/3}. \tag{23}$$

We require $\epsilon \leq \frac{1}{4K}$ so that we can estimate the LHS of (23) with (22). We use the estimate $w_i/n \leq 1/k + 1/n$ from (9) to set a sufficient condition for (23):

$$\left[2\binom{k-1}{\lfloor K\epsilon k\rfloor}\right]^{3/k+3/n} < 1.1. \tag{24}$$

Observe that for fixed $K\epsilon < 1/4$, $\lim_{k\to\infty} \binom{k-1}{\lfloor K\epsilon k\rfloor}^{1/k} = H(\epsilon K)$ (see formulas (16) and (17)). As $\lim_{\epsilon\to 0} H(\epsilon K) = 1$, we can set a sufficently small ϵ and sufficiently large k_0 and n_0, such that all $k \geq k_0$ and $n \geq n_0$ satisfy (24). This finishes the proof of the lemma. $\square$

We are now in a position to prove

Lemma 10. *Select $\epsilon > 0$ according to Lemma 9. $b\Big(G^*\Big) \geq \epsilon nk$ with probability at least $1 - (\frac{1.21}{1.5})^n k^2$.*

Proof. Let $s = \epsilon nk$. Fix an a from the range $n/3 \leq a \leq n/2$, and assume that $(A, V \setminus A)$ is an $a, n-a$-cut of H with at most s edges. Then, there must be an i, such that $(A, V \setminus A)$ is a cut of H_i with at most $\frac{s}{k} = \epsilon n$ edges.

Since by Lemma 9 the number of such cuts of H_i is less than 1.1^n and there are $k \leq n$ choices for i, we have that

$$c^2(a,s,H) < \left(k \cdot 1.1^n\right)^2 \leq \left(1.21k^{2/n}\right)^n < \left(\frac{1.21k^{2/n}}{1.5}\right)^n \frac{1}{n}\binom{n}{a}.$$

This together with Lemmas 4 and 6 gives the required result. ☐

Lemma 11. *$cr\Big(G^*\Big) \geq c_1 m(G^*)^2$ with probability $1 - o(1)$.*

Proof. We require that $n_0 \geq 32(1.58)^2/\epsilon^2$. With this choice, the lower bound for the 1/3-2/3 bisection width of G^*, ϵnk, plugged into Lemma 5, sets an $c_1 m(G^*)^2$ lower bound for $cr(G^*)$, as in G^* every degree is at most $4k$. ☐

With this, we finished the proof of Lemma 2.

Open Problem 4. *We showed in Lemma 10 that $b(G^*)$ is large for a particular graph sequence, $G_1 = G_2 = H(n,k)$. Would this hold (perhaps with mild additional conditions) for all graphs ?*

Finally, we prove Theorem 2. Set $k_1 = 3k_0$, where k_0 is the constant in Lemma 2. Let $m \geq k_1 n$ denote the target edge number. According to the remark above Lemma 2, we may assume $m \leq n^2/20$. Let k denote the largest integer with $2nk < m$. Clearly $n/10 \geq k > k_0$, and we can use Lemma 2 for $\Gamma[H(n,k), H(n,k)]$. In particular, there is a permutation π of n elements, such that the crossing number of $G^* = H(n,k) \cup [H(n,k)]^\pi$ is large. Now think about the graph $H(n,k)$ as a subgraph embedded into $H(n,3k)$, and consider $G^{***} = H(n,3k) \cup [H(n,3k)]^\pi$ with the *very same* π that gave G^*. According to (10) and (11), $m(G^*) < 2n(k-1) < m$, and $(n-3k)(3k-1) < m(G^{***})$. By the choice of k, $m \leq 2n(k+1) \leq (n-3k)(3k-1) < m(G^{***})$, as $n/10 \geq k$. Observe that G^* is sugraph of G^{***}, and therefore there exists a graph G^{**} with exactly m edges such that $G^* \subset G^{**} \subset G^{***}$. Within a constant multiplicative factor, all these 3 graphs have the same number of edges, $cr(G^{**}) \geq cr(G^*)$, and latter crossing number was big according to Lemma 2, and finally $cr_2(G^{**}) \leq cr_2(G^{***})$, and latter crossing number is small according to Lemma 7.

