Abel's binomial theorem

The following was assigned as homework problem: for variables x, y, z the following polynomial identity holds:

$$\sum_{k=0}^{n} \binom{n}{k} x(x+kz)^{k-1}(y+(n-k)z)^{n-k} = (x+y+nz)^n; \tag{1}$$

for nonzero numbers x, y the identity

$$\sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1}(y+(n-k))^{n-k-1} = (\frac{1}{x}+\frac{1}{y})(x+y+n)^{n-1} \tag{2}$$

holds, and finally, the following numerical identity holds:

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}. \tag{3}$$

First note that (1) implies the binomial theorem with $z=0$. The term $k=0$ may cause a suspicion if we deal with a polynomial on the LHS at all, but for $k=0$, the corresponding term is $x \cdot x^{-1}(y+nz)^n$. If one wants to avoid this case, he may prove instead of (1)

$$(y+nz)^n + \sum_{k=1}^{n} \binom{n}{k} x(x+kz)^{k-1}(y+(n-k)z)^{n-k} = (x+y+nz)^n, \tag{4}$$

where both sides are polynomials without any doubt. To prove (1) or (4), we differentiate both sides by y and apply induction on n. Formulas (2), (3) easily will follow from (4). Call $f_n(x,y,z)$ the LHS of 1 or (4), call $g_n(x,y,z)$ the RHS of (1) or (4). Note that for $n=1$ $f_1(x,y,z) = y+z+x = g_1(x,y,z)$. We apply induction on n to prove the identity. For this goal, it is enough to show that the partial derivatives in y are equal polynomials, and in addition, for a certain value of y the identity holds, i.e.

$$\frac{\partial}{\partial y} f_n(x,y,z) = \frac{\partial}{\partial y} g_n(x,y,z); \tag{5}$$

$$f_n(x,-x-nz,z) = g_n(x,-x-nz,z). \tag{6}$$

It is easy to see that

$$\frac{\partial}{\partial y} g_n(x,y,z) = n(x+(y+z)+(n-1)z)^{n-1} = n g_{n-1}(x, y+z, z). \tag{7}$$

On the other hand, using the identity $n\binom{n-1}{k} = \binom{n}{k}(n-k)$, we obtain

$$\frac{\partial}{\partial y} f_n(x,y,z) = \sum_{k=0}^{n} \binom{n}{k} x(x+kz)^{k-1}(n-k)(y+(n-k)z)^{n-k-1}, \tag{8}$$

$$= n\sum_{k=0}^{n-1}\binom{n-1}{k}x(x+kz)^{k-1}[(y+z)+((n-1)-k)z]^{(n-1)-k} = nf_{n-1}(x,y+z,z). \tag{9}$$

It follows from the induction hypothesis that

$$nf_{n-1}(x,y+z,z) = ng_{n-1}(x,y+z,z),$$

and this finishes the proof of (5). We turn to the proof of (6). This will be proved much like in the manner of the proof to (5). Since obviously $g_n(x,-x-nz,z) = 0$, we have to prove $f_n(x,-x-nz,z) = 0$. We are going to do it by induction on n. The base case is trivial again. We have to prove the following two facts:

$$\frac{\partial}{\partial z}f_n(x,-x-nz,z) = 0; \tag{10}$$

$$f_n(x,-x,0) = 0. \tag{11}$$

Let us start with

$$f_n(x,-x-nz,z) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x(x+kz)^{n-1}.$$

To verify (10),

$$\frac{\partial}{\partial z}f_n(x,-x-nz,z) = (n-1)\sum_{k=0}^{n}\binom{n}{k}xk(-1)^{n-k}(x+kz)^{n-2}$$

(using $k\binom{n}{k} = n\binom{n-1}{k-1}$ for $k > 0$)

$$= n(n-1)\sum_{k=1}^{n}\binom{n-1}{k-1}x(-1)^{n-k}(x+kz)^{n-2}$$

$$= n(n-1)\sum_{j=0}^{n-1}\binom{n-1}{j}x(-1)^{n-1-j}(x+(j+1)z)^{n-2}. \tag{12}$$

