# Ramsey Theory

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## 1 König's Lemma

König's Lemma is a basic tool to move between finite and infinite combinatorics. To be concise, we use the notation $[k] = \{1, 2, ..., k\}$, and $[X]^r$ will denote the family of all $r$-element subsets of $X$.

**Lemma 1** *Assume that we have family tree with the following properties:*

- *For every natural number $n$, there is a finite and nonempty set of vertices, the $n^{th}$ generation.*
- *For every natural number $n \geq 1$, every vertex in the $n^{th}$ generation is joined by a unique edge to a vertex in the $(n-1)^{th}$ generation. (It is called his father, and we also say that he is a child of his father.)*
- *There is a single vertex in the $0^{th}$ generation.*

*Then there is an infinite path starting from the $0^{th}$ generation, in other words, a sequence of vertices $a_0, a_1, ....$ such that $a_i$ belongs to the $i^{th}$ generation and $a_i$ is the father of $a_{i+1}$ for all $i = 0, 1, 2, ....$*

Define recursively *descendants* of any vertex $x$ in the family tree: (i) everybody is a descendant of $x$, whose father is $x$; (ii) if $x$ is the father of $y$ and $z$ is an already defined descendant of $y$, then $z$ is a descendant of $x$. Now $a_0$ from the $0^{th}$ generation has infinitely many descendants but only a finite number of children. Therefore he must have a child $a_1$ who has a infinitely many descendants. If you already defined $a_0, a_1, ..., a_i$ such that $a_i$ has infinitely many descendants, you can define a child $a_{i+1}$ that has infinitely many descendants. So we defined the sequence $a_0, a_1, ....$ by induction.

Recall that a (proper) $k$-coloration of a graph $G$ is a function $c : V(G) \to [k]$, such that $xy \in E(G)$ implies $c(x) \neq c(y)$. It is well-known (and not difficult to see) that a finite graph is 2-colorable if and only if it has no odd cycles.

**Theorem 2** Erdős-de Bruijn Theorem *(assuming the Axiom of Choice) For a natural number $k$, a graph $G$ is properly $k$-colorable if and only if all its finite subgraphs are properly $k$-colorable.*

If $G$ is properly $k$-colored, then obviously the same coloration is proper if we restrict it for any finite subgraph. The other direction is the one that requires proof. Here we give a proof only for the case of *countably infinite* $V(G)$. Assume that all finite subgraphs can be colored with colors from $[k]$. Assume $V(G) = \{v_1, v_2, ...\}$ as it is countably infinite. Define a family tree in the following way. Let the $n^{th}$ generation contain all proper colorations of the subgraph of $G$ induced by $\{v_1, v_2, ..., v_n\}$. By our assumptions the $n^{th}$ generation is nonempty and finite. Define a coloration of the vertices $\{v_1, v_2, ..., v_n\}$ to be the father of a coloration of the vertices of $\{v_1, v_2, ..., v_n, v_{n+1}\}$, if disregarding vertex $v_{n+1}$ the colorations are identical. The $0^{th}$ generation has one element, $\emptyset$. König's lemma applies and gives a proper coloration $c_n : \{v_1, v_2, ..., v_n\} \to [k]$, such that $c_n$ is the father of $c_{n+1}$. Consider the coloration $c : V(G) \to [k]$ defined by $c(v_n) = c_n(v_n)$. This is a proper coloration of $G$. Indeed, if $\{v_i, v_j\}$ is an edge of $G$, then without loss of generality $i < j$, $c_j(v_i) \neq c_j(v_j)$ as $c_j$ was proper. Furthermore, $c(v_i) = c_i(v_i) = c_j(v_i) \neq c_j(v_j) = c(v_j)$.

# 2 Infinite Ramsey Theorem

**Theorem 3** *For all $k, r$ positive integers, for all infinite set $X$ and for all $f : [X]^r \to [k]$ coloration, there exists a $Y \subseteq X$ infinite set, such that $f|_{[Y]^r}$ is constant.*

The proof goes by induction on $r$. For $r = 1$ — identifying one-element subsets with their elements — we color the infinite set $X$ with finitely many colors. One color must be used infinitely many times. Let $Y$ be the set of elements getting this color. Assume now that $r \geq 2$ and the hypothesis that the theorem holds for $r - 1$ instead of $r$. We are going to define by induction for $n = 1, 2, ...$ an infinite subset $V_n \subseteq X$ and $v_n \in V_n$. The base case is $V_1 = X$ and $v_1 \in V_1$ arbitrary. If $V_n \subseteq X$ and $v_n \in V_n$ are already defined, consider the coloration $f^* : [V_n \setminus \{v_n\}]^{r-1} \to [k]$ defined by $f^*(A) = f(A \cup \{v_n\})$. As $V_n \subseteq X$ was infinite, so is $V_n \setminus \{v_n\}$, and the induction hypothesis applies to this set and the coloration $f^*$. We have a $Y' \subseteq V_n \setminus \{v_n\}$ infinite set, such that $f^*|_{[Y']^r}$ is constant. Set $V_{n+1} = Y'$ and pick $v_{n+1} \in V_{n+1}$ arbitrary. Define $col(v_n) = f^*(A)$ where $A \subseteq V_{n+1} = Y'$ and $|A| = r - 1$. Clearly $col(v_n)$ is independent of the choice of $A$. Note $col(v_n) \in [k]$. As $v_n$ is an infinite sequence, there is an infinite subsequence of it repeating the same value, say $t \in [k]$. Namely $t = col(v_{i_1}) = col(v_{i_2}) = col(v_{i_3}) = \cdots$. Define now $Y = \{v_{i_1}, v_{i_2}, v_{i_3}, ...\}$, an infinite set. We claim that $f|_{[Y]^r} = t$, a constant function. Indeed, if $B \in [Y]^r$, then $B = \{v_{i_{j_1}}, v_{i_{j_2}}, v_{i_{j_3}}, ..., v_{i_{j_r}}\}$ for some $i_{j_1} < i_{j_2} < i_{j_3} < ... < i_{j_r}$. By the construction $\{i_{j_2}, i_{j_3}, ..., i_{j_r}\} \subseteq V_{i_{j_1}+1}$ and therefore $col(B) = f_{[Y]^r} = t$.

Note that Infinite Ramsey Theorem is the best possible in two senses.

(1) We cannot allow colorations using using a countably infinite set of colors with the same conclusion. Namely, if $X$ is countably infinite, then so is $[X]^r$, and therefore each and every $r$-element set can get a different color.