5 Towards the proof of Theorem 3

Let us be given a nice drawing D' of a graph G' without isolated vertices. Assume that we apply the randomized algorithm that we used in the proof of Theorem 1 to D' to obtain a biplanar drawing on two planes. For e, f edges of G', let $X_{e,f}$ (resp. $Y_{e,f}$) denote the indicator variable that in the random drawing edges e and f cross in the first (resp. second) plane. Set $X = \sum_{e,f} X_{e,f}$ and $Y = \sum_{e,f} Y_{e,f}$ (the summation goes for unordered pairs of edges). Note that if $X_{e,f} = 1$, then the four vertices of $e \cup f$ are all in the same partition of the random bipartition U, W; and if $Y_{e,f} = 1$, then both e and f connect a point of U to a point of W. Let $cr(D')$ denote the number of crossings in the drawing D'.

Our observations in the proof of Theorem 1 amount to

$$\mathbb{E}[X] = \frac{cr(D')}{8} \qquad \text{and} \qquad \mathbb{E}[Y] = \frac{cr(D')}{4}. \tag{25}$$

Our first goal is to study the variance of Y.

Lemma 12.

$$\sigma^2[Y] \leq cr(D')(2m(D') + 12n(D') + 3), \tag{26}$$

where $n(D')$ and $m(D')$ denote the order and size of G', respectively.

Proof. We have

$$\sigma^2[Y] = \mathbb{E}[Y^2] - \mathbb{E}^2[Y] = \sum_{\{e,f\}} \sum_{\{a,b\}} \mathbb{E}[Y_{e,f} Y_{a,b}] - \mathbb{E}[Y_{e,f}]\mathbb{E}[Y_{a,b}]. \tag{27}$$

We think about edges as 2-element sets of vertices. Since D' is nice, $Y_{e,f} \neq 0$ implies that $|e \cup f| = 4$. Observe that if the vertex sets $a \cup b$ and $e \cup f$ are disjoint, then by independence the contribution of the $\{e, f\}$, $\{a, b\}$ terms is zero to (27). We make the following case analysis:

(i) $|\{a \cup b\} \cap \{e \cup f\}| = 1$

(ii) $|\{a \cup b\} \cap \{e \cup f\}| = 2$ and $\{a \cup b\} \cap \{e \cup f\} \notin \{a, b, e, f\}$

(iii) $|\{a \cup b\} \cap \{e \cup f\}| = 2$ and $\{a \cup b\} \cap \{e \cup f\} \in \{a, b\} \cap \{e, f\} \neq \emptyset$

(iv) $|\{a \cup b\} \cap \{e \cup f\}| = 2$ and $\{a \cup b\} \cap \{e \cup f\} \in \{a, b\}$,
but $\{a \cup b\} \cap \{e \cup f\} \notin \{e, f\}$ (or vice versa)

(v) $|\{a \cup b\} \cap \{e \cup f\}| = 3$ and $|\{a, b\} \cap \{e, f\}| = 1$

(vi) $|\{a \cup b\} \cap \{e \cup f\}| = 3$ and $\{a, b\} \cap \{e, f\} = \emptyset$

(vii) $|\{a \cup b\} \cap \{e \cup f\}| = 4$ but $\{a, b\} \neq \{e, f\}$

(viii) $\{a, b\} = \{e, f\}$

(see Fig. 2).