We have to show that (12) is equal to zero. Use the hypothesis

$$0 = f_{n-1}(x,-x,-(n-1)z,z)$$

to prove

$$0 = \sum_{j=0}^{n-1}(-1)^{n-1-j}\binom{n-1}{j}x(x+jz)^{n-2}. \tag{13}$$

Substitute in (13) $x+z$ to the place of x and multiply the resulting formula by $x/(x+z)$ to obtain that (12) is equal to zero.

The proof of (11) is trivial, since

$$f_n(x,-x,0) = x^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = 0,$$

because of the rule on the summation of binomial coefficients with alternating sign in a row of the Pascal triangle.

An alternative proof of (6) goes in a more direct way but uses a basic inclusion-exclusion result:

$$f_n(x,-x-nz,z) = x \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x+kz)^{n-1}$$

$$= x \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{n-1} \binom{n-1}{j} k^j z^j x^{n-j} = x \sum_{j=0}^{n-1} x^{n-j} z^j \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^j.$$

Note that for $j < n$ natural number $\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^j$ = the number of surjections from an j-element set to an n-element set, and hence is zero.

Turning to the proof of (2), apply (1) with $z = 1$:

$$f_n(x,y,1) = \sum_{k=0}^{n} \binom{n}{k} x(x+k)^{k-1} (y+n-k)^{n-k-1} (y+n-k) \quad (14)$$

$$= \sum_{k=0}^{n} \binom{n}{k} x(x+k)^{k-1} y(y+n-k)^{n-k-1} \quad (15)$$

$$+ \sum_{k=0}^{n} \binom{n}{k} x(x+k)^{k-1} (y+n-k)^{n-k-1} (n-k). \quad (16)$$

Here, using $\binom{n}{k}(n-k) = n\binom{n-1}{k}$ for $k \le n-1$, we have that (15) is equal to

$$\sum_{k=0}^{n-1} n \binom{n-1}{k} x(x+k)^{k-1} ((y+1)+(n-1)-k)^{(n-1)-k}$$

$$= n f_{n-1}(x, y+1, 1) = n(x+y+n)^{n-1}.$$

Since $f_n(x,y,1) = (x+y+n)^n$, (14,15,16) imply that (16) is equal to

$$(x+y+n)^n - n(x+y+n)^{n-1} = (x+y)(x+y+n)^{n-1}.$$

Dividing by xy yields the required identity (2).

Finally, in order to prove (3), subtract

$$\frac{1}{x}(y+n)^{n-1} + \frac{1}{y}(x+n)^{n-1}$$

from both sides of (2). We get

$$\sum_{k=1}^{n-1}\binom{n}{k}x(x+k)^{k-1}(y+n-k)^{n-k-1}=$$

$$\frac{1}{x}[(x+y+n)^{n-1}-(y+n)^{n-1}]+\frac{1}{y}[(x+y+n)^{n-1}-(x+n)^{n-1}].$$

Taking the limit when $x, y \to 0$ we obtain the LHS of (1) on the LHS by continuity, on the RHS we have

$$\lim_{x\to 0}\frac{1}{x}[(x+y+n)^{n-1}-(y+n)^{n-1}]=\frac{d}{dx}(x+y+n)^{n-1}|_{x=0}=(n-1)(y+n)^{n-2},$$

and by continuity $\lim_{y\to 0}(n-1)(y+n)^{n-2}=(n-1)n^{n-2}$. Another $(n-1)n^{n-2}$ comes from the other term in the RHS. We proved (3).