(2) For a fixed natural $k$ and a countably infinite $X$, we cannot change $[X]^r$ to the set of infinite subsets of $X$ and keep the same conclusion. We make a counterexample for $k = 2$. The counterexample $G'$ below will be a 2-coloration of the infinite subsets of a countably infinite set, such that every infinite subset contains infinite subsets of different colors. Consider the following graph: vertices of $G$ are subsets of countably infinite $X$, and subsets $A, B$ are joined by an edge if the symmetric difference $A \triangle B$ has exactly one element. (If $X$ were finite, this would be nothing else

but the graph called *hypercube*.) It is easy to see that $G$ has no odd cycles. Its finite subgraphs have no odd cycles either, therefore the finite subgraphs are all 2-colorable. By the Erdős-de Bruijn theorem $G$ is 2-colorable. Delete from $G$ the vertices that correspond to finite subsets of $X$. The remaining graph $G'$ is still 2-colorable. A 2-coloration of $G'$ is a 2-coloration $c$ of all infinite subsets of $X$. Assume that there is a $Y \subseteq X$ such that every infinite subset of $Y$ gets the same color, as the conclusion would require. Take a $y \in Y$. Note that $Y \setminus \{y\}$ is still infinite, and $(Y \setminus \{y\}) \triangle Y = \{y\}$. Therefore $c(Y) \neq c(Y \setminus \{y\})$, a contradiction.

# 3 Finite Ramsey Theorem

**Theorem 4** *For all $k, r, a_1, ..., a_k$ positive integers, there exists an $n_0$ positive integer, such that for all $n \geq n_0$ positive integers, for every set $X$ with $|X| = n$, for all $f : [X]^r \to [k]$ colorations, there exists a $Y \subseteq X$ and an $i$ ($1 \leq i \leq k$) such that $|Y| \geq a_i$ and $f|_{[Y]^r} = i$ constant.*

Note that if an $n_0$ has the required property, all bigger numbers have it as well. The smallest such $n_0$ number is defined as the *Ramsey number* $R_k^r(a_1, ..., a_k)$. Also, $|Y| \geq a_i$ can be changed to $|Y| = a_i$ with no effect on $n_0$. Furthermore, "*there exists an $n_0$ positive integer, such that for all $n \geq n_0$ positive integers*" can be changed to "*there exists an $n$ positive integer, such that*", as the existence of such an $n$ immediately imply that bigger numbers also have the same property. A further equivalent to Theorem 4 is:

**Theorem 5** *For all $k, r, p$ positive integers, there exists an $m$ positive integer, such that for all $f : [m]^r \to [k]$ colorations, there exists a $Y \subseteq [m]$ such that $|Y| \geq p$ and $f_{[Y]^r} = constant$.*

Theorem 5 follows from Theorem 4 by plugging in $X \leftarrow [m]$ and $a_i \leftarrow p$. Theorem 4 follows from Theorem 5 by plugging in $p \leftarrow \max\{a_1, a_2, ..., a_k\}$ and noting that truth of the theorem depends only on the size of $X$, not on the set.

We show that Theorem 3 implies Theorem 4. Assume that Theorem 4 is false. That means that for every $n$, there is an $f_n : [n]^r \to [k]$ coloration such that for every $Y \subseteq [n]^r$ and every $i$ ($1 \leq i \leq k$), $|Y| = a_i$ implies that $f_n|_{[Y]^r}$ is not constant $i$. Make the $n^{th}$ generation of a family tree of such colorations. Clearly there are only a finite number of them. We say that a coloration $f_n$ is the father of the coloration $f_{n+1}$, if $f_{n+1}|_{[n]^r} = f_n$. The only element of the $0^{th}$ generation is $\emptyset$. The conditions of König's Lemma hold, therefore there is an infinite sequence of colorations, $f_0, f_1, f_2, ...$, such that every element in this sequence is an extension of the previous one. For $A = \{i_1 < i_2 < \cdots < i_r\} \in [\mathbb{N}]^r$, define $f(A) = f_{i_r}(A)$, where $\mathbb{N} = \{0, 1, 2, ...\}$. Now $f : [\mathbb{N}]^r \to [k]$ still has the property that for every $Y \subseteq [\mathbb{N}]^r$ and every $i$ ($1 \leq i \leq k$), $|Y| = a_i$ implies that $f|_{[Y]^r}$ is not constant $i$. Were $f|_{[Y]^r}$ constant $i$, with $t = \max Y$, $f_t|_{[Y]^r} = f|_{[Y]^r}$ would be also constant $i$, contradicting the fact that $f_t$ was a counterexample. The existence of such $f$ contradicts Theorem 3.

**Theorem 6 (Paris-Harrington)** *Consider the statement of Theorem 5 with the following added to the conclusion: $|Y| > \min Y$.*
*"For all $k, r, p$ positive integers, there exists an $m$ positive integer, such that for all $f : [m]^r \to [k]$*

*colorations, there exists a $Y \subseteq [m]$ such that $|Y| \geq p$ and $f_{[Y]^r} =$ constant and $|Y| > \min Y$."*
*Then*
*(i) This statement can be proved from Theorem 3 using König's lemma.*
*(ii) This statement can be formulated in the Peano arithmetics, but cannot be proved within the Peano arithmetics.*

We prove (i) only. Assume that Theorem 5 fails for every $m$ with the added requirement $|Y| > \min Y$ for some coloration $f_m : [m]^r \to [k]$. Then, for every $Y \subseteq [m]$ with $|Y| \geq p$, either $f_m|_{[Y]^r}$ is not constant or $|Y| \leq \min Y$. Build a family tree of these $f_m$ functions like above, and define $f : [N]^r \to [k]$ like above. $f$ still has the property that for every finite $Y \subseteq [N]^r$ with $|Y| \geq p$, $f|_{[Y]^r}$ is not constant or $|Y| \leq \min Y$. (If not, for $m = \max Y$ the function $f_m$ was not a counterexample.)

By Theorem 3, there is an infinite $Y'$ such that $f|_{[Y']^r}$ is constant. Select $Y$ to be the first $1 + \min Y'$ elements of $Y'$. Then $\min Y = \min Y'$. Now $|Y| > \min Y$. Observe that the coloration $f_m$ for $m = \max Y$ did not fail Theorem 5 with the added requirement.

# 4 Applications

**Theorem 7** *For every sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers, there is an infinite strictly increasing subsequence or there is an infinite decreasing subsequence.*

Color the 2-subsets of $N$ with two colors, assuming the order $i < j$:

$$col(\{i,j\}) = \begin{cases} Red, & \text{if } a_i < a_j \\ Blue, & \text{if } a_i \geq a_j. \end{cases}$$

An infinite subset in which every pair is colored Red provides the first outcome, and an infinite subset in which every pair is colored Blue provides the second outcome.

Recall some plane geometry. We consider points such that no 3 of them are collinear. 4 points form a convex 4-gon, if its convex hull has 4 vertices. $n$ points form a convex $n$-gon if their convex hull has $n$ vertices. The following fact is needed: $n$ points form a convex $n$-gon if and only if any 4 of the $n$ points form a convex 4-gon. An easy exercise is to check that among 5 points some 4 always form a convex 4-gon.

