Course Announcement

Course:	Math 748 Cohomology of vector bundles and syzygies
Semester:	Spring 2018
Instructor:	Andy Kustin
Potential audience:	If the course sounds interesting to you, then you will learn something from it. There are some very intimidating topics lurking nearby as well as some very accessible topics. Learn as much as you can.
Textbook:	"Cohomology of vector bundles and syzygies" by Jerzy Weyman Cambridge University Press (2003)

Let $R = \boldsymbol{k}[x_1,\dots,x_n]$ be a polynomial ring with n variables over a field $\boldsymbol{k}$ and M be a finitely generated graded R-module (for example, M could be an ideal I of R which is generated by homogeneous polynomials or M could be a quotient ring R/I where again I is an ideal generated by homogeneous polynomials). If one wants to "understand" M, one might want a minimal generating set for M. This would be a set of homogeneous elements $m_1,\dots,m_{\beta_0}$ in M so that every element in M can be written "in terms" of $m_1,\dots,m_{\beta_0}$ (and none of the m_i can be omitted). In particular, every element in M has the form $\sum_{i=1}^{\beta_0} r_i m_i$ for some r_i in R. As soon as one has a minimal generating set for M, the next natural question is "How can I tell when two elements of M are the same?" That is, one wants to know the set

$$\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_0} \end{bmatrix} \in R^{\beta_0} \,\middle|\, \sum_{i=1}^{\beta_0} r_i m_i = 0 \right\}. \tag{1}$$

The set (1) is another finitely generated graded R-module, called the first syzygy module $\operatorname{Syz}_1(M)$ of M. Repeat the above process. If one wants to "understand" $\operatorname{Syz}_1(M)$, one might want a minimal generating set $X_1,\dots,X_{\beta_1}$ for $\operatorname{Syz}_1(M)$ and then one would want to know the relations on $X_1,\dots,X_{\beta_1}$. The set of relations

$$\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_1} \end{bmatrix} \in R^{\beta_1} \,\middle|\, \sum_{i=1}^{\beta_1} r_i X_i = 0 \right\}$$

on $X_1,\dots,X_{\beta_1}$ is called the second syzygy module $\operatorname{Syz}_2(M)$ of M. One continues in this manner to find $\operatorname{Syz}_i(M)$ for all i.

The syzygy modules of M are uniquely determined up to isomorphism. The Hilbert syzygy theorem guarantees that $\mathrm{Syz}_i(M) = 0$ for $n+1 \le i$. The numbers $\{\beta_i\}$ are called the Betti numbers of M. The collection of free modules and induced maps

$$\cdots \to R^{\beta_2} \to R^{\beta_1} \xrightarrow{\begin{bmatrix} X_1 & \dots & X_{\beta_1} \end{bmatrix}} R^{\beta_0}$$

is called the minimal free resolution of M.

In order to get the most mileage out of the Betti numbers and syzygies it is necessary to have the Betti numbers reflect information about the degrees involved. I'll do this with an example. In this example ℓ_* represents polynomials of degree 1, q_* represents polynomials of degree two, c_* represents polynomials of degree three, and f is a polynomial of degree four. Suppose the minimal homogeneous resolution of M looks like

$$0 \to R(-6)^3 \xrightarrow{\begin{bmatrix} \ell_{1,1} & \ell_{1,2} & \ell_{1,3} \\ \ell_{2,1} & \ell_{2,2} & \ell_{2,3} \\ \ell_{3,1} & \ell_{3,2} & \ell_{3,3} \\ \ell_{4,1} & \ell_{4,2} & \ell_{4,3} \\ \ell_{5,1} & \ell_{5,2} & \ell_{5,3} \\ \ell_{6,1} & \ell_{6,2} & \ell_{6,3} \end{bmatrix}} R(-5)^6 \xrightarrow{\begin{bmatrix} q_{1,1} & q_{1,2} & q_{1,3} & q_{1,4} & q_{1,5} & q_{1,6} \\ q_{2,1} & q_{2,2} & q_{2,3} & q_{2,4} & q_{2,5} & q_{2,6} \\ q_{3,1} & q_{3,2} & q_{3,3} & q_{3,4} & q_{3,5} & q_{3,6} \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 & \ell_6 \end{bmatrix}} \begin{matrix} R(-3)^3 \\ \oplus \\ R(-4)^1 \end{matrix} \xrightarrow{\begin{bmatrix} c_1 & c_2 & c_3 & f \end{bmatrix}} R.$$

