WELL-QUASI-ORDERING

1. Quasi-orders and partial orders

This note is a joint work with Éva Czabarka. We assume the Axiom of Choice.

Definition 1. *A* quasi-order *is a binary relation $\trianglelefteq$ on a set X that is reflexive and transitive.*
A partial order *is an antisymmetric quasi-order.*
A total order *$\trianglelefteq$ is a partial order, such that for every a, b in the base set, a and b are related at least one way, i.e. $a \trianglelefteq b$ or $b \trianglelefteq a$).*
A well-ordering *is a total order, in which every non-empty subset $A \subseteq X$ has a* smallest *element, i.e., say $b \in A$, such that for all $a \in A$, $b \trianglelefteq a$.*

The Axiom of Choice is equivalent to the statement that every set has a well-ordering. We often refer to the relation $\trianglelefteq$ as *inequality.*

As a quasi-order is not necessarily antisymmetric, for a quasi-order $\trianglelefteq$ it is possible to have a, b with $a \trianglelefteq b$, $b \trianglelefteq a$ and $a \neq b$.

Definition 2. *In a quasi-order $(X, \trianglelefteq)$ the "strict inequality" relation $x \triangleleft y$ means $x \trianglelefteq y$ but $y \ntrianglelefteq x$.*

We also write the strict inequality relation $x \triangleleft y$ as $y \triangleright x$. In a partial order, if there is an inequality between two different elements, then it is strict. The rest of this section explains that the concept of quasi-order is just marginally more general than the concept of partial order. As all the important applications of this theory are for partial orders, you can skip the rest of this section and later simply think about partial order when we speak about quasi-order.

The following observation is obvious:

Proposition 3. *If $(A, \trianglelefteq)$ is a quasi-ordered set, then the relation $\sim_\trianglelefteq$ defined by $a \sim_\trianglelefteq b$ when $a \trianglelefteq b$ and $b \trianglelefteq a$, is an equivalence relation on A.*

If $\simeq$ is an equivalence relation on A and the equivalence classes of $\simeq$ are denoted by $[a]_\simeq = \{b \in A : b \simeq a\}$, then, as usual, $A/\simeq \ = \{[a]_\simeq : a \in A\}$ is the set of equivalence classes.

Definition 4. *Given a quasi-order $\trianglelefteq$ on A, $\leq_\trianglelefteq$ is the relation on $A/\sim_\trianglelefteq$ defined by $[a]_{\sim_\trianglelefteq} \leq_\trianglelefteq [b]_{\sim_\trianglelefteq}$ iff $a' \trianglelefteq b'$ for some $a' \in [a]_{\sim_\trianglelefteq}$ and $b' \in [b]_{\sim_\trianglelefteq}$.*

Now quasi-orders are essentially just blown-up partial orders, but it will take a little time to fully establish:

Proposition 5. *Given a quasi-ordered set $(A, \trianglelefteq)$, the following are true:*

(a) *If* $[a]_{\sim_\trianglelefteq} \leq_\trianglelefteq [b]_{\sim_\trianglelefteq}$ *then* $a' \trianglelefteq b'$ *for all* $a' \in [a]_{\sim_\trianglelefteq}$ *and* $b' \in [b]_{\sim_\trianglelefteq}$*,*
(b) $\leq_\trianglelefteq$ *is a partial order on* $A/\sim_\trianglelefteq$*.*

Proof. Assume $a' \in [a]_{\sim_\trianglelefteq}$ and $b' \in [b]_{\sim_\trianglelefteq}$, where $[a]_{\sim_\trianglelefteq} \leq_\trianglelefteq [b]_{\sim_\trianglelefteq}$. The definition of $\leq_\trianglelefteq$ gives that there are $a'' \in [a]_{\sim_\trianglelefteq}$ and $b'' \in [b]_{\sim_\trianglelefteq}$ such that $a'' \trianglelefteq b''$. As $a' \trianglelefteq a''$, $b'' \trianglelefteq b'$, and $\trianglelefteq$ is transitive, $a' \trianglelefteq b'$, yielding (a).

Since $\trianglelefteq$ is reflexive and transitive, so is $\leq_\trianglelefteq$. Assume that $[a]_{\sim_\trianglelefteq} \leq_\trianglelefteq [b]_{\sim_\trianglelefteq}$. and $[b]_{\sim_\trianglelefteq} \leq_\trianglelefteq [a]_{\sim_\trianglelefteq}$. By (a) we have $a \trianglelefteq b$ and $b \trianglelefteq a$, therefore $a \in [b]_{\sim_\trianglelefteq}$, giving $[a]_{\sim_\trianglelefteq} = [b]_{\sim_\trianglelefteq}$; so $\leq_\trianglelefteq$ is antisymmetric. This shows (b). $\square$

Therefore we have that every quasi-order $\trianglelefteq$ defines a unique partial order on the set of equivalence classes $A/\sim_\trianglelefteq$ of its natural equivalence relation $\sim_\trianglelefteq$.