**Theorem 8** *For every $n$ there is an $m$ such that among any $m$ points in the plane, such that no 3 are collinear, one finds $n$ points that form a convex $n$-gon.*

We show that $m = R_2^4(n, 5)$ suffices. We color 4-subsets of the $m$ points with colors "convex" and "not convex". By the choice of $m$, we have $n$ points such that every 4-subset of them is colored "convex" (and hence they from a convex $n$-gon), or we have 5 points such that every 4 of them does not from a convex 4-gon. By the exercise mentioned before the theorem, the second outcome is impossible. Therefore we must have the first outcome.

**Theorem 9 (Schur)** *For every positive integer $m$ there exists a positive integer $n$ such that for every coloration $f : [n] \to [m]$, there exists $1 \leq x, y, z \leq n$ integers, such that $x + y = z$ and $x, y, z$ monochromatic (i.e. $f(x) = f(y) = f(z)$.)*

We show that $n = R^2_m(3,3,...,3)$ suffices. Take an arbitrary $f : [n] \to [m]$ coloration. Color 2-element subsets of $[n]$ by the rule $f^*(\{i,j\}) = f(|i-j|) \in [m]$. By the choice of $n$, there is a 3-element subset $\{a,b,c\} \subseteq [n]$ such that every 2-element subset of it gets the same color, in other words $f^*(\{a,b\}) = f^*(\{c,b\}) = f^*(\{a,c\})$. We may assume without loss of generality $a < b < c$. By the definition of $f^*$ the equations above boil down to $f(b-a) = f(c-b) = f(c-a)$. Take $x = b-a, y = c-b, z = c-a$.

This theorem has a nice consequence about Fermat's Last Theorem in a modular version. First recall the following fact from number theory: for any prime number $p$, the non-zero residue classes mod $p$ form a multiplicative group, and this multiplicative group is cyclic (i.e. can be generated by a single element). In other words, there is number $g$ (called a *primitive root* mod $p$) such that for any $x \in \{1,2,...,p-1\}$ there exists $i_x \in \{1,2,...,p-1\}$ (called the *index* of $x$) such that $g^{i_x} \equiv x$ mod $p$.

**Theorem 10 (Schur)** *For every positive integer $m$ there exists an integer $p_0$, such that for all primes $p$ with $p > p_0$, the equation $x^m + y^m \equiv z^m$ mod $p$ has a solution, for which $xyz \not\equiv 0$ mod $p$.*

We show that $p_0 = R^2_m(3,3,...,3)$ suffices. Assume $p > p_0$ is a prime and that $g$ is a primitive root mod $p$. Define a coloration $c : [p-1] \to [m]$ by

$$c(x) = j \Leftrightarrow i_x \equiv j \bmod m.$$

By the first Schur Theorem, there exist $1 \le x, y, z \le p-1$ such that $c(x) = c(y) = c(z)$ and $x + y = z$. Assume that the common color is $j$. Then $i_x = \ell_1 m + j$, $i_y = \ell_2 m + j$, $i_z = \ell_3 m + j$. By the definition of the index,

$$g^{\ell_1 m + j} \equiv x \bmod p,$$
$$g^{\ell_2 m + j} \equiv y \bmod p,$$
$$g^{\ell_3 m + j} \equiv z \bmod p.$$

Combining the three congruences above with $x + y = z$, we conclude

$$g^{\ell_1 m + j} + g^{\ell_2 m + j} \equiv g^{\ell_3 m + j} \bmod p.$$

We can divide through by the congruence class of $g^j$, as it is non-zero, to obtain

$$\left(g^{\ell_1}\right)^m + \left(g^{\ell_2}\right)^m \equiv \left(g^{\ell_3}\right)^m \bmod p.$$

# 5 Higman's Theorem

We consider finite length words over a finite alphabet. The empty word has no letters at all. Two words are identical if they contain the same words in the same order. A word is a *subword* of another word, if we can get it by eliminating some letters. For example, CLUE is a subword of CLUTTER, but ULCER is not. We denote this by CLUE $\sqsubseteq$ CLUTTER.

**Theorem 11** *Let us be given infinitely many finite length words over a finite alphabet.*
*(i) Then there are words $u, w$ among them, such that $u \sqsubseteq w$.*
*(ii) There is an infinite sequence of $u_1 \sqsubseteq u_2 \sqsubseteq \cdots$ among our words.*

First observe that conclusions (i) and (ii) are equivalent to each other. Indeed, (ii) trivially implies (i). For the other direction, assume (i). Let $W$ denote our set of words. We 2-color $[W]^2$ in the following way: the color of the set $\{u, v\}$ is "comparable" if one is a subword of the other, and "non-comparable" otherwise. From the infinite Ramsey theorem, there is an infinite $Y \subset W$ such that every 2-element subset of $Y$ has the same color. This color cannot be "non-comparable" by (i). Therefore this color is "comparable". The words in $Y$ must have different length, therefore they can be put in an increasing sequence as required in (ii).
**Proof 1.** We prove the theorem by induction on the size of of the alphabet. The conclusion is clearly true for a 1-letter alphabet. So assume that we already know the conclusion for alphabets that have fewer letters than ours. Assume now that $W$ is an infinite set of words such that (i) fails. Fix an arbitrary word $w \in W$ and write it as $\ell_1\ell_2....\ell_k$. As (i) holds if $w$ is the empty word, we may assume $k \geq 1$ and that the empty word is not in $W$ at all. Take an arbitrary $u \in W$, as (i) fails, $w \not\sqsubseteq u$. This allows the following *factorization* of $w$

$$w = [...]_1\ell_1[...]_2\ell_2[...]_3 \cdots \ell_i[...]_{i+1}, \tag{5.1}$$

where $[...]_1$ does not contain letter $\ell_1$, $[...]_2$ does not contain letter $\ell_2$, ... , and $[...]_{i+1}$ does not contain letter $\ell_{i+1}$; and $0 \leq i \leq k-1$. (For example, if $u$ =CONTRIVE and $w$ =COMBINATORICA, the factorization is []C[]O[MBI]N[A]T[O]R[]I[CA].) The *type* of a factorization like (5.1) is $i$. There are $k$ possible types. As $W \setminus \{u\}$ is infinite, there is an infinite $W' \subseteq W$, in which every word has the same type of factorization, say $i$.

Consider now the fragments $[...]_1$ that you see when factorizing all $w \in W'$ by $u$. There are two possibilities: either you find finitely many fragments or infinitely many fragments. If there are finitely many fragments, then one of them, say $f_1$ arises from every word in an infinite $W_1 \subseteq W'$. If there are infinitely many fragments (note that they do not use $\ell_1$, so they can be described in smaller alphabet), there is an infinite chain of the fragments that we find in $[...]_1$, and there is an infinite $W_1 \subseteq W'$ that realizes these fragments in their factorizations. If $w$ had type $i = 0$, these fragments are the whole words from $W_1$, and we proved (ii)! If $i > 0$, we go further. Using induction, for $j = 1, 2, ..., i+1$, we define an infinite $W_j \subseteq W'$ such that one of the following alternatives hold:

- every $w \in W_j$ provides the very same fragments on each of $[...]_1$, ... , $[...]_j$.
- factorizing every $w \in W_j$ by $u$, the initial segments $[...]_1\ell_1[...]_2\ell_2[...]_3 \cdots \ell_{j-1}[...]_j$ of their factorizations make an increasing sequence of subwords.

