COMMUTATIVE ALGEBRA II, SPRING 2019, A. KUSTIN, CLASS NOTES

1. Regular sequences

This section loosely follows sections 16 and 17 of [6].

Definition 1.1. Let R be a ring and M be a non-zero R-module.

(a) The element r of R is regular on M if $rm = 0 \implies m = 0$, for $m \in M$.
(b) The elements $r_1, \dots, r_s$ (of R) form a regular sequence on M, if
  (i) $(r_1, \dots, r_s)M \neq M$,
  (ii) r_1 is regular on M, r_2 is regular on $M/(r_1)M$, $\dots$, and r_s is regular on $M/(r_1, \dots, r_{s-1})M$.

Example 1.2. The elements $x_1, \dots, x_n$ in the polynomial ring $R = \boldsymbol{k}[x_1, \dots, x_n]$ form a regular sequence on R.

Example 1.3. In general, order matters.
Let $R = \boldsymbol{k}[x, y, z]$.
The elements $x, y(1-x), z(1-x)$ of R form a regular sequence on R.
But the elements $y(1-x), z(1-x), x$ do not form a regular sequence on R.

Lemma 1.4. *If M is a finitely generated module over a Noetherian local ring R, then every regular sequence on M is a regular sequence in any order.*

Proof. It suffices to show that if x_1, x_2 is a regular sequence on M, then x_2, x_1 is a regular sequence on M.

Assume x_1, x_2 is a regular sequence on M. We first show that x_2 is regular on M. If $x_2 m = 0$, then the hypothesis that x_1, x_2 is a regular sequence on M guarantees that $m \in x_1 M$; thus $m = x_1 m_1$ for some m_1.

But $0 = x_2 m = x_1 x_2 m_1$ and x_1 is still regular on M; so $x_2 m_1 = 0$ and $m_1 = x_1 m_2$.

Proceed in this manner to see that $m \in \cap_i x_1^i M$. Apply the Krull Intersection Theorem which says that if R is a Noetherian local ring, I is a proper ideal, and M is a finitely generated R-module, then $\cap_i I^i M = 0$. Thus m is zero and x_2 is regular on M.

Now we prove that x_1 is regular on $M/x_2 M$. Suppose $x_1 m_1 = x_2 m_2$. The sequence x_1, x_2 is regular on M and $x_2 m_2 \in x_1 M$; hence $m_2 = x_1 m_2'$ for some $m_2' \in M$. Thus,

$$x_1 m_1 = x_1 x_2 m_2'.$$

But, x_1 is regular on M; hence $m_1 = x_2 m_2' \in x_2 M$. □

Theorem 1.5. [Krull Intersection Theorem] *If R is a Noetherian local ring, I is a proper ideal, and M is a finitely generated R-module, then $\cap_i I^i M = 0$.*

Proof. Let $N = \cap_i I^i M$. Recall that the Artin Rees Lemma says that if $A \subseteq B$ are finitely generated modules over the Noetherian ring R and I is a proper ideal of R, then there exists an integer c so that
$$I^n B \cap A = I^{n-c}(I^c B \cap A),$$
for $c \le n$. In our situation,
$$N = I^n M \cap N = I^{n-c}(I^c M \cap N) \subseteq IN.$$
It is clear that $IN \subseteq N$. Thus $N = IN$. Apply Nakayama's Lemma to conclude that $N = 0$. (Of course, Nakayama's Lemma says that if M is a finitely generated module over a local ring $(R, \mathfrak{m})$ and $\mathfrak{m}M = M$, then $M = 0$.) □

Definition 1.6. Let R be a ring, I be an ideal in R, and M be an R-module with $IM \ne M$. The grade in I on M (denoted $\operatorname{grade}(I, M)$) is the length of the longest regular sequence in I on M. We write $\operatorname{grade}(I)$ to mean $\operatorname{grade}(I, R)$. If $(R, \mathfrak{m})$ is Noetherian and local and M is a non-zero finitely generated R module, then $\operatorname{grade}(\mathfrak{m}, M)$ is also denoted $\operatorname{depth} M$; and of course, $\operatorname{depth} R$ is $\operatorname{grade}(\mathfrak{m}, R)$.

Recall that $\dim M$ is a geometric measure of the size of M. We will now show that $\operatorname{depth} M$ is a homological measure of the size of M.