We will now set up the converse of this statement:

Definition 6. *Let* $\simeq$ *be an equivalence relation on the set* A*, and* $\trianglelefteq$ *be a partial order on* $A/\simeq$*. The relation* $\preceq_\trianglelefteq$ *on* A *is defined by* $a \preceq_\trianglelefteq b$ *iff* $[a]_\simeq \trianglelefteq [b]_\simeq$*.*

Note that it should be obvious that every partial order on a set B can be viewed as a partial order on some set $A/\simeq$ as long as $|A/\simeq| = |B|$.

Now we are ready to set up the converse promised earlier (and show that, viewing it the right way, we have defined two mappings between partial orders and quasi-orders that are inverses of each other)

Proposition 7. *Let* $\simeq$ *be an equivalence relation on* A *and* $\trianglelefteq$ *be a partial order on* $A/\simeq$*. The following are true:*
(a) $\preceq_\trianglelefteq$ *is a quasi-order on* A*,*
(b) $\sim_{\preceq_\trianglelefteq} \;=\; \simeq$*,*
(c) $\leq_{\preceq_\trianglelefteq} \;=\; \trianglelefteq$*.*

Proof. The reflexivity and transitivity of $\preceq_\trianglelefteq$ follows from the reflexivity and transitivity of $\trianglelefteq$, giving (a)

(b) follows from $x \sim_{\preceq_\trianglelefteq} y$ iff ($x \preceq_\trianglelefteq y$ and $y \preceq_\trianglelefteq x$) iff ($[x]_\simeq \trianglelefteq [y]_\simeq$ and $[y]_\simeq \trianglelefteq [x]_\simeq$) iff $[x]_\simeq = [y]_\simeq$ iff $x \simeq y$.

(b) and the definitions yield (c). $\square$

The above fairly simply gives

Proposition 8. *For a quasi-order* $\trianglelefteq$ *we have* $\preceq_{\leq_\trianglelefteq} \;=\; \trianglelefteq$*.*

2. Examples

A relevant example for a quasi-order, which is not a partial order, is divisibility in $\mathbb{Z}$. Let $m \trianglelefteq n$ mean $m|n$. The equivalence classes of the relation $\sim_\trianglelefteq$ are the $\{n, -n\}$ sets for $n > 0$ and $\{0\}$.

Let $\mathfrak{A}$ be a finite alphabet. Let the relation $\trianglelefteq$ simply be the $=$ relation on $\mathfrak{A}$, i.e., $A \trianglelefteq A$, $B \trianglelefteq B$,...,$Z \trianglelefteq Z$. The equivalence classes of the relation $\sim_\trianglelefteq$ are singleton sets, containing a single letter. Hence $\trianglelefteq$ is a partial order.

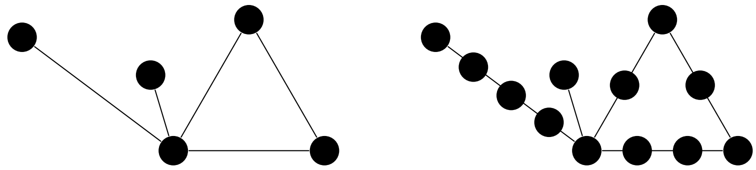

Figure 1. Graph G on the left, one of its subdivisions on the right.

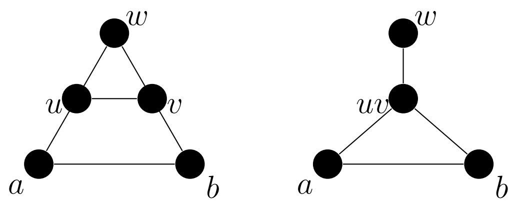

Figure 2. Graph G on left, and G^* on right. The graph G^* is obtained from the graph G by contracting the edge $\{u, v\}$.

Words over a finite alphabet $\mathfrak{A}$ make a quasi-order for the subword relation. A word is a *subword* of another word, if we can obtain it by deleting some letters. For example, CLUE is a subword of CLUTTER, but ULCER is not. This is a partial order.

We define two procedures of graphs. A *subdivision* of a graph is created by substituting some edges with paths, see Fig. 1. A *contraction* of an $e = \{u, v\}$ edge of a graph G creates a new uv vertex, and joins it to w, if $uw \in E(G)$ or $uv \in E(G)$; and then removes vertices u, v and the edges incident to them, see Fig. 2.