Simple calculations show that in cases (i), (ii), (iv) $cov(Y_{e,f}, Y_{a,b}) = 0$. Otherwise $cov(Y_{e,f}, Y_{a,b}) \leq 1$, so $\sigma^2(Y)$ is bounded above by the number of ordered pairs of unordered edge-pairs $(\{e, f\}, \{a, b\})$ that are in one of the configurations covered by (iii) and (v)-(viii). Since $\{e, f\}$ can be chosen in at most $cr(D')$ ways, it is enough to bound the number of ways $\{a, b\}$ may be chosen once $\{e, f\}$ is fixed. In particular, once $\{e, f\}$ is given, $\{a, b\}$ is fixed in configuration (viii), and there are only 2 ways $\{a, b\}$ can be chosen for configuration (vii), so there are at most $3cr(D')$ pairs in configurations (vii) and (viii).

In configurations (v) and (vi), $|(a \cup b) - (e \cup f)| = 1$, therefore, if $\{e, f\}$ is already given, there are no more than $n(D')$ ways to choose $(a \cup b) - (e \cup f)$. Once both $\{e, f\}$ and the vertex in $(a \cup b) - (e \cup f)$ is chosen, there is at most 4 ways to choose $\{a, b\}$ for configuration (v) and at most $4 \times 2 = 8$ ways to choose it for (vi), therefore we have at most $12n(D')cr(D')$ pairs in configurations (v) and (vi).

To estimate the number of pairs in configuration (iii), let $cr(e)$ denote the number of crossings of edge e in D'. Since D' is a nice drawing, $cr(e), cr(f) \leq m(D')$, therefore if $\{e, f\}$ is fixed, the number of ways to choose $\{a, b\}$ is at most $2m(D')$, so the number of pairs in configuration (iii) is at most $2cr(D')m(D')$.