Remark 1.7. If R is a Noetherian ring, M is finitely generated R-module, and I is an ideal of R with $IM \ne M$, then every regular sequence in I on M is part of a finite maximal regular sequence. Indeed, if $x_1, x_2, \dots$ is a regular sequence in I on M, then
$$(x_1)M \subsetneq (x_1, x_2)M \subsetneq \cdots$$
If equality occurred at spot i, then $x_i M$ would be contained in $(x_1, \dots, x_{i-1})M$ with x_i regular on $M/(x_1, \dots, x_{i-1})M$. This would force $M \subseteq (x_1, \dots, x_{i-1})M$ which has been ruled out.

Observation 1.9 is designed to show that $\operatorname{grade} I$ is connected to the functor
$$\operatorname{Hom}_R(R/I, -).$$
The proof uses Emmy Noether's Theory of primary decomposition.

Fact 1.8. *If M is a non-zero finitely generated module over a Noetherian ring R, I is an ideal of R, and every element of I is a zero divisor on M, then $Im = 0$ for some non-zero $m \in M$.*

Proof. Observe that
$$I \subseteq \operatorname{ZeroDivisors}(M) = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}.$$
Therefore, $I \subseteq \mathfrak{p} = \operatorname{ann} m$ for some $\mathfrak{p} \in \operatorname{Ass} M$. □

Observation 1.9. *Let M be a non-zero finitely generated module over the Noetherian ring R, and let I be an ideal in R. Then $\operatorname{Hom}_R(R/I, M) = 0$ if and only if there is an element $x \in I$ with x regular on M.*

Proof. ($\Leftarrow$) Assume x is an element of I with x regular on M. Prove $\operatorname{Hom}_R(R/I, M) = 0$.

Apply $\operatorname{Hom}_R(R/I, -)$ to the exact sequence

$$0 \to M \xrightarrow{x} M \to M/(x)M \to 0$$

to get the exact sequence

$$0 \to \operatorname{Hom}_R(R/I, M) \underbrace{\xrightarrow{x}}_{0} \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, M/(x)M).$$

Conclude $\operatorname{Hom}_R(R/I, M) = 0$.

($\Rightarrow$) Assume I is contained in the zero divisors on M. Prove $\operatorname{Hom}_R(R/I, M) \neq 0$.

The ideal I is contained in the set of zero divisors on M; hence there is a non-zero element m of M with $Im = 0$. Observe that $1 \mapsto m$ is a non-zero element of $\operatorname{Hom}_R(R/I, M)$. □

Class on Jan. 17. Let M be a finitely generated module over the Noetherian ring R, and let I be an ideal in R with $IM \neq M$.

- We proved that every regular sequence in I on M is finite.
- We proved that $\operatorname{Hom}_R(R/I, M) \neq 0$ if and only if every element of I is a zero divisor on M.
- We let $\operatorname{grade}(I, M)$ denote the length of the longest regular sequence in I on M.

In the next result we show that each maximal regular sequence in I on M has the same length, and we interpret this length homologically.

Theorem 1.10. *Let M be a finitely generated module over the Noetherian ring R, and let I be an ideal in R with $IM \neq M$. The following statements hold:*

(a) $\operatorname{grade}(I, M) = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$,

(b) *every maximal regular sequence in I on M has the same length,*

(c) $\operatorname{grade}(I, M)$ *is finite, and*

(d) $\operatorname{grade}(I, M) \leq \operatorname{pd}_R R/I$.

Recall that the projective dimension ($\operatorname{pd}_R M$) of M as an R-module (denoted $\operatorname{pd}_R M$) is the length of the shortest resolution of M by projective R-modules.

Recall also that the R-module P is projective if every picture of R-module homomorphisms of the form

$$\begin{array}{ccc} & & P \\ & & \downarrow \\ A & \twoheadrightarrow & B \end{array}$$

gives rise to a commutative diagram of R-module homomorphisms of the form

$$\begin{array}{ccc} & & P \\ & \exists \swarrow & \downarrow \\ A & \twoheadrightarrow & B \end{array}$$

Similarly, the R-module E is injective if every picture of R-module homomorphisms of the form

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \\ E & & \end{array}$$

gives rise to a commutative diagram of R-module homomorphisms of the form

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & \swarrow \exists & \\ E & & \end{array}$$

Every free R-module is a projective R-module. If R is a domain, the the fraction field Q of R is an injective R-module. Here is a quick sketch of the proof. Suppose

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow f & & \\ Q & & \end{array}$$

is a picture of R-module homomorphisms. Let $f' : A' \to Q$ be a maximal extension of f with $A \subseteq A' \subseteq B$. (Use Zorn's Lemma to establish the existence of f'.) We claim that $A' = B$. Otherwise, there exits $b \in B \setminus A'$. We will extend $f' : A' \to Q$ to be a homomorphism $f'' : A' + Rb \to Q$, with $f''|_{A'} = f'$. (Of course such an extension is not possible because f' is a maximal extension of f.) Consider the ideal $I = (A' :_R b)$. If $I = 0$, then define $f''(b) = 0$. If i is a non-zero element of I, then define $f''(b) = \frac{f'(ib)}{i}$. **Verify that f'' is a well-defined homomorphism.**

Background 1.11. Here are a few comments about Ext.