Consider three quasi-orders on the set of finite simple graphs, whose vertices are selected from $\mathbb{N}$:

Definition 9. *H is a* subgraph *of G, if there is an injection $i : V(H) \to V(G)$, such that for all $\{a, b\} \in E(H)$, we have $\{i(a), i(b)\} \in E(H)$.*
H is a topological minor *of G, if a subdivision of H is a subgraph of G.*
H is a minor *of G, if H is a subgraph of G^*, where G^* is obtained from G by contracting edges.*

It is easy to see that these relations are quasi-orders. Furthermore, it is clear that subgraphs are topological minors, and topological minors are minors. We refer to the notation introduced in Section 1. If $H \trianglelefteq G$ means that H is a subgraph of G, then the equivalence classes $\sim_{\trianglelefteq}$ are exactly the isomorphism classes of graphs. These equivalence classes can be taken for the definition of "unlabelled graphs". The relation $\leq_{\trianglelefteq}$ can be taken as the partial order of "subgraph relation of unlabelled graphs".

The topological minor and minor relations can be extended to unlabelled graphs, where they turn into partial orders, in the following way. An unlabelled graph U is a topological minor (resp. minor) of an unlabelled graph W, if there is a $H \in U$ and $G \in W$, such that H is a topological minor (resp. minor) of G. It is easy to see that this definition is proper, i.e., it does not depend on the representatives selected from the equivalence classes.

Among unlabelled graphs, subgraphs are still topological minors, and topological minors are still minors.

3. Further definitions

As every total order is a partial order and every partial order is a quasi-order, defining something for quasi-orders makes the definition for all three.

Definition 10. *Let* $(A, \trianglelefteq)$ *be a quasi-ordered set and* $S \subseteq A$
S *is an* antichain*, if for all* $s_1, s_2 \in S$*,* $s_1 \trianglelefteq s_2$ *implies* $s_1 = s_2$*,*
S *is a* chain*, if for all* $s_1, s_2 \in S$*,* $s_1 \trianglelefteq s_2$ *or* $s_2 \trianglelefteq s_1$*,*
a chain S *is a* strict chain*, if for all* $s_1, s_2 \in S$*,* $(s_1 \trianglelefteq s_2$ *and* $s_2 \trianglelefteq s_1)$ *implies* $s_1 = s_2$*.*

Note that in partial orders every chain is a strict chain.

Definition 11. *Let* $(A, \trianglelefteq)$ *be a quasi-ordered set and* $S \subseteq A$*.*
The down-set of S *is* $\mathcal{D}(S) = \{y \in A : \exists s \in S\ y \trianglelefteq s\}$*.*
The up-set of S *is* $\mathcal{U}(S) = \{y \in A : \exists s \in S\ s \trianglelefteq y\}$*.*
S *is* down-closed *(or is a* down-ideal*) if* $\mathcal{D}(S) = S$*.*
S *is* up-closed *(or is an* up-ideal*) if* $\mathcal{U}(S) = S$*.*

It is easy to see that down-sets are down-closed and up-sets are up-closed. Furthermore, complements of up-closed sets are down-closed sets, and vice versa, allowing $\emptyset$ to be both down-closed and up-closed.

Definition 12. *Let* $(A, \trianglelefteq)$ *be a quasi-ordered set and* $Y \subseteq A$*. We say that* y *is a* minimal element *of* Y*, if*
$y \in Y$*, and*
for all $z \in Y$*,* $z \trianglelefteq y$ *implies* $y \trianglelefteq z$*.*

Note that in a quasi-order, even if a minimal element exists, it is not necessarily unique. E.g., if $(A, \trianglelefteq)$ is a quasi-order then for any $a \in A$, the set $Y = [a]_{\sim_{\trianglelefteq}}$ is a chain, whose every element is a minimal element of Y. If Y has a smallest element (see Definition 1), then it is a minimal element of Y, but a minimal element is not necessarily smallest.

Proposition 13. *Let* $(A, \trianglelefteq)$ *be a quasi-ordered set. The following four facts are equivalent:*
(i) every non-empty subset $X \subseteq A$ *has a minimal element,*
(ii) every non-empty chain C *of* A *has a smallest element,*
(iii) every strict decreasing chain $a_1 \trianglerighteq a_2 \trianglerighteq a_3 \trianglerighteq \cdots$ *is finite.*
(iv) every up-ideal U *is generated by the set of its minimal elements.*

Proof. (i) $\to$ (ii): C must have a minimal element. It is a smallest element in the chain, as it is comparable to every element of the chain. (ii) $\to$ (iii): assume (iii) fails, hence there is an infinite strict decreasing chain. This chain does not have a smallest element.
(iii) $\to$ (iv): As $\mathcal{U}(\emptyset) = \emptyset$, and the set of minimal elements of $\emptyset$ is empty, (iv) is obvious for

the empty up-ideal. Let U be a non-empty up-ideal, and let X be the set of its minimal elements. Clearly, $\mathcal{U}(X) \subseteq U$. If $U \neq \mathcal{U}(X)$, take $x_1 \in U \setminus \mathcal{U}(X)$. As x_1 is not a minimal element of U, there is an $x_2 \in U$ such that $x_2 \lhd x_1$; and, as $x_1 \notin \mathcal{U}(X)$, $x_2 \notin \mathcal{U}(X)$. Repeating this process we buld a strict downwards infinite chain $x_1, x_2, \dots$ in $U \setminus \mathcal{U}(X)$, contradicting (iii).
(iv) $\to$ (i) is obvious. □