Note that we already provided above $W_1$ for this alternative in the case of $j = 1$, this is the base case of the induction.
Assume that we already defined $W_j$ according the alternatives. If $j = i+1$, then we are done: the first alternative provides a contradiction by showing $|W_j| = 1$, while the second alternative provides (ii). Hence $j < i+1$, and we are going to define $W_{j+1}$. We consider cases.

If words from $W_j$ create only finitely many different fragments $[...]_{j+1}$, then one fragment belongs to an infinite subset of them. Call this infinite subset $W_{j+1}$. $W_{j+1}$ satisfies the same of the two alternatives that $W_j$ satisfied.
If words from $W_j$ create infinitely many different fragments in $[...]_{j+1}$, as those fragments are free of the letter $\ell_{j+1}$, using the inductive hypothesis for the alphabet size provides an infinite $W'_j \subset W_j$, such that their fragments on $[...]_{j+1}$ make an infinite chain of subwords $y_1 \sqsubseteq y_2 \sqsubseteq \cdots$. If $W_j$ satisfied the first alternative, then $W_{j+1} \leftarrow W'_j$ satisfies the second alternative. If $W_j$ satisfied the second alternative, then say the initial segments $[...]_1\ell_1[...]_2\ell_2[...]_3 \cdots [...]_j$ of the factorization of words of $W'_j$ reads like $z_1 \sqsubseteq z_2 \sqsubseteq \cdots$. Combining them, the factorization of every element of $W'_j$ reads like $z_p\ell_j y_q$, for some $p, q$ indices. If a $z_p$ occurs with infinitely many $q$ values, select the words from $W'_j$ realizing them to define $W_{j+1}$. Do similarly, if some $y_q$ occurs with infinitely many $p$ values. If every $p$ occurs only with finitely many $q$'s and every $q$ occurs with finitely many $p$'s, select a $p_1, q_1$ pair that occurs. Now there are only finitely many words in $W'_j$ with the form $z_p\ell_j y_q$ with $p \le p_1$ or $q \le q_1$. Hence we can select a $p_2, q_2$ pair that occurs with $p_2 > p_1$ and $q_2 > q_1$, and similarly for every $n$, a $p_n, q_n$ pair that occurs, with $p_n > p_{n-1}$ and $q_n > q_{n-1}$. Define $W_{j+1}$ as the infinite subset of words from $W'_j$ that reads like $z_{p_n}\ell_j y_{q_n}$ on $[...]_1\ell_1[...]_2\ell_2[...]_3 \cdots [...]_j\ell_j[...]_{j+1}$.
This definition maintains the validity of the alternatives, so the proof is completed.

**Proof 2.** Assume that Higman's theorem fails conclusion (i). Then there are counterexamples showing it. We define the *type* of a counterexample by a sequence $n_1 \le n_2 \le \cdots$, listing the sequence lengths of words in the counterexample in increasing order. (Note that any word length can occur only finitely many times, due to the finite alphabet size. Also, $n_1 > 0$, as the empty word is a subword of any word.) We are going to show that there is a counterexample whose type is lexicographically minimal.

Let $n_1^*$ denote the smallest $n_1$ that occurs in countexamples. If $n_1^*, n_2^*, ..., n_i^*$ are already defined, define $n_{i+1}^*$ to be the the smallest $n_{i+1}$ among counterexamples, for which $n_j = n_j^*$ for $j = 1, 2, , , , , i$. This way we defined a sequence of positive integers $n_i^*$ for all $i \ge 1$. Next we show that there exists a counterexample of type $n_i^* : i \ge 1$. We do this with König's lemma. The $i^{th}$ generation will contain ordered $i$-tuples of words $(w_1, w_2, ..., w_i)$, such that there is a counterexample to (i) which contains all these words, and in addition, the length of $w_j = n_j^*$ for $j = 1, 2, ..., i$. By our assumptions, all generations are finite and non-empty. An ordered $i$-tuple is the father of an ordered $(i+1)$-tuple, if we get it from the ordered $(i+1)$-tuple by deleting $w_{i+1}$. The $0^{th}$ generation is the 0-tuple, a single entry. König's lemma guarantees an infinite path: $w_i : i \ge 0$. If one word of these would be a subword of the other, say at generations $p < q$, then this already happens among $w_i : q \ge i \ge 0$ contradicting $(w_1, w_2, ..., w_q)$ being listed in the $q^{th}$ generation.

Consider now a counterexample $w_1, w_2, ...,$ in which $w_i$ has length $n_i^*$. There should be a letter $\ell$ such that infinitely many words start with letter $\ell$. Let those words be $\ell u_1$, $\ell u_2$, .... . Assume that the words before $\ell u_1$ are $w_1, w_2, ..., w_n$. Consider now the infinite sequence of words $w_1, w_2, ..., w_n, u_1, u_2, .....$ Clearly there cannot be a subset relation between $u$-words, or between $w$-words. A $u$-word cannot coincide with a $w$-word either. The $w$'s cannot be subwords of $u$'s. So the only possibility is that a $u$-word is shorter by one than a $w$-word and is a proper subword of it. Without loss of generality we may assume that they are $w_n$ and $u_1$. (Changing the order to achieve this does not change the sequence of lengths of these words.) Eliminate $w_n$ from the sequence, and repeat this elimination step if necessary. After finitely many such steps we arrive at counterexample that is lexicographically smaller than our choice with lengths $n_i^*$.

# 6 Bounds on Ramsey numbers

Clearly $R_k^1(a_1, ..., a_k) = (a_1 - 1) + (a_2 - 1) + ... + (a_k - 1) + 1$, and this is essentially the well-known Pigeonhole Principle. It is obvious that $R_1^2(a) = a$.