(a) Let L and M be modules over the ring R. I will tell you the first way to compute $\operatorname{Ext}^i_R(L, M)$. Let

$$P : \quad \cdots \to P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_0 \to 0$$

be a projective resolution of L. Apply $\operatorname{Hom}_R(-, M)$ to obtain the complex

$$\operatorname{Hom}_R(P, M) : \quad 0 \to \operatorname{Hom}_R(P_0, M) \xrightarrow{d_1^*} \operatorname{Hom}_R(P_1, M) \xrightarrow{d_2^*} \operatorname{Hom}_R(P_2, M) \xrightarrow{d_3^*} \cdots .$$

Then

$$\operatorname{Ext}^i_P(L, M) = \mathrm{H}^i(\operatorname{Hom}_R(P, M)) = \frac{\ker d_{i+1}^*}{\operatorname{im} d_i^*}.$$

In particular, the functor $\operatorname{Hom}_R(-, M)$ is left exact; so,

$$\operatorname{Ext}^0_R(L, M) = \operatorname{Hom}_R(L, M).$$

One should make sure that

(i) Ext is well-defined, in the sense that if P' is another projective resolution of P, then $\mathrm{H}^i(\mathrm{Hom}_R(P,M))$ is isomorphic to $\mathrm{H}^i(\mathrm{Hom}_R(P',M))$. This is a three step process. One proves the comparison theorem to exhibit a map of complexes $\alpha : P' \to P$ and a map of complexes $\beta : P \to P'$. Each of these maps of complexes extends the identity map on L. Then one proves that $\beta \circ \alpha : P' \to P'$ is homotopic to the identity map. Then one proves to two homotopic maps of complexes induce the same map on homology. I can say more.

Class on Jan. 22, 2019

Let L and M be modules over the ring R. Last time gave two ways to compute $\mathrm{Ext}^i_R(L,M)$:

- Let (P,d) be a projective resolution of L. Then
$$\mathrm{Ext}^i_R(L,M) = \mathrm{H}^i(\mathrm{Hom}_R(P,M)).$$
We gave some idea of why this object does not depend on the choice of P.
- Let (E,δ) be an injective resolution of M. Then
$$\mathrm{Ext}^i_R(L,M) = \mathrm{H}^i(\mathrm{Hom}_R(N,E)).$$
The previous explanation does work here to see that the object does not depend on the choice of E.
- Now I will tell you a second way to compute $\mathrm{Ext}^i(L,M)$. Let
$$E: \quad 0 \to E_0 \xrightarrow{d^0} E_1 \xrightarrow{d^1} E_2 \xrightarrow{d^2} \cdots$$
be an injective resolution of M. Apply $\mathrm{Hom}_R(L,-)$ to obtain the complex
$$\mathrm{Hom}_R(L,E): \quad 0 \to \mathrm{Hom}(L,E_0) \xrightarrow{d^0_*} \mathrm{Hom}(L,E_1) \xrightarrow{d^1_*} \mathrm{Hom}(L,E_2) \xrightarrow{d^2_*} \cdots .$$
Then
$$\mathrm{Ext}_R(L,M) = \mathrm{H}^i(\mathrm{Hom}_R(L,E)) = \frac{\ker d^i_*}{\operatorname{im} d^{i-1}_*}.$$

Topic one. Today's first project is to answer the following question.

Question 1.12. *Why is* Ext *built using* (P,d) *related to* Ext *built using* (E,δ)*?*

Answer. One looks at $\mathrm{Hom}_R(P,E)$ which is a double complex (or if you prefer a big commutative picture). There is a natural complex (called $\mathrm{Tot}(\mathrm{Hom}_R(P,E))$) and there are natural maps

$$\mathrm{H}^i(\mathrm{Tot}(\mathrm{Hom}_R(P,E)) \to \mathrm{H}^i(\mathrm{Hom}(L,E)) \tag{1.12.1}$$

and

$$\mathrm{H}^i(\mathrm{Tot}(\mathrm{Hom}_R(P,E)) \to \mathrm{H}^i(\mathrm{Hom}(P,M)). \tag{1.12.2}$$

One uses the long exact sequence of (co)-homology which is associated to a short exact sequence of complexes in order to see that the maps (1.12.1) and (1.12.2) both are isomorphisms.