Definition 14. *A quasi-ordered set $(A, \trianglelefteq)$ is called* well-founded*, if it satisfies the equivalent properties from Proposition 13.*

Theorem 15. *Assume that $(A, \trianglelefteq)$ is a well-founded quasi-order. The following properties are equivalent:*
(i) there is no infinite antichain in A,
(ii) for every infinite sequence $a_1, a_2, a_3, \dots$ elements of A, there exists $i < j$, such that $a_i \trianglelefteq a_j$,
(iii) for every infinite sequence $a_1, a_2, a_3, \dots$ elements of A, there exists an infinite strictly increasing sequence of indices $i_1 < i_2 < i_3 < \cdots$, such that $a_{i_1} \trianglelefteq a_{i_2} \trianglelefteq a_{i_3} \trianglelefteq \cdots$
(iv) for all $X \subseteq A$, if $X \neq \emptyset$, then there exists a finite *$Y \subseteq X$, such that for all $x \in X$, there exists a $y \in Y$ with $y \trianglelefteq x$.*
(v) for all $U \subseteq A$ up-ideals, if $U \neq \emptyset$, then there exists a finite *$Y \subseteq U$, such that for all $x \in U$, there exists a $y \in Y$ with $y \trianglelefteq x$.*

Proof. (iii)$\to$(ii)$\to$(i) is trivial. Assume (i) and prove (iii). Color $[\mathbb{Z}^+]^2$ with 3 colors as follows. Assume $i < j$, and set

$$color(\{i,j\}) = \begin{cases} Red & \text{if } a_i \trianglelefteq a_j \\ Blue & \text{if } a_i \trianglerighteq a_j \text{ but } a_j \ntrianglerighteq a_i \\ Green & \text{otherwise.} \end{cases}$$

The Infinite Ramsey Theorem implies that there is an infinite $Y \subset \mathbb{Z}^+$, in which every pair received the same color. Write $Y = \{i_1 < i_2 < i_3 < \cdots\}$. If all pairs of Y are Red, we have the required conclusion (iii). If all pairs of Y are Green, then $\{a_i : i \in Y\}$ is an infinite antichain, contradicting assumption (i). If all pairs of Y are Blue, then we have an infinite strict decreasing chain, namely $\{a_i : i \in Y\}$, contradicting the well-foundedness.
(iv)$\to$(v) is trivial. Assume that (v) holds and prove (iv). Take the up-ideal $\mathcal{U}(X)$ generated by X and apply (v) to find a finite set Y' for $\mathcal{U}(X)$. The only possible problem is that Y' may not be a subset of X. But for every $y \in Y'$, there is an $x_y \in X$, such that $x_y \trianglelefteq y$. Now $Y = \{x_y : y \in Y'\}$ is a finite subset of X that has the required property.
(iv)$\to$(ii): if the sequence $a_1, a_2, a_3, \dots$ repeats elements, we are done. If not, let $X = \{a_1, a_2, a_3, \dots\}$. There is a finite $Y \subseteq X$, such that every element of X has a lower bound from Y. Hence, there exists a $y \in Y$, which is lower bound for infinitely many a_ℓs. Set $a_i = y$ and let a_j be a term from the sequence, for which y is a lower bound, and its index is bigger than i.
(ii)$\to$(iv): do proof by contradiction. Assume that a set $X \subseteq A$, there is no finite Y as

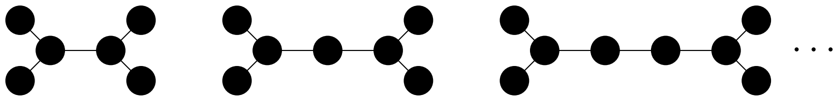

FIGURE 3. Infinite sequence of trees, none of them is subgraph of another.

required. Take $a_0 \in X$ arbitrarily. Define a sequence by induction as follows: if $a_0, a_1, ..., a_n$ was already defined, select $a_{n+1} \in X \setminus \mathcal{U}(\{a_0, a_1, ..., a_n\})$ arbitrarily. The set from which we select a_{n+1} is non-empty, as $Y = \{a_0, a_1, ..., a_n\}$ does not provide a lower bound for all elements of X. $\square$

Definition 16. *Let $(A, \trianglelefteq)$ be a well-founded quasi-ordered set. $\trianglelefteq$ is a well-quasi-order (in short, WQO) if the 5 equivalent properties of Theorem 15 hold.*

Notice the analogy between König's Lemma and WQO. The infinite tree in König's Lemma is a partial order under the relation "ancestor $\trianglelefteq$ descendant". The conclusion is an infinite chain in both cases. The forefather in König's Lemma is substituted by the well-foundedness is WQO. The finiteness of every generation in König's Lemma is substituted by the non-existence of infinite antichains. (However, the partial order of the infinite tree defined above can have an infinite antichain!)

