Math 142: Series Test Proofs

Theorem: (The Monotone Convergence Theorem) If a_n is a decreasing sequence that is bounded below, then it converges. Similarly, is a_n is increasing and bounded above, then it converges.

Proof: Suppose a_n is decreasing and bounded below. Let $\epsilon > 0$, and consider the greatest lower bound L of the sequence (this exists by the completeness axiom). Then by definition of greatest lower bound, $L + \epsilon$ is not a lower bound of a_n. Let N be the smallest value such that $a_N < L + \epsilon$. Then since a_n is decreasing, we know that $a_n < L + \epsilon$ for all $n \geq N$. Finally, this says that $a_n - L < \epsilon$ for all $n \geq N$, and since L is a lower bound of a_n, we know that $a_n - L \geq 0$. Thus, $|a_n - L| < \epsilon$, so $\lim_{n\to\infty} a_n = L$ by definition.

Suppose a_n is increasing and bounded below. The proof is identical, except this time we let L be the least upper bound of the sequence, note that $L - \epsilon$ is not an upper bound of a_n, and find an N such that $a_n > L - \epsilon$ for all $n \geq N$. Since $L - a_n \geq 0$, we get that $|a_n - L| < \epsilon$. ■

Theorem: (Geometric Series) The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1-r}$ when $|r| < 1$ and diverges when $|r| \geq 1$.

Proof: First, we'll get an expression for s_n:

$$\begin{aligned} s_n &= a + ar + ar^2 + ar^3 + ... + ar^{n-1} \\ rs_n &= \qquad ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n \end{aligned}$$

Subtracting these two equations, we get that $s_n - rs_n = a - ar^n$, so $s_n(1-r) = a(1-r^n)$, and finally, we get an expression for s_n: $\frac{a(1-r^n)}{1-r}$. We now proceed to take the limit of s_n.

If $|r| < 1$, $\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$, so it converges to $\frac{a}{1-r}$.
If $|r| > 1$, $\lim_{n\to\infty} r^n$ diverges, so s_n diverges and hence the series diverges.

If $r = 1$, then the series is simply $\sum_{n=1}^{\infty} a = a + a + a + ...$, which diverges.

If $r = -1$, then the series is simply $\sum_{n=1}^{\infty} a(-1)^{n-1} = a - a + a - a + ...$, which diverges. ■

Theorem: (The Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: We'll prove the contrapositive: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Notice that $a_n = s_n - s_{n-1}$, where s_n is the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} a_n$ converges, $s_n \to s$. Clearly, this means that $s_{n-1} \to s$ as well. So, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$.
■

Theorem: (Constant Multiples of Series) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} ca_n$ converges to $c\sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ca_n$ diverges.

Proof: Let s_n be the partial sums of $\sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} a_n$ converges, then say $s_n \to s$. The n^{th} partial sum of $\sum_{n=1}^{\infty} ca_n$ is $ca_1 + ca_2 + ... + ca_n = c(a_1 + a_2 + ... + a_n) = cs_n$. So, $\lim_{n\to\infty} cs_n = c \lim_{n\to\infty} s_n = cs$. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} s_n$ diverges. Thus, $\lim_{n\to\infty} cs_n$ diverges, so $\sum_{n=1}^{\infty} ca_n$ diverges. ■

Theorem: (Sum of Series) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. If <u>one</u> of $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges. Finally, if $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} b_n$ both diverge to ∞ or both diverge to $-\infty$, then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges to the same value.