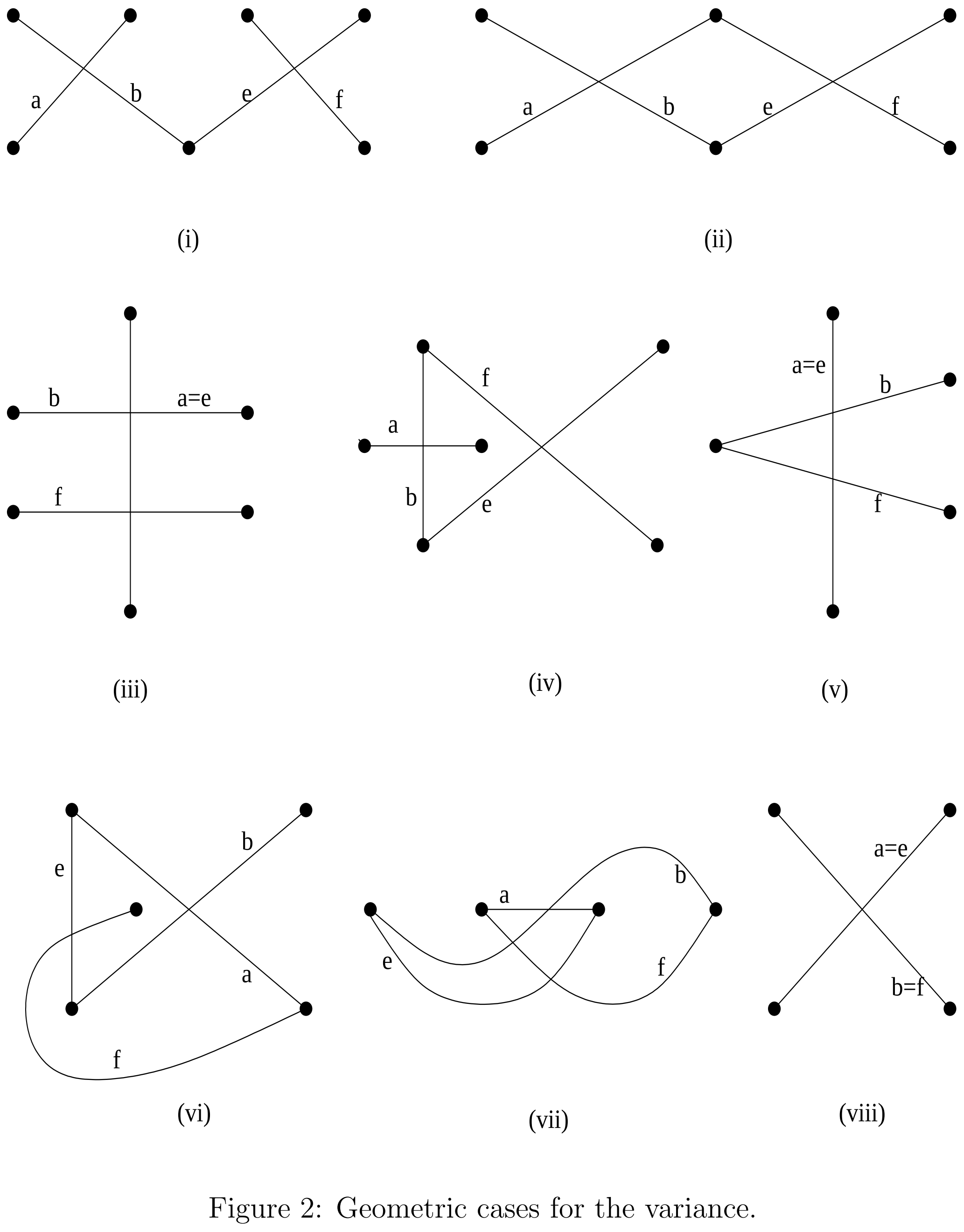

Figure 2: Geometric cases for the variance.

Combining all these results yields

$$\sigma^2[Y] \leq cr(D')(2m(D') + 12n(D) + 3).$$

$\square$

Next we examine the probability that Y is much larger that its expectation, if the nice drawing D' has many crossings:

Lemma 13. *Let $\epsilon > 0$. If $cr(D') > K(2m(D') + 12n(D') + 3)$ then*

$$\mathbb{P}\Big[Y > (1+\epsilon)\mathbb{E}(Y))\Big] < \frac{16}{K\epsilon^2} \tag{28}$$

Proof. Using Chebyshev's inequality, Lemma 12 and equation (25), we get that

$$\begin{aligned} \mathbb{P}\Big[Y > (1+\epsilon)\mathbb{E}(Y)\Big] &\leq \frac{\sigma^2(Y)}{\epsilon^2\mathbb{E}^2(Y)} \\ &\leq \frac{16(2m(D') + 12n(D') + 3)}{\epsilon^2 cr(D')}, \end{aligned} \tag{29}$$

from which the statement follows. $\square$

Now we are ready to show that when $cr(D')$ is large, the probability that *both* X and Y stay below $(1+\epsilon)$ times their respective expectations is bounded away from 0:

Lemma 14. *Let $\epsilon > 0$. If there is a $K > \frac{16(1+\epsilon)}{\epsilon^2}$ such that $cr(D') > K(2m(D') + 12n(D') + 3)$, then*

$$\mathbb{P}\Big[X \leq (1+\epsilon)\mathbb{E}(X) \text{ and } Y \leq (1+\epsilon)\mathbb{E}(Y)\Big] \geq r(\epsilon, K) > 0, \tag{30}$$

where $r(\epsilon, K) = \frac{\epsilon}{1+\epsilon} - \frac{16}{K\epsilon^2}$. In particular, if $\epsilon \leq 0.34$, and $K = 100/\epsilon^3$, we have

$$r\left(\epsilon, \frac{100}{\epsilon^3}\right) \geq \frac{\epsilon}{2}. \tag{31}$$

Proof. Since

$$\mathbb{P}\Big[X \le (1+\epsilon)\mathbb{E}(X) \text{ and } Y \le (1+\epsilon)\mathbb{E}(Y)\Big] \ \ge \\ \mathbb{P}\Big[X \le (1+\epsilon)\mathbb{E}(X)\Big] - \mathbb{P}\Big[Y > (1+\epsilon)\mathbb{E}(Y)\Big],$$

equation (30) follows directly from Markov's inequality for X (Lemma 3, equation (14)) and Lemma 13. The remaining part is straightforward. □