4. More on the examples

The equality relation on a finite alphabet $\mathfrak{A}$ is well-founded, and is a WQO.

In 1950, Higman showed that words over the finite alphabet $\mathfrak{A}$ with the "subword" partial order make a WQO

All three 3 quasi-orders defined on (finite) graphs are well-founded. Indeed, a strict inequality in any of them requires that the smaller graph has strictly fewer vertices or edges.

Consider any surface S and graphs that can be drawn on S without crossing edges. In each of the 3 quasi-orders, such graphs make down-ideal. Indeed, subgraphs can be represented by the same drawing with the deletion of some edges and vertices, while contraction of an edge of a graph drawn on a surface can be done on the surface without creating crossings, see Fig. 2.

If S is the plane, Kuratowski's Theorem asserts that graphs that can be drawn without edge crossings are exactly those, of which neither $K_{3,3}$ nor K_5 are topological minors. Wagner's Theorem asserts that that graphs that can be drawn without edge crossings in the plane are exactly those, of which neither $K_{3,3}$ nor K_5 are minors.

Conversely, those graphs that cannot be drawn in the plane, in each of the 3 quasi-orders, make an up-ideal. $K_{3,3}$ or K_5 is a topological minor of them, and therefore $K_{3,3}$ or K_5 is a minor of them.

The "subgraph" quasi-order is not WQO among graphs, as the cycles C_3, C_4, C_5,... make an infinite antichain. There is an even stronger counterexample: the "subgraph" quasi-order is not WQO among trees, see Fig. 3.

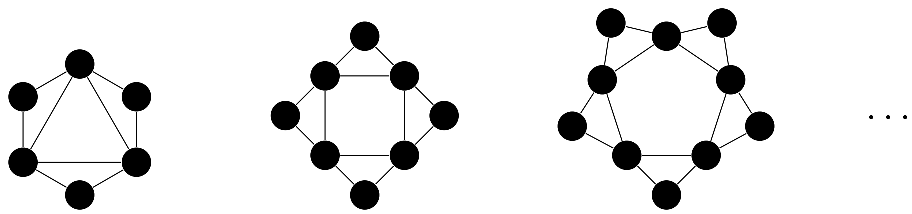

FIGURE 4. Infinite sequence of graphs, none of them is topological minor of another.

However, Erdős and Vázsonyi conjectured in the 1930's that the "topological minor" quasi-order is WQO among finite trees. This conjecture was proved by Kruskal in 1960, a much simpler proof will be shown in a later section.

The "topological minor" quasi-order is not WQO among graphs, see Fig. 4. This means Kuratowski's Theorem is an "accident", as a such a finite characterization is not guaranteed by Theorem 15 (iv).

The proposition that the "minor" quasi-order of graphs is WQO has been known as Wagner's Conjecture. It was proved by Robertson and Seymour in a sequence of 20 papers between 1983 and 2004. This implies that a finite characterization of graphs that cannot be drawn in the plane without crossings as in Theorem 15 (iv) is a necessity, just it does not say which graphs (and how many graphs) are forbidden as minors.

5. Bad sequences

Definition 17. *Let $(A, \trianglelefteq)$ be a well-founded quasi-ordered set. A* bad sequence *is an infinite sequence $a_0, a_1, a_2, ...$ of elements of A, such that for all $0 \le i < j$ indices $a_i \ntrianglelefteq a_j$, in other words, Theorem 15 part (ii) fails for this sequence.*

It is clear that a well-founded quasi-ordered set is WQO if and only if it does not have bad sequences.

Definition 18. *Let $(A, \trianglelefteq)$ be a well-founded quasi-ordered set. An infinite sequence $a_0, a_1, a_2, ...$ of elements of A is a* minimal bad sequence*, if*

- *it is a bad sequence, and*
- *for all $n \ge 0$, there is no bad sequence $a_0, a_1, a_2, ..., a_{n-1}, b_n, b_{n+1}, \ldots$ with $b_n \triangleleft a_n$ strict inequality.*

Lemma 19. *If the $(A, \trianglelefteq)$ well-founded quasi-ordered set is not WQO, then it has a minimal bad sequence.*

Proof. Consider $X = \{x \in A : x \text{ is first term in a bad sequence}\}$. As there are bad sequences, $X \neq \emptyset$. By Proposition 13, there is a minimal element x_0 of X. If $x_0, x_1, ..., x_n$ were already defined, consider $X = \{x \in A : x_0, x_1, ..., x_n, x \text{ are first } n+2 \text{ terms in a bad sequence}\}$. $X \neq \emptyset$, and there is a minimal element x_{n+1} in X. We defined with induction an infinite sequence, which is a bad sequence, and by its construction it is a minimal bad sequence. □