**Theorem 12** *For any $k \geq 2, a_1, a_2, ..., a_k$ positive integers,*

$$R_k^2(a_1, a_2, ..., a_k) \leq \binom{(a_1 - 1) + (a_2 - 1) + ... + (a_k - 1)}{a_1 - 1, a_2 - 1, ..., a_k - 1}. \tag{6.2}$$

Note that if any $a_i = 1$, $R_k^2(a_1, a_2, ..., a_k) = 1$, and the RHS is at least 1. Take now for some fixed $k \geq 2$, and prove the theorem by induction on $a_1 + a_2 + ... + a_k$. For the base case, the theorem holds if the sum is at most $2k - 1$. Assume that for all $i$, $a_i \geq 2$, and we have a complete graph $K_n$ with edges colored with $1, 2, ..., k$, such that there are no $a_i$ vertices with all their pairs colored with color $i$, for any $i$. Fix a vertex $v$ and consider $N_i(v) = \{u : colour(\{u, v\}) = i\}$. Clearly $N_i(v)$ cannot have $a_i - 1$ vertices such that every pair among them is colored $i$; and for every $j \neq i$, it cannot have $a_j$ vertices such that all of its pairs are colored $j$. Therefore the induction hypothesis applies and

$$|N_i(v)| \leq R_k^2(a_1, a_2, ..., a_i - 1, ..., a_k) - 1 \leq \binom{(a_1 - 1) + (a_2 - 1) + ... + (a_i - 2) + ... + (a_k - 1)}{a_1 - 1, a_2 - 1, ..., a_i - 2, ..., a_k - 1} - 1.$$

We conclude

$$n \leq 1 + \sum_{i=1}^{k} \left[ \binom{(a_1 - 1) + (a_2 - 1) + ... + (a_i - 2) + ... + (a_k - 1)}{a_1 - 1, a_2 - 1, ..., a_i - 2, ..., a_k - 1} - 1 \right]$$

$$= \binom{(a_1 - 1) + (a_2 - 1) + ... + (a_k - 1)}{a_1 - 1, a_2 - 1, ..., a_k - 1} - (k - 1),$$

and therefore 1+ the RHS of the last equation is an upper bound for $R_k^2(a_1, a_2, ..., a_k)$. (Note that the summation identity for multinomial coefficients used above is the same that we needed to prove Menon's Theorem.)

**Corollary 13**

$$R_2^2(a, b) \leq \binom{a + b - 2}{a - 1}, \qquad R_2^2(a, a) \leq \binom{2a - 2}{a - 1} < 4^a.$$

**Theorem 14** *For $a$ sufficiently large, $2^{\frac{a}{2}} < R_2^2(a, a)$.*

**Proof.** Consider red-blue edge colorations of $K_n$. There are $2^{\binom{n}{2}}$ red-blue edge colorations. The number of edge colorations that have all edges red among some $a$ vertices is at most $\binom{n}{a}2^{\binom{n}{2}-\binom{a}{2}}$. Similarly, the number of edge colorations that have all edges blue among some $a$ vertices is at most $\binom{n}{a}2^{\binom{n}{2}-\binom{a}{2}}$. Hence, if

$$2^{\binom{n}{2}} > 2\binom{n}{a}2^{\binom{n}{2}-\binom{a}{2}}, \tag{6.3}$$

then some edge colorations have neither $a$ vertices with all edges red among them nor $a$ vertices with all edges blue among them, hence $n < R_2^2(a,a)$. Clearly (6.3) is equivalent to

$$2^{\binom{a}{2}-1} > \binom{n}{a}. \tag{6.4}$$

It is easy to see that for $a$ sufficiently large,

$$2^{\binom{a}{2}-1} > \binom{\lceil 2^{\frac{a}{2}} \rceil}{a},$$

as needed.

Paul Erdős offered \$ 500 for proving that $\lim_{a\to\infty}\Big(R_2^2(a,a)\Big)^{\frac{1}{a}}$ exists, and \$ 100 for the proof that it does not exist.

**Theorem 15** *For any $r \geq 3, k \geq 2, a_1, a_2, ..., a_k$ positive integers,*

$$R_k^r(a_1,a_2,...,a_k) \leq 1 + R_k^{r-1}\Bigg(R_k^r(a_1-1,a_2,...,a_k),...,R_k^r(a_1,a_2,...,a_i-1,...,a_k),...,R_k^r(a_1,a_2,...,a_k-1)\Bigg).$$

*This theorem, combined with Theorem 12 gives recursive upper bounds on Ramsey numbers. (For initialization of the recursion, if an argument $a_i$ is 1, take the corrresponding Ramsey number 1.)*

**Proof.** Assume that we have a set $X$ with a coloration $f : [X]^r \to [k]$ that does not have any $a_i$-size subset with all $r$-subsets colored $i$. Consider an arbitrary $x \in X$ and color $r-1$-subsets of $X \setminus \{x\}$ by $f^*(A) = f(A \cup \{x\})$. If $X \setminus \{x\}$ had an $R_k^r(a_1,a_2,...,a_i-1,...,a_k)$-size subset $Y$, in which every $r-1$-subset is colored $i$, then either we find within $Y$ an $a_j$-size subset ($j \neq i$), such that all its $r$-subsets are colored $j$, contradicting the definition of $f$, or find an $a_i - 1$-size subset such that all its $r$-subsets are colored $i$. Adding $x$ to this set, we contradict again the definition of $f$. The only way to escape the contradiction is the inequality

$$|X \setminus \{x\}| < R_k^{r-1}\Bigg(R_k^r(a_1-1,a_2,...,a_k),...,R_k^r(a_1,a_2,...,a_i-1,...,a_k),...,R_k^r(a_1,a_2,...,a_k-1)\Bigg).$$

Note that the proof of Theorem 15 also works for $r = 2$, the proof of Theorem 12 recurses in the same way to $r = 1$. The proof of Theorem 15 is the "finitized" version of the proof of the Infinite Ramsey Theorem, Theorem 3.

Regarding the growth of Ramsey numbers, we note without proof that for $r > 1, k > 1$, the following bound holds:

$$R_k^{r+1}(a, a, ..., a) \leq k^{[R_k^r(a,a,...,a)]^r}.$$

From this—with some algebra—one can derive that $R_k^r(a, a, ..., a)$ has an upper bound in the form of an iterated power, where $r$ levels of 2's are are followed by $a \cdot k^{r^2}$.

# 7 The chromatic number of $\mathbb{R}^d$

It is a long-standing open problem what is the minimum number of colors $\chi(\mathbb{R}^d)$ needed to color the points of $\mathbb{R}^d$ to avoid monochromatic points at unit distance apart. This problem is known as *the chromatic number of* $\mathbb{R}^d$. The explanation of this terminology is that one can make graph, whose vertices are points of $\mathbb{R}^d$, and edges are pairs of points at unit distance apart, and we really talk about the chromatic number of this graph. As this number is easily seen to be finite, the Erdős—de Bruijn Theorem applies, and the chromatic number is already needed to color properly some finite subset of $\mathbb{R}^d$. Even for $d = 2$, after 70 years of research, the best bounds were $4 \leq \chi(\mathbb{R}^2) \leq 7$. Very recently Aubrey de Grey improved the lower bound to 5. In view of the Erdős-de Bruijn Theorem, the value of $\chi(\mathbb{R}^d)$ may depend on which axioms we are willing to use. There are exponential lower and upper bounds for $\chi(\mathbb{R}^d)$.
*Definition.* A finite $S \subset \mathbb{R}^n$ is an $(M, D, \delta)$-critical configuration, if $|S| = M$, and among any $D + 1$ points of $S$, there are two, whose distance is $\delta$. It is clear that scaling $S$ up or down by a factor $c$, we obtain a configuration $cS$, which is $(M, D, c\delta)$-critical.