Next we describe and analyze a procedure that we will recursively use in the algorithm for (6) in Theorem 3:

`PROCEDURE`

`Input`: An ϵ with $.34 > \epsilon > 0$, a non-negative integer N and a nice one-plane drawing D of order $n(D)$ and size $m(D)$, with $cr(D) > 0$ crossings, without isolated vertices.
`Output`: Either FAIL, or a partitioning of all edges of D into at most $\frac{23200}{\epsilon^3}\log_2 n$ planar drawings (i.e. without crossings) and two other drawings D_1 and D_2, such that $cr(D_1) < (\frac{1}{8}+\epsilon)cr(D)$ and $cr(D_2) < (\frac{1}{4}+\epsilon)cr(D)$. D_1 and D_2 have no isolated vertices.

Case 1: `IF` $cr(D) > \frac{100}{\epsilon^3}(2m(D) + 12n(D) + 3)$, `THEN` make (at most) N runs of the `Splitting Algorithm`.

`IF` a drawing on two planes is achieved that satisfy the requirements for D_1 and D_2, then remove their isolated vertices and output two new drawings D_1 and D_2 using two new planes, such that $cr(D_1) < (\frac{1}{8}+\epsilon)cr(D)$ and $cr(D_2) < (\frac{1}{4}+\epsilon)cr(D)$.

`OTHERWISE` output `FAIL`

`END PROCEDURE`

Case 2: `IF` $cr(D) \le \frac{100}{\epsilon^3}(2m(D)+12n(D)+3)$, `THEN` introduce $\frac{11600}{\epsilon^3}$ new planes, copy the vertices of D to each, and then use the greedy algorithm to move as many edges of D as possible to the new planes, so that no crossings on the new planes arise. When the greedy algorithm stops, eliminate the isolated vertices from the rest of D, and call the leftover drawing D'. Output the planar drawings. Run the `PROCEDURE` on D'.

Lemma 15. *In every application of Case 2, D' inherits at most half of the edges of D, and consequently* `PROCEDURE` *executes Case 2 at most $2\log_2 n$ times. Moreover, as $\epsilon \leq 0.34$, the probability that* `PROCEDURE` *results in a* `FAIL` *is at most $(1-\frac{\epsilon}{2})^N$.*

Proof. We are going to use Markov's inequality. Consider a probability space, whose elements are are the edges of D, each with probability $1/m(D)$. Consider the function $cr(e)$ which assigns to every edge e in D the number of crossings that this edge makes, as a random variable. Its expectation is $2cr(D)/m(D)$, which is, by the definition of Case 2, $\leq \frac{100}{\epsilon^3}(4+24\frac{n(D)}{m(D)}+6)$, and that, since D has no isolated vertices and therefore $m(D) \geq n(D)/2$, is bounded by $\frac{5800}{\epsilon^3}$. After the stopping of the greedy algorithm, only such edges may remain in D' which crossed at least $\frac{11600}{\epsilon^3}$ other edges in D. Using (14) (with 1 instead of ϵ in it), we obtain $\mathbb{P}[cr(e) \leq \frac{11600}{\epsilon^3}] \geq \mathbb{P}[cr(e) \leq 2\frac{2cr(D)}{m(D)}] \geq \frac{1}{2}$. Hence, at most half of the edges stay after the use of the greedy algorithm. This sets a limit of $\log_2 m(D) < 2\log_2 n$ for the number of consecutive runs of the greedy algorithm.

By Lemma 14, if $\epsilon \leq 0.34$ then a single run of the `Splitting Algorithm` results in failure with probability at most $1-\epsilon/2$. Therefore the probability that all of N independent trials result in `FAIL` in the Case 1 step of the `PROCEDURE` is at most $(1-\epsilon/2)^N$. □

6 Proof to Theorem 3

Proof. To prove Theorem 3, let us be given an arbitrary $\gamma > \frac{\ln x_0}{\ln 2}$ and $\delta > 0$ (recall that x_0 is the positive real root of $x^3 = x+1$), and also a graph G with a nice drawing D in one (two) planes that we have to partition into planar graphs. We will choose an appropriate ϵ (depending on γ), and a positive integer N (which will depend on ϵ, and on n, the number of vertices of the graph G), as specified below.