Definition 20. *If $R \subseteq A \times A$ is a binary relation on A, and $\emptyset \neq Y \subseteq A$, then the* restriction $R|_Y$ *is defined as $R \cap (Y \times Y)$.*

The following claims are easy to see:

Proposition 21. *(i) If $(A, \trianglelefteq)$ is a quasi-order (resp. partial order), then $(Y, \trianglelefteq|_Y)$ is a quasi-order (resp. partial order).*
(ii) If the quasi-order $(A, \trianglelefteq)$ is well-founded, then so is $(Y, \trianglelefteq|_Y)$.
(iii) Assume that $(A, \trianglelefteq)$ is well-founded quasi-order. If an infinite sequence $y_0, y_1, y_2, \ldots$ is a bad sequence in $(Y, \trianglelefteq|_Y)$, then it is also a bad sequence in $(A, \trianglelefteq)$.
(iv) If $(A, \trianglelefteq)$ is WQO, then so is $(Y, \trianglelefteq|_Y)$.

Lemma 22. *Assume that $(A, \trianglelefteq)$ is well-founded quasi-order, in which $x_0, x_1, x_2, \ldots$ is a minimal bad sequence. Consider*

$$Y = \left\{ x \in A : \exists i \text{ such that } x \triangleleft x_i \right\}$$

(note the strict inequality in the formula!). Then $Y = \emptyset$ or $(Y, \trianglelefteq|_Y)$ is a WQO.

Proof. If $Y \neq \emptyset$ and $(Y, \trianglelefteq|_Y)$ is not a WQO, then, based on Lemma 19, select a minimal bad sequence $y_0, y_1, y_2, \ldots$ for $(Y, \trianglelefteq|_Y)$. By the construction of Y,

$$\forall i \geq 0 \, \exists i' \geq 0 \text{ such that } y_i \triangleleft x_{i'}.$$

Select the smallest natural number i' that comes up in this way, and select the smallest index i that produces this i'. Now $y_i, y_{i+1}, y_{i+2}, \ldots$ is still a bad sequence for $(Y, \trianglelefteq|_Y)$, as it is an infinite subsequence of a bad sequence. By part (iii) of the Proposition above, $y_i, y_{i+1}, y_{i+2}, \ldots$ is also a bad sequence in $(A, \trianglelefteq)$. If $i' = 0$, the bad sequence $y_i, y_{i+1}, y_{i+2}, \ldots$ contradicts the minimality of the bad sequence $x_0, x_1, x_2, \ldots$ by $y_i \triangleleft x_0$.
Assume now $i' \geq 1$. We are going to show that

$$x_0, x_1, \ldots, x_{i'-1}, y_i, y_{i+1}, y_{i+2}, \ldots$$

is a bad sequence in $(A, \trianglelefteq)$, which will contradict the minimality of the bad sequence $x_0, x_1, x_2, \ldots$, as $y_i \triangleleft x_{i'}$. Indeed, no pair in the x part of the sequence and no pair in the y part of the sequence would fail badness. The only possible problem is if there is an x_m with $m \leq i' - 1$ and y_n with $n \geq i$ such that $x_m \trianglelefteq y_n$. As $y_n \in Y$, there is an n' such that $y_n \triangleleft x_{n'}$. By the choice of i', $m \leq i' - 1 < i' \leq n'$ and $x_m \trianglelefteq x_{n'}$, contradicting the badness of the $x_0, x_1, x_2, \ldots$ sequence. $\square$

6. Words over quasi-orders

Let $(X \trianglelefteq)$ be a quasi-order. Let $X^{<\omega}$ denote the set of finite words over the alphabet X. The empty word — is in $X^{<\omega}$. This is a generalization of the concept of finite words over a finite alphabet. We still speak about the *length* of words. We define the *subword* relation as follows:

Definition 23. *Assume $\bar{x}, \bar{y} \in X^{<\omega}$ and $\bar{x} = (x_0, x_1, \ldots, x_{m-1})$, $\bar{y} = (y_0, y_1, \ldots, y_{n-1})$, where $x_i, y_j \in X$. We say that $\bar{x} \trianglelefteq \bar{y}$ if there is an $f : \{0, 1, \ldots, m-1\} \rightarrow: \{0, 1, \ldots, n-1\}$ monotone increasing injection, such that for all $0 \leq i \leq m-1$, we have $x_i \trianglelefteq y_{f(i)}$.*

This is similar to the definition of subwords over a finite alphabet, except that letters must be simply the same in that case. Using the same relation sign comparing subwords is just a slight abuse, as this is the extension of the quasi-order of X, as the relation between 1-letter words is exactly the same as the relation between the letters in X. The empty word is a subword of every word. The length of a subword is at most the length of the word. The following lemma is not difficult to see:

Lemma 24. *(i) The subword relation is a quasi-order on $X^{<\omega}$.*
(ii) If $(X, \trianglelefteq)$ is a partial order, then so is $(X^{<\omega}, \trianglelefteq)$.
(iii) If $(X, \trianglelefteq)$ is well-founded, then so is $(X^{<\omega}, \trianglelefteq)$.