The graded Betti numbers of M are

$$\beta_{3,6} = 3, \quad \beta_{2,5} = 6, \quad \beta_{1,3} = 3, \quad \beta_{1,4} = 1, \quad \beta_{0,0} = 1.$$

The computer algebra system Macaulay2 would report that the Betti table for M is:

	0	1	2	3
0	1	–	–	–
1	–	3	–	–
2	–	1	6	3.

Example. This is the first example in Eisenbud's book "The geometry of syzygies" [4, Theorem 2.4]. It shows dramatically that syzygies encode significant information about geometry. (I think that the field $\boldsymbol{k}$ is allowed to be an arbitrary field, but surely the assertion holds if $\boldsymbol{k}$ is the field of complex numbers.) Let X be a set of 7 points in projective 3-space $\mathbb{P}^3$ over $\boldsymbol{k}$. Assume no more than 2 points of X are on any line and no more than 3 points of X are on any plane. Let $I(X)$ be the set of polynomials in $R = \boldsymbol{k}[x_1, x_2, x_3, x_4]$ that vanish on X and S be the homogeneous coordinate ring of X, in other words, let $S = R/I(X)$. View S as an R-module. Then there are exactly two distinct Betti diagrams possible for the homogeneous coordinate ring S:

	0	1	2	3
0	1	–	–	–
1	–	3	–	–
2	–	1	6	3

and

	0	1	2	3
0	1	–	–	–
1	–	3	2	–
2	–	3	6	3.

In the first case the points do not lie on any curve of degree 3. In the second case, the ideal J generated by the three quadratic generators of $I(X)$ is the ideal of the unique curve of degree 3 which contains X and this curve is irreducible. (An algebraist would say that the curve defined by the three quadratic forms of J has multiplicity 3; a geometer says that the curve has degree 3. It is not important to me what this invariant is called; but it is important to point out that this use of the word "degree" is much more subtle than any of the other uses in this course announcement.)

Example. This example shows that the **form** of the syzygies and not just the graded Betti numbers affects the geometry. Consider three homogeneous polynomials g_1, g_2, g_3 of the same degree in $\boldsymbol{k}[x,y]$. Assume that the only polynomials that divide all three g's are the constants. The morphism

$$\begin{array}{cccc} [g_1 : g_2 : g_3] : & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & [a:b] & \mapsto & [g_1(a,b) : g_2(a,b) : g_3(a,b)] \end{array}$$

defines a rational curve C in the projective plane. Information about the singularities of C may be read from the syzygies of $[g_1, g_2, g_3]$. See [7, sections 6 and 7] and [2, sections 4 and 9].

So much for the questions "What is a syzygy?" and "Why do I care?" Now it is time to answer, "So, what is the course about?" I have borrowed the following description of Weyman's book from the review written by Laurent Manivel on MathSciNet.

"The subject of this book is the computation of syzygies of certain types of algebraic varieties by geometric techniques. It provides a synthesis of the work of the author and his collaborators over the last twenty years, on problems which were first addressed a very long time ago but on which progress has been slow and scarce. The question of computing the syzygies of the Plücker ideals (the ideals defining the Grassmann varieties in their Plücker embeddings) was raised by Study as early as 1880, and remains open except for Grassmannians of planes and a few low-dimensional cases. Varieties defined by minors of matrices with polynomial entries were also a classical subject of interest. In the projective setting, the problem of computing their dimensions and degrees was tackled by Macaulay and the Italian geometers, notably Giambelli. Their work has been an important source of inspiration for the modern theory of Schubert calculus. But the computation of the syzygies of determinantal varieties, those varieties defined by minors of a given size of a generic matrix, is a different story. For maximal minors, the answer was given only in 1962, by J. A. Eagon and D. G. Northcott [3]. The general solution was obtained by A. Lascoux in his thesis [9].