Take an α such that $x_0 < \alpha < \gamma$, an α is so close to $\frac{\ln x_0}{\ln 2}$, that the $\epsilon = \epsilon(\alpha)$

positive solution of

$$1 = \left(\frac{1}{4} + \epsilon\right)^{\alpha} + \left(\frac{1}{8} + \epsilon\right)^{\alpha} \tag{32}$$

is less than .34. (This can be achieved by continuity arguments, as $\epsilon \to 0$ in equation (32) implies the corresponding α exponent decreasing to $\frac{\ln x_0}{\ln 2}$. Note that 2^{x_0} is the solution of the equation $1 = (\frac{1}{4})^{x_0} + (\frac{1}{8})^{x_0}$.) Set $N = \frac{2}{\epsilon} \ln \frac{1}{\delta} + \frac{2}{\epsilon} \ln(n^2)$.

The main algorithm that obtains a partition of G into $c_\gamma cr(D)^\gamma \log_2 n$ plus 1 (plus 2) planar graphs from a planar (biplanar) drawing of G, would output the input drawing, if it has no crossings, otherwise it will be the simultaneous recursive call of the `PROCEDURE` described in the previous section for every current drawing which has at least one crossing, starting with the drawing of D. (You may keep the original N or may redefine it with the decreasing n.)

First we show that the main algorithm partitions G into planar drawings with probability at least $1 - \delta$. A Case 1 step of `PROCEDURE` yields `FAIL` with probability $(1 - \epsilon/2)^N \leq e^{-\epsilon N/2}$. For an n-vertex graph, clearly $\binom{n}{2}$ is an upper bound for getting into Case 1 in the algorithm. Therefore the probability that we get `FAIL` is $< \binom{n}{2} e^{-\epsilon N/2} < \delta$ with our choice for ϵ and N. If we never get `FAIL` in the algorithm, then we end the main algorithm when there are no more crossings in the current graph drawing. We will call such runs of the algorithm *successful*.

Next, we are going to estimate the number of planar drawings obtained in a successful run. Let us denote by $f_\epsilon(c)$ the largest number of planes coming from the *first case* of the procedure for some initial drawing with at most c crossings, for all possible input drawings and successful runs. (This definition is *independent of* n, so one has to pause if $f_\epsilon(c)$ exists! But indeed $f_\epsilon(c) \leq c$.) The algorithm implies the recurrence relation

$$f_\epsilon(c) \leq f_\epsilon\left(\lfloor(\frac{1}{4} + \epsilon)c\rfloor\right) + f_\epsilon\left(\lfloor(\frac{1}{8} + \epsilon)c\rfloor\right). \tag{33}$$

It is easy to prove by induction from (33) that

$$f_\epsilon(c) \leq Kc^\alpha, \tag{34}$$

where α is the solution of the equation (32), and the constant K depends on the initial condition of the recurrence. By Lemma 15, the number of planes output by the whole algorithm is at most $\log_2 n$ times more than the number of planes output in a Case 1 step of the `PROCEDURE`. We verified the claim about the number of planes output by any successful run.

Finally, the polynomiality of the algorithm in n and $1/\delta$ follows from the polynomiality of N and the output size in these variables (as $cr(D) < n^4$), since every output plane takes polynomial time to compute.

□

References

[1] M. Ajtai, V. Chvátal, M. Newborn, E. Szemerédi, Crossing-free subgraphs, *Annals of Discrete Mathematics* **12** (1982), 9–12.

[2] N. Alon, J. H. Spencer, *The Probabilistic Method,* John Wiley and Sons, New York, 1992.

[3] J. Battle, F. Harary, Y. Kodama, The thickness of the complete graph, *Bull. AMS* **68** (1962), 569–571.