Proof: Let s_n be the partial sums of $\sum_{n=1}^{\infty} a_n$ and t_n be the partial sums of $\sum_{n=1}^{\infty} b_n$. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then say $s_n \to s$ and $t_n \to t$. The n^{th} partial sum of $\sum_{n=1}^{\infty} (a_n + b_n)$ is $(a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n) = (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n) = s_n + t_n$. So $\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t$. If one of $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} b_n$ diverges, say without loss of generality that s_n diverges and $t_n \to t$. Then $\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + t$. This limit diverges, so $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges. Finally, if $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} b_n$ both diverge to ∞, then $\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = \infty$. Thus, $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges to ∞, and the $-\infty$ case is identical. ■

Note 1: $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} (a_n + (-b_n))$, so we can treat differences in the same manner as the theorem above. In particular, if both series converge, then $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

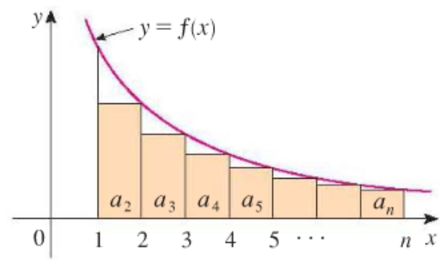
Note 2: All other cases not covered in this theorem must be treated with further analysis, and they cannot be split easily.

Theorem: (The Integral Test) Suppose a_n is a positive decreasing sequence and $f(x)$ is a continuous function such that $f(n) = a_n$. Then,

(1) If $\int_1^\infty f(x)\ dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

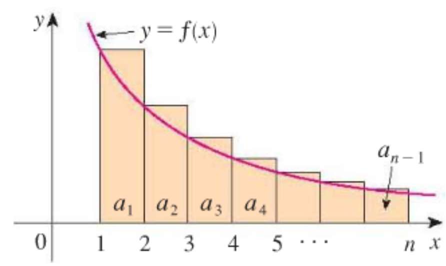
(2) If $\int_1^\infty f(x)\ dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: (1) Consider the following figure:



It is clear from the figure that $\sum_{k=2}^{n} a_k < \int_1^n f(x)\ dx$. If $\int_1^\infty f(x)\ dx$ is convergent, then $\sum_{k=2}^{n} a_k < \int_1^n f(x)\ dx \leq \int_1^\infty f(x)\ dx$ since $f(x) \geq 0$. So, $s_n = a_1 + \sum_{k=2}^{n} a_k \leq a_1 + \int_1^\infty f(x)\ dx$. Let $a_1 + \int_1^\infty f(x)\ dx = M$, which gives that $s_n \leq M$ for all n. Thus, $\{s_n\}$ is bounded above. Also, $s_{n+1} = s_n + a_{n+1} \geq s_n$ since $a_{n+1} = f(n+1) \geq 0$. Thus, by the Monotone convergence theorem, $\{s_n\}$ converges, so $\sum_{n=1}^{\infty} a_n$ converges.

(2) Consider the following figure:



It is clear from the figure that $\int_1^n f(x)\ dx \leq \sum_{k=1}^{n-1} a_k$. If $\int_1^\infty f(x)\ dx$ is divergent, then $\int_1^n f(x)\ dx \to \infty$ as $n \to \infty$ because $f(x) \geq 0$. So, $\int_1^n f(x)\ dx \leq \sum_{k=1}^{n-1} a_k = s_{n-1}$ implies that $s_{n-1} \to \infty$ as $n \to \infty$.

Therefore, $s_n \to \infty$ as $n \to \infty$, so $\sum_{n=1}^{\infty} a_n$ diverges. ■

Theorem: (P-Series Test) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges with $p \leq 1$.

Proof: When $p = 0$, we have the series $\sum_{n=1}^{\infty} 1$, which is obviously divergent. When $p < 0$, the terms $\frac{1}{n^p}$ are increasing, so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is again divergent. When $p > 0$, $\frac{1}{n^p}$ is positive and decreasing, so we'll apply the Integral Test. If $p = 1$, then $\int_1^\infty \frac{1}{x}\ dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\ dx = \lim_{t\to\infty} \ln x|_1^t = \lim_{t\to\infty} \ln t = \infty$. Thus, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

If $p > 0$ and $p \neq 1$, then we have $\int_1^\infty \frac{1}{x^p}\ dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\ dx = \lim_{t\to\infty} \left. \frac{x^{1-p}}{1-p} \right|_1^t = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$. So, if $\lim_{t\to\infty} t^{1-p}$ converges, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, and if $\lim_{t\to\infty} t^{1-p}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