**Theorem 16** *Suppose there is a finite $S \subset \mathbb{R}^n$, which is an $(M, D, 1)$-critical configuration. Then*
*(i) $\chi(\mathbb{R}^d) \geq M/D$.*
*(ii) Assume $X \subseteq \mathbb{R}^n$ is Lebesgue measurable, and no two points of $X$ are at unit distance apart. Then*

$$\limsup_{r\to\infty} \frac{\mu(B(r, 0) \cap X)}{\mu(B(r, 0))} \leq D/M,$$

*where $B(r, 0)$ is the ball of radius $r$ around the origin, and $\mu$ is the Lebesgue measure.*

Part (i) follows from the fact that the finite subgraph of the unit distance graph induced by the vertices of $S$ has independence number at most $D$.
To obtain part (ii), fix $B(r, 0)$, and mark a point $s \in S$. Translate $S$ in all possible ways into $t(S)$, such that the translate $t(s)$ of $s$ falls into $B(r, 0)$, and integrate $|t(S) \cap X|$ for those translation. The integral is $M\mu(B(r, 0) \cap X)$ and a small error term, depending on the boundary, while $D\mu(B(r, 0) \cap X)$ is an upper bound for the integral.

Assume that there exists an $(M, D, 1)$ critical configuration in $\mathbb{R}^n$, and that $\mathbb{R}^n = V_1 \cup V_2 \cup ... \cup V_m$ is a partition into Lebesgue measurable sets, with $m < M/D$. There must be a $V_i$ partition class, for which $\limsup_{r\to\infty} \frac{\mu(B(r,0)\cap V_i)}{\mu(B(r,0))} \geq 1/m > D/M$. By part (ii) of the theorem, this $V_i$ partition class contains two points at unit distance apart. By the same argument, for every $0 \leq d$, this $V_i$ partition class contains two points at distance $d$ apart. Surprisingly, Lebesgue measurability of the partition classes is not needed to this conclusion. Larman and Rogers proved the following theorem, generalizing a particular result of Raiskii:

**Theorem 17** *Assume $n \geq 2$ and that there exists an $(M, D, 1)$ critical configuration $S$ in $\mathbb{R}^n$, and that $\mathbb{R}^n = V_1 \cup V_2 \cup ... \cup V_k$ is a partition with $k < M/D$. Then there exists a class $V_i$, such that for every $0 \leq d$, this $V_i$ partition class contains two points at distance $d$ apart.*

To prove the theorem, we need a lemma:

**Lemma 18** *Assume $n \geq 2$ and $S_1, S_2, ..., S_k$ finite point configurations in $\mathbb{R}^n$. Then there are configurations $R_1, R_2, ..., R_k$ such that*
*(i) for every $i$, there is an isometry of $\mathbb{R}^n$ that maps $S_i$ to $R_i$.*
*(ii) the vector sums $\left\{\mathbf{v}_1 + \ldots + \mathbf{v}_k : \mathbf{v}_i \in R_i \text{ for all } 1 \leq i \leq k\right\}$ are all distinct.*

*Proof of the theorem.* Assume that the theorem fails, and for every partition class $V_i$ there is distance $\alpha_i$ not present between points of $V_i$. Now $S_i = \alpha_i S$ is an $(M, D, \alpha_i)$ critical configuration. Take $R_i$ according the lemma, $R_i$ is also $(M, D, \alpha_i)$ critical. Consider the set $P = \left\{\mathbf{v}_1 + \ldots + \mathbf{v}_k : \mathbf{v}_i \in R_i \text{ for all } 1 \leq i \leq k\right\}$. Using the lemma, $|P| = M^k$. On the other hand, $|P \cap V_i| \leq M^{k-1}D$, as $P$ is the union of $M^{k-1}$ translates of $R_i$. Hence $M^k = |P| \leq kM^{k-1}D$, a contradiction.
*Proof of the Lemma.* We may assume without loss of generality that the $S_i$ configurations are pairwise disjoint and do not contain the origin $O$. First we prove the lemma for $n = 2$ by induction on $k$. The induction step is obvious, the base case $k = 2$ needs proof. Let $S_1 = \{\mathbf{a}_1, ..., \mathbf{a}_\ell\}$ and $S_2 = \{\mathbf{b}_1, ..., \mathbf{b}_m\}$. Let $R_\phi$ denote the rotation with angle $\phi$ around $O$. We want to find an $R_\phi$ such that the vectors pointing to the points of $R_1 = S_1$, $R_2 = R_\phi(S_2)$ have $\ell m$ distinct sums. We have to avoid equalities $\mathbf{a}_i + R_\phi(\mathbf{b}_p) = \mathbf{a}_j + R_\phi(\mathbf{b}_q)$ for $i \neq j$ and $p \neq q$. Observe that the equality is equivalent to $\mathbf{a}_i - \mathbf{a}_j = R_\phi(\mathbf{b}_q) - R_\phi(\mathbf{b}_p) = R_\phi(\mathbf{b}_q - \mathbf{b}_p)$. In other words, an angle $\phi$ is needed, such that the $m(m-1)$ $R_\phi$-images of the $\mathbf{b}_q - \mathbf{b}_p$ vectors avoid the $\ell(\ell - 1)$ vectors of the form $\mathbf{a}_i - \mathbf{a}_j$. Only finitely many $\phi$ angles are to be avoided.
We claim that for $n \geq 3$, we can write $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$, such that the orthogonal projections of the points of $S_1, S_2, ..., S_k$ to $\mathbb{R}^2$ are all distinct and none of them is the origin. (We are going to make orthogonal projections of the configuration into a space of one lower dimension $n-2$ times, reaching dimension 2. The composition of these projections is what we wanted. First observe that there are finitely many connecting lines of the points of $S_1, S_2, ..., S_k$. Find a line not parallel to any of those, and project the points to an $(n-1)$-dimensional subspace orthogonal to the line. The projection is injective on our points. Repeat this for the images in the lower dimensional subspace, until we have distinct projected images of $S_1, S_2, ..., S_k$ in $\mathbb{R}^2$.)
Let the projection of $S_i$ to $\mathbb{R}^2$ be $p(S_i)$. Assume that isometries $\psi_i$ of $\mathbb{R}^2$ map $p(S_i)$ to $T_i$, such that the vector sums $\{\mathbf{u}_1 + \ldots + \mathbf{u}_k : \mathbf{u}_i \in T_i \text{ for all } 1 \leq i \leq k\}$ are all distinct. It is easy to see that

$$R_i = \left\{\Big(\psi_i(p(\mathbf{v})), \mathbf{v} - p(\mathbf{v})\Big) : \mathbf{v} \in S_i\right\}$$

is as required. (Indeed, $R_i$ is isomorphic to $S_i$.)