[4] L. W. Beineke, Biplanar graphs: a survey, *Computers. Math. Applic.* **34** (1997), 1–8.

[5] É. Czabarka, O. Sýkora, L. A. Székely, and I. Vrťo, Crossing numbers and biplanar crossing numbers I: a survey of problems and results, to appear in: *Finite and Infinite Combinatorics*, Bolyai Studies, Budapest.

[6] P. Erdős, R. K. Guy, Crossing number problems, *American Mathematical Monthly* **80** (1973), 52–58.

[7] A. M. Dean, J. P. Hutchinson, E. Scheinerman, On the thickness and arboricity of a graph, *J. Comb. Theory* **B 52** (1991), 147–151.

[8] R. K. Guy, Crossing numbers of graphs, in: *Graph Theory and its Applications*, *Lecture Notes in Mathematics* **303**, Springer Verlag, Berlin 1972 , 111–124.

[9] J. Halton, On the thickness of graphs of given degree, *Info. Sci.* **54** (1991), 219–238.

[10] F. T. Leighton, *Complexity Issues in VLSI.* MIT Press, Cambridge, 1983.

[11] A. Liebers, Methods for planarizing graphs — a survey and annotated bibliography, *J. Graph Algorithms and Applications* **5** (2001), 1–74. `http://www.cs.brown.edu/publications/jgaa/`

[12] A. Owens, On the biplanar crossing number, IEEE Transactions on Circuit Theory **CT-18** (1971), 277–280.

[13] S. M. Malitz, Genus g graphs have pagenumber $O(\sqrt{g})$, *J. Algorithms* **17** (1994), 85–109.

[14] A. Mansfield, Determining the thickness of graphs is NP-hard, *Math. Proc.Cambridge Philos. Soc.* **9** (1983), 9–23.

[15] P. Mutzel, T. Odenthal, M. Scharbrodt, The thickness of graphs: a survey, *Graphs Combin.* **14** (1998), 59–73.

[16] J. Pach, F. Shahrokhi and M. Szegedy, Applications of crossing numbers, *Algorithmica* **16** (1996), 111–117.

[17] F. Shahrokhi, O. Sýkora, L. A. Székely and I. Vrťo, Crossing numbers: bounds and applications, in: *Intuitive Geometry*, eds. I. Bárány and K. Böröczky, Bolyai Society Mathematical Studies **6**, János Bolyai Mathematical Society, Akadémiai Kiadó, Budapest, 1997, 179–206.

[18] F. Shahrokhi, O. Sýkora, L. A. Székely and I. Vrťo, Bounds and methods for k-planar crossing numbers, *Graph Drawing* 2004, Lecture Notes in Computer Science Vol. 2912, Springer Verlag, Berlin, 2004, 37–46. (The journal version is to appear in *Discrete Appl. Math.*)

[19] The biplanar crossing number of the random graph, in: *Towards a Theory of Geometric Graphs*, ed. J. Pach, *Contemporary Mathematics* **342**, Amer. Math. Soc. 2004, 269–271.

[20] O. Sýkora, L. A. Székely, and I. Vrťo, Two counterexamples in graph drawing, in: *Proc. 27th Intl. Workshop on Graph-Theoretic Concepts in Computer Science*, Lecture Notes in Computer Science 2573, Springer Verlag, Berlin, 2002, 391–398.

[21] O. Sýkora, I. Vrťo, I., On VLSI layouts of the star graph and related networks, *Integration* **17** (1994), 83–93.

[22] L. A. Székely, A successful concept for measuring non-planarity of graphs: the crossing number, *Discrete Math.* **276** (2004) (1–3), 331–352.

[23] J. Pach, J. Spencer and G. Tóth, New bounds on crossing numbers, *Discrete Comput. Geom.* **24** (2000), 623–644.

[24] A. T. White and L. W. Beineke, Topological graph theory, in: *Selected Topics in Graph Theory,* (L. W. Beineke and R. J. Wilson, eds.), Academic Press, 1978, 15–50.