If $0 < p < 1$, then $1 - p > 0$, and hence $\lim_{t\to\infty} t^{1-p}$ diverges. This completes the proof that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $p \leq 1$. Finally, if $p > 1$, then $1-p < 0$. So, $\lim_{t\to\infty} t^{1-p} = 0$, and hence, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$. ■

Theorem: (The Comparison Test) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. Then

(1) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: Let $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$, $t = \sum_{n=1}^{\infty} b_n$.

(1) Since both series have positive terms, $\{s_n\}$ and $\{t_n\}$ are increasing. ($s_{n+1} = s_n + a_{n+1} \geq s_n$). Also, $t_n \to t$ as $n \to \infty$, so $t_n \leq t$ for all n. Since $a_n \leq b_n$ for all n, we have $s_n \leq t_n$ for all n. Thus, $s_n \leq t$ for all n. So, by the Monotone Convergence Theorem, s_n converges, so $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\sum_{n=1}^{\infty} b_n$ is divergent, then $t_n \to \infty$ as $n \to \infty$ (since $\{t_n\}$ is increasing). But $a_n \geq b_n$ for all n, so $s_n \geq t_n$ for all n. Thus, $s_n \to \infty$ as $n \to \infty$, so $\sum_{n=1}^{\infty} a_n$ diverges. ■

Theorem: (The Limit Comparison Test) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$ and c is finite, then either both series converge or both series diverge.

Proof: Let $0 < \epsilon < c$. Then by definition of $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, there exists an $N > 0$ such that $\left|\frac{a_n}{b_n} - c\right| < \epsilon$ when $n > N$. Rewriting this, we have:

$$\begin{aligned}
\left|\frac{a_n}{b_n} - c\right| &< \epsilon \\
-\epsilon < \frac{a_n}{b_n} - c &< \epsilon \\
c - \epsilon < \frac{a_n}{b_n} &< c + \epsilon \\
(c - \epsilon)\, b_n < a_n &< (c + \epsilon)\, b_n
\end{aligned}$$

Let $m = c - \epsilon > 0$, $M = c + \epsilon > 0$. Then we have that $mb_n < a_n < Mb_n$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} Mb_n$ converges, so by the Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} mb_n$ diverges, so by the Comparison Test, $\sum_{n=1}^{\infty} a_n$ diverges.■

Theorem: (The Alternating Series Test) If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - ...$ ($b_n > 0$) satisfies (1) $b_{n+1} \leq b_n$ for all n (b_n decreasing) and (2) $\lim_{n\to\infty} b_n = 0$, then the series converges.

Proof: First consider the even partial sums: $s_2 = b_1 - b_2 \geq 0$ (since $b_2 \leq b_1$), $s_4 = s_2 + (b_3 - b_4) \geq s_2$ (since $b_4 \leq b_3$), and in general, $s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2}$ (since $b_{2n} \leq b_{2n-1}$). So, we have that $0 \leq s_2 \leq s_4 \leq ... \leq s_{2n} \leq ...$, which tells us that the even partial sums are positive and increasing. Also, $s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - ... - (b_{2n-2} - b_{2n-1}) - b_{2n}$. b_{2n} and every term in parentheses is positive, so $s_{2n} \leq b_1$ for all n. Thus, the sequence $\{s_{2n}\}$ of even partial sums is bounded above, so by the Monotone Convergence Theorem, $\{s_{2n}\}$ converges. Let $\lim_{n\to\infty} s_{2n} = s$.