What made Lascoux's breakthrough possible? First, the idea to use Kempf's technique of collapsing to compute syzygies. The main observation of G. R. Kempf [5] was that many interesting varieties can be desingularized by a vector bundle: for instance, determinantal varieties, and many other types of orbit closures. He deduced that these varieties are normal with rational singularities, and the group action of course helps a lot: when the desingularization is a homogeneous vector bundle on a rational homogeneous variety, Bott's theorem gives a good control of the cohomology of this bundle. Representation theory thus provides the second key to Lascoux's approach. One could add that this is an obvious limit to its potential scope. Weyman's book should nevertheless convince the reader of the wealth of its applications. The first four chapters are introductory. Chapter 1 discusses tableaux and a few useful statements from homological and commutative algebra. Chapter 2 develops the theory of Schur functors and complexes, following the characteristic-free approach of K. Akin, D. A. Buchsbaum and the author [1]. In positive characteristics, Weyl functors are defined in terms of divided differences; they are dual to Schur functors. In zero characteristic, Weyl and Schur functors coincide and provide the irreducible modules of the general linear group. Geometry enters into play in Chapter 3, where Grassmannians and flag varieties are introduced and their relevant properties discussed. Chapter 4 is devoted to Bott's theorem, which is proved for the general linear group and shown to extend to certain homogeneous vector bundles.

Chapter 5 is the heart of the book. The setting of Kempf's method is explained: a subbundle of a trivial bundle on a projective variety is given; by projection, the total space Z of this subbundle maps to a subvariety Y of the fiber of the trivial bundle. When this map is birational, an explicit resolution of the coordinate ring of the normalization of Y can be deduced by pushing forward a Koszul complex, and one can possibly check from that resolution that Y is indeed normal with rational singularities. The main problem at that point is that the terms of the resolution are given by cohomology groups which may be hard to compute explicitly.

The last four chapters apply this geometric technique in different contexts. In Chapter 6, Lascoux's resolution for determinantal varieties in characteristic zero is obtained, as well as its extensions to symmetric and skew-symmetric matrices. An example due to Hashimoto is included, showing that the minimal resolutions of the determinantal ideals are not characteristic-free.

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The book is full of concrete examples that help one to grasp the combinatorics – which can be quite involved – of the complexes built from the Schur-Weyl functors. Each chapter ends with interesting problems, further opening the perspective. Several questions and directions for future research are mentioned. All this should help in making this beautiful circle of ideas more accessible, and probably enlarge its already vast field of applications."

The course will mainly be about chapters two and six of Weyman's book. Chapter two concerns irreducible $\mathrm{GL}_n(\boldsymbol{k})$-modules (or, if you prefer, $\mathrm{GL}_n(\boldsymbol{k})$-representations). These Schur modules and Weyl modules are the building blocks of the free resolutions and they have the wonderful property that there is a non-zero $\mathrm{GL}_n(\boldsymbol{k})$-module homomorphism from one irreducible $\mathrm{GL}_n(\boldsymbol{k})$-module to another irreducible $\mathrm{GL}_n(\boldsymbol{k})$-module only if the two irreducible $\mathrm{GL}_n(\boldsymbol{k})$-modules are equal; furthermore if the two irreducible modules are equal, then every homomorphism between them is multiplication by a scalar. (Everything I know about this technique was taught to me by Weyman. Once he got me to understand the preceding sentence, I became much less intimidated by the whole process.) Chapter six is a collection of resolutions built using the geometric technique. My goal is to get everyone in the room to understand these resolutions. If we happen to pick up information about vector bundles, or cohomology, or the collapsing of homogeneous bundles, or the Bott isomorphism theorem, that is a bonus. I am more interested in applying the technique than understanding the intricate details of the technique. (It might be amusing to compare the resolutions in [6] and [8]. These are the same resolution, one built using ad hoc methods, the other built using the geometric method.)

REFERENCES

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