Here are some of the details. The double complex $\operatorname{Hom}_R(P, E)$ looks like

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \delta_{2*}} & & \uparrow{\scriptstyle \delta_{2*}} & & \uparrow{\scriptstyle \delta_{2*}} & & \\
\operatorname{Hom}(P_0, E_2) & \xrightarrow{d_1^*} & \operatorname{Hom}(P_1, E_2) & \xrightarrow{d_2^*} & \operatorname{Hom}(P_2, E_2) & \xrightarrow{d_3^*} & \cdots \\
\uparrow{\scriptstyle \delta_{1*}} & & \uparrow{\scriptstyle \delta_{1*}} & & \uparrow{\scriptstyle \delta_{1*}} & & \\
\operatorname{Hom}(P_0, E_1) & \xrightarrow{d_1^*} & \operatorname{Hom}(P_1, E_1) & \xrightarrow{d_2^*} & \operatorname{Hom}(P_2, E_1) & \xrightarrow{d_3^*} & \cdots \\
\uparrow{\scriptstyle \delta_{0*}} & & \uparrow{\scriptstyle \delta_{0*}} & & \uparrow{\scriptstyle \delta_{0*}} & & \\
\operatorname{Hom}(P_0, E_0) & \xrightarrow{d_1^*} & \operatorname{Hom}(P_1, E_0) & \xrightarrow{d_2^*} & \operatorname{Hom}(P_2, E_0) & \xrightarrow{d_3^*} & \cdots
\end{array}$$

The total complex is obtained by adding along the diagonals of slope -1; so the complex $\operatorname{Tot}(\operatorname{Hom}(P, E))$ is

$$0 \to \operatorname{Hom}(P_0, E_0) \xrightarrow{\begin{bmatrix} \delta_{0*} \\ d_1^* \end{bmatrix}} \begin{array}{c} \operatorname{Hom}(P_0, E_1) \\ \oplus \\ \operatorname{Hom}(P_1, E_0) \end{array} \xrightarrow{\begin{bmatrix} \delta_{1*} & 0 \\ d_1^* & -\delta_{0*} \\ 0 & d_2^* \end{bmatrix}} \begin{array}{c} \operatorname{Hom}(P_0, E_2) \\ \oplus \\ \operatorname{Hom}(P_1, E_1) \\ \oplus \\ \operatorname{Hom}(P_2, E_0) \end{array} \xrightarrow{\begin{bmatrix} \delta_{2*} & 0 & 0 \\ d_1^* & -\delta_{1*} & 0 \\ 0 & d_2^* & \delta_{0*} \\ 0 & 0 & d_2^* \end{bmatrix}} \cdots$$

In general, the signs are not mysterious. They follow the following pattern:

$$\operatorname{Hom}(P_i, E_j) \xrightarrow{\begin{bmatrix} (-1)^i \delta_{j*} \\ d_{i+1}^* \end{bmatrix}} \begin{array}{c} \operatorname{Hom}(P_i, E_{j+1}) \\ \oplus \\ \operatorname{Hom}(P_{i+1}, E_j) \end{array} \xrightarrow{\begin{bmatrix} (-1)^i \delta_{j+1*} & 0 \\ d_{i+1}^* & (-1)^{i+1} \delta_{j*} \\ 0 & d_{i+2}^* \end{bmatrix}} \begin{array}{c} \operatorname{Hom}(P_i, E_{j+2}) \\ \oplus \\ \operatorname{Hom}(P_{i+1}, E_{j+1}) \\ \oplus \\ \operatorname{Hom}(P_{i+2}, E_j). \end{array}$$