Now, looking at the odd partial sums, $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + b_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1}$. Now, since $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} b_{2n+1} = 0$. Thus, we have that $\lim_{n\to\infty} s_{2n+1} = s$, so $\lim_{n\to\infty} s_n = s$. Finally, by definition, $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges. ■

Theorem: (The Absolute Convergence Test) If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof: First, note that $a_n \leq |a_n|$ for all n. So, $0 \leq a_n + |a_n| \leq 2|a_n|$ ($|a_n|$ is either a_n or $-a_n$, so $a_n + |a_n| \geq 0$ for all n). Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, $\sum_{n=1}^{\infty} (a_n + |a_n|)$ is convergent by the Comparison Test. Finally, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$, so since both of these converge, $\sum_{n=1}^{\infty} a_n$ converges. ■

Theorem: (The Ratio Test) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive, meaning we cannot determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges using this test.

Proof: Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, and suppose that $L < 1$. Let r be a value such that $L < r < 1$ (r is like $L + \epsilon$). Then by the definition of a limit, there is an $N > 0$ such that for all $n \geq N$, $\left|\frac{a_{n+1}}{a_n}\right| < r$. Equivalently, $|a_{n+1}| < |a_n| r$. This gives us a few facts:

$$\begin{aligned}
&|a_{N+1}| < |a_N| r \\
&|a_{N+2}| < |a_{N+1}| r < |a_N| r^2 \\
&|a_{N+3}| < |a_{N+2}| r < |a_N| r^3 \\
&\qquad \vdots \\
&|a_{N+k}| < |a_N| r^k \text{ for all } k \geq 1
\end{aligned}$$

So, using the comparison test,

$$\begin{aligned}\sum_{n=1}^{\infty}|a_n| &= \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n| \\ &= \sum_{n=1}^{N}|a_n| + \sum_{k=1}^{\infty}|a_{N+k}| \\ &\leq \sum_{n=1}^{N}|a_n| + \sum_{k=1}^{\infty}|a_N|\, r^k\end{aligned}$$

Since $0 < r < 1$, $\sum_{k=1}^{\infty}|a_N|\, r^k$ converges, and hence $\sum_{n=1}^{N}|a_n| + \sum_{k=1}^{\infty}|a_N|\, r^k$ converges. Thus, by the comparison test, $\sum_{n=1}^{\infty}|a_n|$ converges, so $\sum_{n=1}^{\infty}a_n$ converges absolutely.

Suppose that $L > 1$. Then there exists an $N > 0$ such that for all $n \geq N$, $\left|\frac{a_{n+1}}{a_n}\right| > 1$ (follows from letting $L < \epsilon < 1$ and using the definition). Thus $|a_{n+1}| > |a_n|$, so $\lim_{n\to\infty} a_n$ is either infinite or does not exist. Therefore, $\sum_{n=1}^{\infty}a_n$ diverges by the Divergence Test. ■

Theorem: (The Root Test) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=1}^{\infty}a_n$ is absolutely convergent. If $\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$, then the series $\sum_{n=1}^{\infty}a_n$ is divergent. If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, then the test is inconclusive, meaning we cannot determine whether $\sum_{n=1}^{\infty}a_n$ converges or diverges using this test.

Proof: Let $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, and suppose that $L < 1$. Let r be a value such that $L < r < 1$. Then by the definition of a limit, there is an $N > 0$ such that for all $n \geq N$, $\sqrt[n]{|a_n|} < r$. Equivalently, $|a_n| < r^n$. So, using the comparison test,

$$\begin{aligned}\sum_{n=1}^{\infty}|a_n| &= \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n| \\ &< \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}r^n\end{aligned}$$

Since $0 < r < 1$, $\sum_{n=N+1}^{\infty}r^n$ converges, and hence $\sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}r^n$ converges. Thus, by the comparison test, $\sum_{n=1}^{\infty}|a_n|$ converges, so $\sum_{n=1}^{\infty}a_n$ converges absolutely.

Suppose that $L > 1$. Then there exists an $N > 0$ such that for all $n \geq N$, $\sqrt[n]{|a_n|} > 1$. Thus $|a_n| > 1$, so $\lim_{n\to\infty} a_n \neq 0$ or does not exist. Therefore, $\sum_{n=1}^{\infty}a_n$ diverges by the Divergence Test. ■