Math 142: Taylor Series Proof Example

To show that a function has a power series expansion, it is generally easier to show that it is equal to its Taylor Series expansion. Let $T_n(x)$ be the degree n partial Taylor series of $f(x)$ centered at a. That is, $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{n!}(x-a)^k$. Next, let $R_n(x) = f(x) - T_n(x)$ so that $f(x) = T_n(x) + R_n(x)$. We'll now need two theorems to show this for most functions.

Theorem 1: If $\lim_{n\to\infty} R_n(x) = 0$, for $|x-a| < R$, then $f(x)$ is equal to its Taylor series expansion on the interval $|x-a| < R$. ■

Theorem (Taylor's Inequality): If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$.

Example: Prove that e^x is equal to its Taylor series expansion, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. First, note that if $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$. Let d be any positive number, and assume that $|x| \leq d$. Then $\left|f^{(n+1)}(x)\right| = e^x \leq e^d$. So, using $a = 0$ and $M = e^d$, by Taylor's Inequality, $|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$. Thus, $\lim_{n\to\infty} |R_n| \leq \lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \frac{|x|^{n+1}}{(n+1)!} = 0$. By the squeeze theorem, this means that $\lim_{n\to\infty} R_n(x) = 0$ for all values of x (since d was arbitrary), so by Theorem 1, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

Below are the proofs of the above theorems, included for reference.

Proof of Theorem 1: Consider $\lim_{n\to\infty} f(x)$ for $|x-a| < R$. By our setup, $\lim_{n\to\infty} f(x) = \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} T_n(x) + 0$. Now, $\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{n!}(x-a)^k$, which is equal to the Taylor expansion of $f(x)$ by the definition of a convergent series. ■

Proof of Taylor's Inequality: Since $\left|f^{(n+1)}(x)\right| \leq M$, we have that $-M \leq f^{(n+1)}(x) \leq M$. We'll use $f^{(n+1)}(x) \leq M$ for right now, and we'll assume that $a \leq x \leq a+d$ ($a-d \leq x \leq a$) is similar. With this in mind,

$$\int_a^x f^{(n+1)}(t)\ dt \leq \int_a^x M\ dt$$
$$f^{(n)}(x) - f^{(n)}(a) \leq M(x-a)$$
$$f^{(n)}(x) \leq f^{(n)}(a) + M(x-a)$$

Now, we integrate again:

$$\int_a^x f^{(n)}(t)\ dt \leq \int_a^x f^{(n)}(a) + M(x-a)\ dt$$

$$f^{(n-1)}(x) - f^{(n-1)}(a) \leq f^{(n)}(a)(x-a) + M\frac{(x-a)^2}{2}$$

$$f^{(n-1)}(x) \leq f^{(n-1)}(a) + f^{(n)}(a)(x-a) + M\frac{(x-a)^2}{2}$$

Continuing this process, we have:

$$f^{(n-2)}(x) \leq f^{(n-2)}(a) + f^{(n-1)}(a)(x-a) + f^{(n)}(a)\frac{(x-a)^2}{2} + M\frac{(x-a)^3}{3!}$$

$$f^{(n-3)}(x) \leq f^{(n-3)}(a) + f^{(n-2)}(a)(x-a) + f^{(n-1)}(a)\frac{(x-a)^2}{2} + f^{(n)}(a)\frac{(x-a)^3}{3!} + M\frac{(x-a)^4}{4!}$$

$$\vdots$$

$$f'(x) \leq f'(a) + f''(a)(x-a) + f'''(a)\frac{(x-a)^2}{2} + ... + f^{(n)}(a)\frac{(x-a)^{n-1}}{(n-1)!} + M\frac{(x-a)^n}{n!}$$

$$f(x) \leq f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + ... + f^{(n)}(a)\frac{(x-a)^n}{n!} + M\frac{(x-a)^{n+1}}{(n+1)!}$$

So, we have that $f(x) - f(a) - f'(a)(x-a) - f''(a)\frac{(x-a)^2}{2} - ... - f^{(n)}(a)\frac{(x-a)^n}{n!} \leq M\frac{(x-a)^{n+1}}{(n+1)!}$, which is the same as $f(x) - T_n(x) \leq \frac{M}{(n+1)!}(x-a)^{n+1}$, and finally, $R_n(x) \leq \frac{M}{(n+1)!}(x-a)^{n+1}$. By a similar argument, using $f^{(n+1)}(x) \geq -M$, we can show that $R_n(x) \leq \frac{-M}{(n+1)!}(x-a)^{n+1}$. Thus, $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$. ■