Proof. We only prove (iii). For contradiction, assume that $\bar{x}_1 \triangleright \bar{x}_2 \triangleright \bar{x}_3, \ldots$ is an infinite strictly decreasing chain in $(X^{<\omega}, \trianglelefteq)$. The length of these words is a decreasing sequence of natural numbers, hence the same length occurs infinitely many times. Assume without loss of generality that $length(x_i)$ is the same number h for every i, so $\bar{x}_i = (x^i_0, x^i_1, ..., x^i_{h-1})$. If $\bar{x}_i \triangleright \bar{x}_j$, then there must a letter position ℓ, such that $x^i_\ell \triangleright x^j_\ell$. Therefore there is infinite sequence $i_1 < i_2 < i_3 < \cdots$ and a letter position ℓ such that $x^{i_1}_\ell \triangleright x^{i_2}_\ell \triangleright x^{i_3}_\ell \triangleright \cdots$, contradicting the well-foundedness of $(X, \trianglelefteq)$. □

Theorem 25. [Higman] *If $(X, \trianglelefteq)$ is a WQO, then so is $(X^{<\omega}, \trianglelefteq)$.*

This is a far-reaching generalization of the theorem that for infinitely many words over a finite alphabet, there is a subword relation between two words.

Proof. We already know from Lemma 24 that $(X^{<\omega}, \trianglelefteq)$ is well-founded. For contradiction, if $(X^{<\omega}, \trianglelefteq)$ is not WQO, then by Lemma 19 it has a minimal bad sequence $\bar{x}_0, \bar{x}_1, \bar{x}_2, ...$, where $\bar{x}_i = (x_{i,0}, x_{i,1}, x_{i,2}, ..., x_{i,j_i-1}) \in X^{<\omega}$, and $x_{i,\ell} \in X$. We have $length(\bar{x}_i) \geq 1$, as the empty word is a subword of every word. Define $tail(\bar{x}_i) = (x_{i,1}, x_{i,2}, ..., x_{i,j_i-1})$, a length $j_i - 1$ word that remains after after deleting the first letter of $\bar{x}_i$. Observe that $tail(\bar{x}_i) \triangleleft \bar{x}_i$, a strict inequality in $(X^{<\omega}, \trianglelefteq)$. As $(X, \trianglelefteq)$ is a WQO, the bad sequence $\bar{x}_0, \bar{x}_1, \bar{x}_2, ...$ may contain only finitely many words of length 1, and consequently $Y = \left\{ tail(\bar{x}_i) : i \geq 0 \right\}$ must contain some non-empty words. By Lemma 22, $Z = \{\bar{y} : \exists i \;\; \bar{y} \triangleleft \bar{x}_i\}$ is a WQO. As $Y \subseteq Z$, $(Y, \trianglelefteq_Y)$ is a WQO. By Theorem 15 (iii), Y has an infinite chain:

$$tail(\overline{x_{i_0}}) \trianglelefteq tail(\overline{x_{i_1}}) \trianglelefteq tail(\overline{x_{i_2}}) \trianglelefteq \cdots .$$

Consider now the $x_{i_1,0}, x_{i_2,0}, x_{i_3,0}, ...$ infinite sequence in X. As $(X, \trianglelefteq)$ is a WQO, there exists $i_\ell < i_k$ indices, such that $x_{i_\ell,0} \trianglelefteq x_{i_k,0}$. As $\overline{x_{i_\ell}}$ is the concatenation of letter $x_{i_\ell,0}$ and word $tail(\overline{x_{i_\ell}})$, and $\overline{x_{i_k}}$ is the concatenation of letter $x_{i_k,0}$ and word $tail(\overline{x_{i_k}})$, we have that $\overline{x_{i_\ell}} \trianglelefteq \overline{x_{i_k}}$, so the minimal bad sequence was not bad. □

7. Kruskal's theorem

It is easy to see that finite trees for the topological minor relation form a quasi-order, and that this quasi-order is well founded.

Theorem 26. [Kruskal] *The quasi-order of finite trees for the topological minor relation is a WQO.*

We are going to prove a stronger theorem. Our objects will be rooted trees. Assume that T is a tree with root r. There is a partial order on the vertex set $V(T)$: $x \preceq_T y$ if vertex x is on the unique ry path. For vertices u, v, define $u \wedge v$ as the unique intersection point of the three paths connecting uv, ur and vr. The partial order $\preceq_T$ has the following special property: if $uv \in E(T)$ and $u \preceq_T v$ and $w \prec v$, then $w \preceq_T u$.