A two cocycle in $\operatorname{Tot}(\operatorname{Hom}(P, E))$ looks like

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow & & & & \\
x_{0,2} & \longrightarrow & \text{same} & & \\
 & & \uparrow & & \\
 & & x_{1,1} & \longrightarrow & \text{same} \\
 & & & & \uparrow \\
 & & & & x_{2,0} \longrightarrow 0
\end{array}$$

in $\mathrm{Hom}(P,E)$. **In particular,** $x_{2,0}$ represents a 2 cocycle in $\mathrm{Hom}(P,M)$ and $x_{0,2}$ represents a 2 cocycle in $\mathrm{Hom}(L,E)$. From this we get homomorphisms

$$\mathrm{H}^i(\mathrm{Hom}(L,E)) \leftarrow \mathrm{H}^i(\mathrm{Tot}(\mathrm{Hom}(P,E))) \rightarrow \mathrm{H}^i(\mathrm{Hom}(P,M)). \tag{1.12.3}$$

To show that the maps of (1.12.3) are isomorphisms one considers two short exact sequence of complexes:

$$0 \rightarrow \mathrm{Hom}(P,E) \rightarrow \mathrm{Tot}(\mathrm{Hom}(P^{\text{extended}},E)) \rightarrow \mathrm{Hom}(L,E_0)^{\text{shifted}} \rightarrow 0$$

and

$$0 \rightarrow \mathrm{Hom}(P,E) \rightarrow \mathrm{Tot}(\mathrm{Hom}(P,E^{\text{extended}})) \rightarrow \mathrm{Hom}(P,M)^{\text{shifted}} \rightarrow 0$$

and uses the long exact sequence of homology which corresponds to a short exact sequence of complexes. Keep in mind that $\mathrm{Tot}(\mathrm{Hom}(P^{\text{extended}},E))$ and $\mathrm{Tot}(\mathrm{Hom}(P,E^{\text{extended}}))$ are exact! Their cohomology is zero. Of course, P^{extended} is

$$\cdots P_1 \rightarrow P_0 \rightarrow L \rightarrow 0$$

and E^{extended} is

$$0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots$$

Topic Three. The long exact sequence of homology which corresponds to a short exact sequence of complexes.

Let

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

be a short exact sequence of complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle c_2} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle c_1} & & \\
0 & \longrightarrow & A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(The columns are complexes. The rows are exact.)

Then there is a long exact sequence of homology

$$\cdots \rightarrow \mathrm{H}_2(A) \xrightarrow{f_*} \mathrm{H}_2(B) \xrightarrow{g_*} \mathrm{H}_2(C) \xrightarrow{\partial} \mathrm{H}_1(A) \xrightarrow{f_*} \mathrm{H}_1(B) \xrightarrow{g_*} \mathrm{H}_1(C) \xrightarrow{\partial} \mathrm{H}_0(A) \xrightarrow{f_*} \mathrm{H}_0(B) \xrightarrow{g_*} \mathrm{H}_0(C) \rightarrow 0 \tag{1.12.4}$$

The maps f_* and g_* are straightforward. For example, if x is a one cycle in A, then $f(x)$ is a one cycle in B. One can easily show that

$$\mathrm{H}_1(A) \to \mathrm{H}_1(B),$$

given by the class of x is sent to the class of $f(x)$ is a well-defined homomorphism.

The "connecting homomorphism" ∂ is a little more complicated. I will describe $\partial : \mathrm{H}_i(C) \to \mathrm{H}_{i-1}(A)$. Let x be an i-cycle in C. The map b_i is onto, so there is a y in B_i which maps to x. Observe that $b_i(y)$ is a cycle. The exactness of row $i-1$ guarantees that there is a z in A_{i-1} with $z \mapsto b_i(z)$. Observe that z is a cycle because f_{i-2} is one-to-one:

$$\begin{array}{ccccc}
 & & y & \longrightarrow & x \\
 & & \downarrow & & \downarrow \\
z & \longrightarrow & b_1(y) & \longrightarrow & 0 \\
\downarrow & & & & \\
? & \longrightarrow & 0 & &
\end{array}$$

One must verify that ∂ of the class of x is equal to the class of z is a well-defined function. Then one must verify that (1.12.4) is an exact sequence.

Class on Jan. 24, 2019

Let L and M be modules over the ring R. Last time:

- We saw that one can compute $\mathrm{Ext}_R(L, M)$ using either a projective resolution of L or an injective resolution of M.
- We saw that a short exact sequence of complexes gives rise to a long exact sequence of homology or cohomology.

Today:

1. Ext is a functor.
2. One long exact sequence of Ext
3. The other long exact sequence of Ext.
4. The Theorem about regular sequences that we want to prove. The statement of the Lemma about Ext that we will use. The Lemma does indeed establish the Theorem.
5. The proof of the Lemma.