We mention here, that Abel's binomial theorem has been further genereralized by Hurwitz into the following polynomial identity in variables $x, y, z_1, ..., z_n$:

Hurwitz' Binomial Theorem:

$$\sum_{\epsilon_i=0,1} x(x+\sum_{i=1}^{n}\epsilon_i z_i)^{\sum_{i=1}^{n}\epsilon_i-1}(y+\sum_{i=1}^{n}(1-\epsilon_i)z_i)^{n-\sum_{i=1}^{n}\epsilon_i}=$$

$$(x+y+z_1+...+z_n)^n;$$

or, in alternative description, corresponding to (2), we have

$$\sum_{\epsilon_i=0,1} x(x+\sum_{i=1}^{n}\epsilon_i z_i)^{\sum_{i=1}^{n}\epsilon_i-1}y(y+\sum_{i=1}^{n}(1-\epsilon_i)z_i)^{n-1-\sum_{i=1}^{n}\epsilon_i}=$$

$$(x+y)(x+y+z_1+...+z_n)^{n-1}.$$

The summation means 2^n terms. The simplest proof of Hurwitz' Binomial Theorem — what a surprise! — goes by counting trees. It is an easy to see how Hurwitz' Binomial Theorem implies Abel's Binomial Theorem. You may note that the second version of Hurwitz' Binomial Theorem generalizes a version of (2), where both sides are multiplied by xy. Hence (2) also has a polynomial identity version:

$$\sum_{k=0}^{n}\binom{n}{k}x(x+k)^{k-1}y(y+(n-k)z)^{n-k-1}=(x+y)(x+y+nz)^{n-1}.$$

There is a subtle point here that has to be mentioned. What we really proved is that (1) holds for all real values of x, y, z, but we have not proved the polynomial identity yet. By definition, polynomial identity means that for all α, β, γ the coefficients of $x^\alpha y^\beta z^\gamma$ are identical in the RHS and in the LHS of (1). However, the following theorem holds:

Theorem If $f(x_1, x_2, ..., x_n)$ and $g(x_1, x_2, ..., x_n)$ are multivariate polynomials, so that we have infinite sets $A_1, A_2, ..., A_n$ such that $f(a_1, a_2, ..., a_n) = g(a_1, a_2, ..., a_n)$ whenever $a_i \in A_i$, then $f(x_1, x_2, ..., x_n) = g(x_1, x_2, ..., x_n)$ as polynomials.

You know this theorem for $n = 1$ from elementary algebra: two different polynomials of degree at most d can agree in at most d places. The proof of the theorem goes by induction on n. Write

$$f(x_1, x_2, ..., x_n) = \sum_\alpha f_\alpha(x_1, x_2, ..., x_{n-1}) x_n^\alpha$$

and

$$g(x_1, x_2, ..., x_n) = \sum_\alpha g_\alpha(x_1, x_2, ..., x_{n-1}) x_n^\alpha.$$

We are going to show that for all $a_i \in A_i$ $(i = 1, 2, ..., n-1)$ and all α,

$$f_\alpha(a_1, a_2, ..., a_{n-1}) = g_\alpha(a_1, a_2, ..., a_{n-1}).$$

Using the hypothesis for $n - 1$, we obtain that for all α,

$$f_\alpha(x_1, x_2, ..., x_{n-1}) = g_\alpha(x_1, x_2, ..., x_{n-1})$$

as polynomials, and this will imply the theorem.

Fix $a_i \in A_i$ in an arbitrary way. Define the single variable polynomials

$$f^*(x_n) = \sum_\alpha f_\alpha(a_1, a_2, ..., a_{n-1}) x_n^\alpha$$

and

$$g^*(x_n) = \sum_\alpha g_\alpha(a_1, a_2, ..., a_{n-1}) x_n^\alpha.$$

Polynomials f^* and g^* agree on infinetely many places for x_n (namely, on the elements of A_n), and therefore by the $n = 1$ base case these polynomials are identical polynomials, i.e. agree termwise, hence

$$f_\alpha(a_1, a_2, ..., a_{n-1}) = g_\alpha(a_1, a_2, ..., a_{n-1})$$

holds.