Definition 27. *Assume that T and S are rooted trees. An injection $f : V(T) \to V(S)$ is a* decent embedding *if*
(i) $u \preceq_T v$ implies $f(u) \preceq_S f(v)$, and
(ii) $f(u) \wedge f(v) = f(u \wedge v)$.
For two rooted trees, T and S, we say $T \trianglelefteq S$, if there is a decent embedding of T into S.

Note that a decent embedding is not required to map root to root.

Proposition 28. *(i) $\trianglelefteq$ is a well-founded quasi-order on rooted trees.*
(ii) if $T \trianglelefteq S$, then T is a topological minor of S.
(iii) the one-vertex tree decently embeds into every tree.

Proof. (i): It is easy to see that the identity function is a decent embedding of any tree T to itself; and if f is a decent embedding of T into S and g is a decent embedding of S into R then $g \circ f$ is a decent embedding of T into R, so $\trianglelefteq$ is a quasi-order. As in a strict decreasing chain decently embedded trees the number of vertices of the trees decrease, the well-foundedness is obvious.
(ii) Let f be a decent embedding of T into S, and let $X = f(V(T))$. Let T' be the subtree of S that we obtain from X by adding the unique $f(u)f(v)$ paths in S for each adjacent vertex pair u, v in T. We will show that T' is a subdivision of T. Let $\{u, v\}, \{w, t\}$ be different edges of T, where without loss of generality $u \preceq_T v$ and $w \preceq_T t$. Since the edges are different, $t \neq v$. Since f is a decent embedding, $f(u) \preceq_S f(v)$ and $f(w) \preceq_S f(t)$. Assume that the $f(u)f(v)$ and $f(w)f(t)$ paths in S intersect. Let x be any vertex in the intersection of the two paths. Then $f(u) \preceq_S x \preceq_S f(v)$ and $f(w) \preceq_S x \preceq_S f(t)$, consequently $x \preceq_S f(v) \wedge f(t) = f(v \wedge t)$. If v, t are incomparable in $\preceq_T$, then $v \wedge t \preceq_T u, w$ by the special property, which implies that $f(v \wedge t) \preceq_S f(u), f(w) \preceq_S x$, therefore $x = f(u) = f(w)$, consequently $u = w$, and $x = f(u) = f(w)$ is the only intersection of the $f(u)f(v)$ and $f(w)f(t)$ paths in S. If v, t are comparable in $\preceq_T$, then without loss of generality $v \preceq_T t$, and $v = v \wedge t \preceq_T w$ by the special property, which implies that $x \preceq_S f(v \wedge t) = f(v) \preceq_S f(w) \preceq_S x$, so $x = f(v) = f(w)$, $v = w$ and $f(v) = f(w)$ is the only intersection point of the $f(u)f(v)$ and $f(w)f(t)$ paths.
(iii) is obvious as mapping the single vertex anywhere yields a decent embedding. □

Theorem 29. *The relation $T \trianglelefteq S$ among rooted trees is WQO.*

Proof. For contradiction, assume it is not WQO. Then by Lemma 19, there is a minimal bad sequence $T_0, T_1, T_2, \ldots$. In view of the proposition above, $|V(T_i)| \geq 2$. As rooted trees are built recursively, T_i has a root r_i, and r_i is connected to roots of smaller rooted trees,

$T_{i,0}, T_{i,1}, ..., T_{i,n_i-1}$, where $n_i \geq 1$. Observe that $T_{i,j} \triangleleft T_i$ (strict inequality), as the natural embedding of $T_{i,j}$ into T_i, is decent. By Lemma 22, $Y = \left\{ T_{i,j} : i \geq 0, 0 \leq j \leq n_i \right\}$ is WQO, as it is non-empty. By Higman's theorem, $(Y^{<\omega}, \trianglelefteq)$ is also WQO. Consider the following sequence of words in $(Y^{<\omega}, \trianglelefteq)$: $\overline{F_0}, \overline{F_1}, \overline{F_2}, ...$, where $\overline{F_i} = (T_{i,0}, T_{i,1}, ..., T_{i,n_i-1})$. By Theorem 15(ii), there is an $i < k$ such that

$$\overline{F_i} = (T_{i,0}, T_{i,1}, ..., T_{i,n_i-1}) \trianglelefteq (T_{k,0}, T_{k,1}, ..., T_{k,n_k-1}) = \overline{F_k}.$$

This means the existence of a monotone increasing injection $g : \{0, 1, ..., n_i - 1\} \to \{0, 1, ..., n_k - 1\}$ and decent embeddings $f_j : T_{i,j} \to T_{i,g(j)}$ for $j = 0, 1, ..., n_i - 1$. The following $f : V(T_i) \to V(T_k)$ is a decent embedding, contradicting the badness of the minimal bad sequence $T_0, T_1, T_2, ...$:

$$f(x) = \begin{cases} r_k & \text{if } x = r_i \\ f_j(x) & \text{if } x \in V(T_{i,j}). \end{cases}$$

□