Topic One. We discuss the fact that $\mathrm{Ext}^i(-, M)$ and $\mathrm{Ext}^i(L, -)$ are functors. That is, if $f : L \to L'$ and $g : M \to M'$ are an R-module homomorphisms, then there are well-defined homomorphisms

$$f^* : \mathrm{Ext}^i_R(L', M) \to \mathrm{Ext}^i_R(L, M)$$

and

$$g_* : \mathrm{Ext}^i_R(L, M) \to \mathrm{Ext}^i_R(L, M').$$

Resolve L and L'. Use the comparison theorem to find a map of complexes

$$\text{extended resolution of } L \to \text{extended resolution of } L'$$

Apply $\operatorname{Hom}(-, M)$. Etc.

Topic Two. If

$$0 \to L' \xrightarrow{f} L \xrightarrow{g} L'' \to 0$$

is a short exact sequence of R-modules, then

$$0 \to \operatorname{Ext}^0_R(L'', M) \xrightarrow{g^*} \operatorname{Ext}^0_R(L', M) \xrightarrow{f^*} \operatorname{Ext}^0_R(L, M) \xrightarrow{\delta} \operatorname{Ext}^1_R(L'', M) \xrightarrow{g^*} \operatorname{Ext}^1_R(L', M)$$
$$\xrightarrow{f^*} \operatorname{Ext}^1_R(L, M) \xrightarrow{\delta} \operatorname{Ext}^2_R(L'', M) \xrightarrow{g^*} \operatorname{Ext}^2_R(L', M) \xrightarrow{f^*} \operatorname{Ext}^2_R(L, M) \to \cdots$$

is a long exact sequence of homology. The proof is long but not hard. One first shows that there is a short exact sequence of complexes

$$0 \to P' \to P \to P'' \to 0$$

(with P' a projective resolution of L', P a projective resolution of L, and P'' a projective resolution of L'' and each row $0 \to P'_i \to P_i \to P''_i \to 0$ exact) which extends the original short exact sequence

$$0 \to L' \to L \to L'' \to 0.$$

(This is called the Horseshoe Lemma.) One then applies $\operatorname{Hom}_P(-, M)$ and obtains a short exact sequence of complexes:

$$0 \to \operatorname{Hom}_R(P'', M) \to \operatorname{Hom}_R(P, M) \to \operatorname{Hom}_R(P', M) \to 0. \tag{1.12.5}$$

(The critical point here is that each row $0 \to P'_i \to P_i \to P''_i \to 0$ is automatically **split** exact because P''_i is projective; hence each row of (1.12.5) is also split exact; hence exact.) We have seen that every short exact sequence of complexes gives rise to a long exact sequence of (co-)homology.

Again, I can do more. My favorite book on Homological Algebra is Rotman [8]; a few years ago Adela taught a course in Homological Algebra from Weibel [10].

Topic Three. If

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

is a short exact sequence of R-modules, then

$$0 \to \operatorname{Ext}^0_R(L, M') \xrightarrow{\alpha_*} \operatorname{Ext}^0_R(L, M) \xrightarrow{\beta_*} \operatorname{Ext}^0_R(L, M'') \xrightarrow{\delta} \operatorname{Ext}^1_R(L, M') \xrightarrow{\alpha_*} \operatorname{Ext}^1_R(L, M)$$
$$\xrightarrow{\beta_*} \operatorname{Ext}^1_R(L, M'') \xrightarrow{\delta} \operatorname{Ext}^2_R(L, M') \xrightarrow{\alpha_*} \operatorname{Ext}^2_R(L, M) \xrightarrow{\beta_*} \operatorname{Ext}^2_R(L, M'') \to \cdots$$

is a long exact sequence of homology.

Topic Four. We want to prove

Theorem (1.10)**.** *Let M be a finitely generated module over the Noetherian ring R, and let I be an ideal in R with $IM \neq M$. The following statements hold:*

(a) $\operatorname{grade}(I, M) = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$,
(b) *every maximal regular sequence in I on M has the same length,*
(c) $\operatorname{grade}(I, M)$ *is finite, and*
(d) $\operatorname{grade}(I, M) \leq \operatorname{pd}_R R/I$.

Recall that $\operatorname{grade}(I, M)$ is equal to the length of the longest regular sequence in I on M. (Before we prove the Theorem, this could be infinity.) We had two warm-up Lemmas:

- Every regular sequence in I on M is finite.
- $\operatorname{Hom}_R(R/I, M) \neq 0 \iff I \subseteq \operatorname{ZeroDivisors}(M)$.