# 8 Geometric Ramsey theorems

Two configurations are *congruent*, if there is a distance preserving transformation of the space that maps one of them into the other. The exercises showed many instances, when coloring the plane or the space with a given number of colors yields a finite points set, whihc is (1) congruent to a prescribed configuration, (2) is monochromatic in the coloration i.e. all of its points have the same color.

*Definition.* A finite point set $R \subset \mathbb{R}^m$ is called *Ramsey*, if for all $k$ positive integer, there exists a $d(k)$, such that for all $f : \mathbb{R}^{d(k)} \to [k]$ coloration, there exists a monochromatic copy of $R$ in $\mathbb{R}^{d(k)}$. Under a *copy* we understand the following: there is a distance preserving embedding of $i : \mathbb{R}^m \to \mathbb{R}^{d(k)}$ such that $f$ is constant on $i(R)$. (It is clear that if a number passes for $d(k)$, then all bigger numbers pass as well.)

For example, two points in $\mathbb{R}^1$ at distance $a$ have the Ramsey property. Indeed, $d(k) = k$ suffices. The reason is that there is a configuration in $\mathbb{R}^k$ a with $k+1$ points, such that any pair of them has distance $a$, called *regular simplex*. Call a point configuration *spheric*, if there is point in the space, from which all points of the configuration have the same distance.

One can easily construct the regular simplex in $\mathbb{R}^d$ with induction on $d$ starting from the base case $d = 1$. It is easy to show that any two regular simplices with $k$ points and distance $a$ are congruent, and that regular simplices are spheric.

**Theorem 19** *The four vertices of a rectangle $R$ with sides $a, b$ are Ramsey.*

Set $N = k^{k+1}$. We are going to show that the $d(k) = N + k$ suffices. Let us be given an arbitrary coloration $f : \mathbb{R}^{N+k} \to [k]$.

Let $v_0, v_1, ..., v_N$ be the vertices of a regular simplex in $\mathbb{R}^N$ with pairwise distance $a$ and let $u_0, u_1, ..., u_k$ be the vertices of a regular simplex in $\mathbb{R}^k$ with pairwise distance $b$. Consider $v_i$ as an ordered $N$-tuple of real numbers and $u_j$ as an ordered $k$-tuple of real numbers using coordinates. We focus on just $(N + 1)(k + 1)$ points in $\mathbb{R}^{N+k}$ and find the four vertices of a monochromatic rectangle among them. The point $(v_i, u_j) \in \mathbb{R}^{N+k}$ is coordinatized by $N + k$ real numbers, where the first $N$ are the coordinates of $v_i$, while the last $k$ are the coordinates of $u_j$. As $f$ colors $\mathbb{R}^{N+k}$, we can assign to $i$ $(0 \le i \le N)$ the color vector

$$\Big(f(v_i, u_0), f(v_i, u_1), ..., f(v_i, u_k)\Big).$$

This vector has $k+1$ entries, each of them is one of $k$ colors. There are at most $k^{k+1}$ such vectors. As $i$ ranges over $N+1 = k^{k+1}+1$ values, by pigeonhole principle, there should be $0 \le i_1 < i_2 \le N$, such that their color vectors agree in every component:

$$\text{for all } j \ f(v_{i_1}, u_j) = f(v_{i_2}, u_j).$$

Consider the $f(v_{i_1}, u_j)$ colors for $j = 0, 1, ..., k$. As we have $k$ colors and $j$ ranges over $k + 1$ values, by the pigeonhole principle there are $0 \le j_1 < j_2 \le k$, such that

$$f(v_{i_1}, u_{j_1}) = f(v_{i_1}, u_{j_2}).$$

Comparing this with the next to last displayed formula,

$$f(v_{i_1}, u_{j_1}) = f(v_{i_1}, u_{j_2}) = f(v_{i_2}, u_{j_1}) = f(v_{i_2}, u_{j_2}),$$

the four points that we needed. This is a rectangle indeed, as the $u$-coordinate and the $v$-coordinate are orthogonal to each other in $\mathbb{R}^{N+k}$.

Next we show that any non-rectangle parallelogram is *not Ramsey*. Without loss of generality assume that the side vectors are $\mathbf{a}, \mathbf{b}$, and that $\mathbf{a} \cdot \mathbf{b} = 1$. (To achieve this we only need to change the scale.) For every $n$, we give a 4-coloration of $\mathbb{R}^n$ that does not have monochromatic copy of this parallelogram $R$. This coloration will be $\mathbf{u} \mapsto \lfloor \mathbf{u} \cdot \mathbf{u} \rfloor$ modulo 4, where the modulo expresses an operation, resulting in one of the mod 4 resudue classes, rather than a relation.

First observe that any $R'$ copy of this parallelogram $R$ in $\mathbb{R}^n$ would look like $\mathbf{w}, \mathbf{w}+\mathbf{a}', \mathbf{w}+\mathbf{b}', \mathbf{w}+\mathbf{a}' + \mathbf{b}'$, for some $\mathbf{a}', \mathbf{b}'$ vectors, such that $\mathbf{a}' \cdot \mathbf{b}' = 1$. Assume now that we have a monochromatic $R'$:

$$\lfloor \mathbf{w} \cdot \mathbf{w} \rfloor \equiv \lfloor (\mathbf{w} + \mathbf{a}') \cdot (\mathbf{w} + \mathbf{a}') \rfloor \equiv \lfloor (\mathbf{w} + \mathbf{b}') \cdot (\mathbf{w} + \mathbf{b}') \rfloor \equiv \lfloor (\mathbf{w} + \mathbf{a}' + \mathbf{b}') \cdot (\mathbf{w} + \mathbf{a}' + \mathbf{b}') \rfloor \bmod 4.$$

Hence

$$\lfloor \mathbf{w} \cdot \mathbf{w} \rfloor - \lfloor (\mathbf{w} + \mathbf{a}') \cdot (\mathbf{w} + \mathbf{a}') \rfloor - \lfloor (\mathbf{w} + \mathbf{b}') \cdot (\mathbf{w} + \mathbf{b}') \rfloor + \lfloor (\mathbf{w} + \mathbf{a}' + \mathbf{b}') \cdot (\mathbf{w} + \mathbf{a}' + \mathbf{b}') \rfloor \equiv 0 \bmod 4.$$

In other words, the value of the LHS expression is $4k$ for some integer $k$. Drop now the floors in the LHS. As there are two positive and two negative terms with floor functions, the value of the LHS increases or decreases by *strictly less than two*. Expanding the LHS without floors, one obtains $2\mathbf{a}' \cdot \mathbf{b}' = 2$. This number cannot be reached from $4k$ by moving less than 2.

The essential problem with the paralellogram was that the parallelogram is not spheric. One can show that a Ramsey configuration must be spheric, and there is a long-standing conjecture that every finite point set on any sphere is Ramsey ($ 1000). It is not even known whether all 4-element subsets of a circular arc are Ramsey ($ 100).