Lemma 1.13. *Let L and M be finitely generated modules over the Noetherian ring R. If $x_1, \dots x_n$ is a regular sequence on M in $\operatorname{ann} L$, then*

$$\operatorname{Ext}^i_R(L, M) \cong \begin{cases} 0, & \text{if } 0 \leq i \leq n-1, \text{ and} \\ \operatorname{Hom}_R(L, M/(x_1, \dots, x_n)M), & \text{if } i = n. \end{cases}$$

Assume Lemma 1.13. *Prove Theorem* 1.10. Let $x_1, \dots x_n$ be a maximal regular sequence in I on M. (Such a sequence exists because every regular sequence in I on M is finite.) Apply Lemma 1.13 to see that

$$\operatorname{Ext}^i_R(R/I, M) = \begin{cases} 0, & \text{if } 0 \leq i \leq n-1 \\ \operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M), & \text{if } i = n. \end{cases}$$

The regular sequence is maximal; so, Observation 1.9 yields that

$$\operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M) \neq 0.$$

This completes the proof of (a) and (b).

(c) We know that there exists a maximal regular sequence. This maximal regular sequence has finite length. Therefore, every maximal regular sequence has the same finite length by (b). Thus $\operatorname{grade}(I, M)$ is finite.

(d) Let $\mathbb{F}$ be a projective resolution of R/I of length $\operatorname{pd}_R R/I$. Use

$$\operatorname{Ext}^i_R(R/I, M) = \operatorname{H}^i(\operatorname{Hom}(\mathbb{F}, M))$$

to compute

$$\operatorname{Ext}^i_R(R/I, M) = 0 \quad \text{for } \operatorname{pd}_R R/I + 1 \leq i.$$

We know from (c) that $\operatorname{grade}(I, M)$ is finite and from (a) and (b) that

$$\operatorname{Ext}^{\operatorname{grade}(I,M)}_R(R/I, M) \neq 0.$$

Conclude $\operatorname{grade}(I, M) \leq \operatorname{pd}_R R/I$. □

Proof of Lemma 1.13. The proof is by induction on n. We start with $n = 1$. Apply $\operatorname{Hom}_R(L, -)$ to the short exact sequence of R-modules

$$0 \to M \xrightarrow{x_1} M \to M/x_1M \to 0$$

to obtain the long exact sequence

$$0 \to \operatorname{Hom}_R(L,M) \underbrace{\xrightarrow{x_1}}_{0} \operatorname{Hom}_R(L,M) \to \operatorname{Hom}_R(L,M/x_1M)$$

$$\to \operatorname{Ext}^1_R(L,M) \underbrace{\xrightarrow{x_1}}_{0} \operatorname{Ext}^1_R(L,M) \to \operatorname{Ext}^1_R(L,M/x_1M) \cdots .$$

Conclude that $\operatorname{Hom}_R(L,M)=0$ and

$$0 \to \operatorname{Ext}^i_R(L,M) \to \operatorname{Ext}^i_R(L,M/x_1M) \to \operatorname{Ext}^{i+1}_R(L,M) \to 0 \tag{1.13.1}$$

is exact for $0 \le i$. In particular, when $i=0$ in (1.13.1) one obtains

$$\operatorname{Hom}_R(L,M/x_1M) \cong \operatorname{Ext}^1_R(L,M).$$

We have established the case $n=1$.

Suppose, by induction, that the assertion holds for $n-1$. Apply the induction hypothesis to the regular sequence $x_2 \dots, x_n$ on the module M/x_1M to conclude that

$$\operatorname{Ext}^i_R(L,M/x_1M) \cong \begin{cases} 0, & \text{if } 0 \le i \le n-2, \text{ and} \\ \operatorname{Hom}_R(L, \underbrace{(M/x_1)/(x_2,\dots,x_n)(M/x_1)}_{M/(x_1,\dots,x_n)M}), & \text{if } i = n-1. \end{cases}$$

Plug $\operatorname{Ext}^i_R(L,M/x_1M)=0$ for $0 \le i \le n-2$ into (1.13.1) to see $\operatorname{Ext}^i_R(L,M)=0$ for $0 \le i \le n-1$. At $i=n-1$, (1.13.1) gives

$$0 \to \underbrace{\operatorname{Ext}^{n-1}_R(L,M)}_{0} \to \underbrace{\operatorname{Ext}^{n-1}_R(L,M/x_1M)}_{\operatorname{Hom}_R(L,M/(x_1,\dots,x_n)M)} \to \operatorname{Ext}^n_R(L,M) \to 0,$$

and this concludes the proof of the Lemma.