# 9 A compactness theorem

The following theorem can easily be proved from Tychonoff's Compactness Theorem:

**Theorem 20** *Let $X$ be a set and $\mathcal{F}$ be a set of some finite subsets of $X$. Let $r$ be a fixed positive integer. The following facts are equivalent:*
*(i) For all $f : X \to [r]$ colorations, there is a monochromatic element $F \in \mathcal{F}$;*
*(ii) There is a finite $Y \subset X$ such that for all $f : Y \to [r]$ colorations, there is an $F \in \mathcal{F}$ with $F \subseteq Y$, which is monochromatic.*

The Erdős—de Bruijn Theorem is a special case of this theorem: $X \leftarrow V(G)$, $[X]^2 \leftarrow E(G)$. The theorem also implies that if a configuration $R$ is Ramsey, then it is enough to color a certain finite subset of the space and still have a guarantee to obtain a monochromatic copy of $R$.

If $K_1 \subset \mathbb{R}^a$ and $K_2 \subset \mathbb{R}^b$, then the product configuration is defined as

$$K_1 * K_2 = \{(x, y) : x \in K_1, y \in K_2\} \subset \mathbb{R}^{a+b},$$

using the coordinatized notation from the proof of Theorem 19. It is not difficult to see that if $K_1$ is congruent to $K_1'$ and $K_2$ is congruent to $K_2'$, then $K_1 * K_2$ is congruent to $K_1' * K_2'$, verifying that the definition of the product configuration is independent of the coordinatization of the factors, i.e., the product is well defined.

Using this compactness theorem, we can show that the product configuration of two Ramsey configurations is also Ramsey. The proof largely repeats the line of arguments that showed rectangles were Ramsey. A corollary of this result is that hypercubes, bricks, and their subsets are Ramsey.

**Theorem 21** *If $K_1 \subset \mathbb{R}^a$ and $K_2 \subset \mathbb{R}^b$ are Ramsey, then so is $K_1 * K_2$.*

Fix $r > 0$, the number of colors, and show that in a sufficiently large dimensional space, for any $r$-coloration of points, there is a monochromatic copy of $K_1 * K_2$. Let $u$ be such an integer, that for every $r$-coloration of points of $\mathbb{R}^u$ there is a monochromatic copy of $K_1$. By the Compactness Theorem, there is a finite $T \subset \mathbb{R}^u$, such such that for every $r$-coloration of points of $T$ there is a monochromatic copy of $K_1$ in $T$. Assume that $T$ has $t$ elements and $T = \{x_1, x_2, ..., x_t\}$.

As $K_2$ is Ramsey, there is an integer $v$ such that every $r^t$-coloration of $\mathbb{R}^v$ has a monochromatic copy of $K_2$.

Next we show that every $r$-coloration $f : \mathbb{R}^{u+v} \to [r]$ has a monochromatic copy of $K_1 * K_2$. Define

$$f' : \mathbb{R}^v \to [r^t] \text{ by } f'(y) = \Bigg( f(x_1 * y), f(x_2 * y), ..., f(x_t * y), \Bigg).$$

There must be a monochromatic copy $K_2'$ of $K_2$ in $\mathbb{R}^v$ for the coloration $f'$.

Define a coloration

$$f'' : T \to [r] \text{ by } f''(x_i) = f(x_i * y) \text{ for some } y \in K_2'.$$

Note that the color $f''(x_i)$ does not depend on which $y \in K_2'$ was used to define it. There must be a monochromatic copy $K_1'$ of $K_1$ in $T$ for the coloration $f''$. Now $K_1' * K_2'$ is monochromatic for $f$.

There is an issue in quantum mechanics that is closely related to Euclidean Ramsey theorems. A photon has an attribute called *spin*. Spin is a 3-dimensional vector of magnitute $\sqrt{2}$. Regarding the components of the spin, if one measures it in 3 pairwise orthogonal directions, one always observes a $+1$, a $-1$, and a 0. Measurement in orthogonal directions can be done one after the other. However, quantum mechanics say, if you make a measurement in a direction not orthogonal to all those measured before, you disturb the photon and it changes its spin. Furthermore, the spin (and location) of the photon comes from a probability distribution, and only the measurement forces on it a certain spin and location.

Make an experiment of thought. Is it possible that the photon has in a deterministic way well-defined components of the spin in every direction in the 3-space, before the measurement happens? This means an assignment of $+1$, $-1$, or 0 values to every point of the unit sphere (vectors from the origin to points of the unit sphere identify directions) such that every 3 pairwise orthogonal directions show exactly one of a $+1$, a $-1$, and a 0.

This experiment of thought is called the *Kochen—Specker paradox* as such a coloration does not exist. To simplify the problem, call 0 Red, and $\pm 1$ Blue.

**Theorem 22** *One cannot color the points of the sphere with Red and Blue, such that in every 3 orthogonal direction one finds 2 Blue and 1 Red point.*

There is a 117 point configuration on the sphere that cannot be colored in this way. (A slight generalization of the compactness theorem applies here: if the sphere cannot be colored in this way, then some finite configuration cannot be colored.)

Here we show Peter Cameron's simple proof to the Kochen—Specker paradox. The proof goes by contradiction, assume that such a Red-Blue coloration exists.

(i) first observe that antipodal points in the sphere must have the same color.

(ii) for every Blue point, visualize this point as a pole, and consider the corresponding equator. On this equator, there must 2 Red points at 45° angle looking from the center of the sphere. Proof: consider 5 points on the equator, following each other at 45° angle looking from the center of the sphere, $P_1, P_2, P_3, P_4, P_5$. $P_2$ and $P_4$ have different colors. If $P_3$ is Red, we are done. If $P_3$ is Blue, then $P_1$ and $P_5$ are Red, and either $P_1, P_2$ are both Red or $P_3, P_4$ are both Red.

(iii) Red poles have Blue equators.

(iv) on every Blue point, two Blue main circles pass through that cross at 45°. Proof: use for the poles of the two main circles the Red points guaranteed by (ii).

(v) if a third main circle intersects the two Blue main circles in (iv) in two points that are at 90° from the center, then this third main circle is all Blue. Proof: the pole if this main circle must be Red.

(vi) there is an $\epsilon > 0$, such that the $\epsilon$-radius cap around every Blue point is Blue. Proof: let the Blue point be $P$. The two Blue main circles passing through $P$ by (iv) define 4 spherical lunes. For each of the lunes, we find points $Q_1$, $Q_2$ of the two Blue main circles at 90° angle from the center. This puts a Blue main circle arc into the lune, connecting the two Blue boundary main circles. This Blue main circle arc can be moved over $P$, while we do change its length and keep its endpoints on the two Blue boundary main circles. Doing this for each of the 4 lunes, we swept with Blue main circle arcs a neighborhood of $P$. $\epsilon$ does not even depend on $P$.

(vii) all points of the sphere are Blue. Proof: if there is a Red and a Blue point, a main circle connects them. By (vi), all points of this main circle must be Blue.