We prove the next Generalization for two reasons. First of all, it uses many of the technical ideas from the first semester. It is fun to see them used. Secondly, it leads to a quick proof of

$$\operatorname{grade}\operatorname{ann} L \le \operatorname{pd}_R L$$

(for any finitely generated module L over a Noetherian ring R) which is a very important inequality to me. (In my world, the module L is called perfect when equality holds.)

Corollary 1.14. *If L and M are finitely generated modules over the Noetherian ring R, with $(\operatorname{ann} L)M \ne M$ and $\operatorname{ann} L \ne 0$, then*

(a) $\operatorname{grade}(\operatorname{ann} L, M) = \min\{i \mid \operatorname{Ext}^i_R(L,M) \ne 0\}$*, and*

(b) $\operatorname{grade}(\operatorname{ann} L) \le \operatorname{pd}_R L$.

Proof. (a) We already saw, in Lemma 1.13, that if $x_1,\dots,x_n$ are elements in $\operatorname{ann} L$ which form a regular sequence on M, then

$$\operatorname{Ext}^i_R(L,M) = \begin{cases} 0 & \text{if } i \le n-1 \\ \operatorname{Hom}_R(L,\overline{M}) & \text{if } i = n, \end{cases}$$

with $\overline{M} = M/(x_1, \ldots, x_n)M$. We only need to prove that

$$\operatorname{Hom}_R(L, \overline{M}) \neq 0 \iff \text{every element of } \operatorname{ann} L \text{ is a zero divisor on } \overline{M}.$$

($\Rightarrow$) Let f be a non-zero element of $\operatorname{Hom}_R(L, \overline{M})$. In this case $f(L)$ contains a non-zero element of M. On the other hand

$$0 = f((\operatorname{ann} L)L) = (\operatorname{ann} L) \cdot f(L).$$

Thus every element of $\operatorname{ann} L$ is a zero divisor on $\overline{M}$.

($\Leftarrow$) We prove that

$$\operatorname{ann} L \subseteq \operatorname{ZeroDivisors} \overline{M} \implies \operatorname{Hom}_R(L, \overline{M}) \neq 0.$$

If

$$\operatorname{ann} L \subseteq \operatorname{ZeroDivisors} \overline{M},$$

then there exists $\mathfrak{p} \in \operatorname{Ass} \overline{M}$ with $\operatorname{ann} L \subseteq \mathfrak{p}$.

We proved

$$\operatorname{Hom}_R(L, \overline{M})_{\mathfrak{p}} \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}, \overline{M}_{\mathfrak{p}})$$

last semester. (The proof uses a finite presentation of $\overline{M}$, the exactness properties of localization and Hom, and the fact that the result is obvious if $\overline{M}$ is a free R-module.) The map $L_{\mathfrak{p}} \to \frac{L_{\mathfrak{p}}}{\mathfrak{p}L_{\mathfrak{p}}}$ is non-zero by Nakayama's Lemma. Here is a non-zero map in

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}, \overline{M}_{\mathfrak{p}}):$$

$$L_{\mathfrak{p}} \to \frac{L_{\mathfrak{p}}}{\mathfrak{p}L_{\mathfrak{p}}} = \bigoplus \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}} \to \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}} \subseteq \overline{M}_{\mathfrak{p}}.$$

For the last map, recall that $\mathfrak{p} \in \operatorname{Ass} \overline{M} \implies \mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass} \overline{M}_{\mathfrak{p}}$. It follows that $\operatorname{Hom}_R(L, \overline{M})$ is also non-zero.

(b) The proof is the same as the proof of Theorem 1.10.(d). The ideal $\operatorname{ann} L$ has a non-negative finite grade. This grade is equal to $\min\{i \mid \operatorname{Ext}^i_R(L, R) \neq 0\}$ and $\operatorname{Ext}^i_R(L, R) = 0$ for $\operatorname{pd}_R L < i$. Thus, $\operatorname{grade} L \leq \operatorname{pd}_R L$. □

If L is a perfect R-module of projective dimension g, and P is a projective resolution of L of length g, then the dual of P (that is, $\operatorname{Hom}_R(P, R)$) is a resolution of $\operatorname{Ext}^g_R(L, R)$. If R is a "Gorenstein ring" and $L = R/I$, then $\operatorname{Ext}^g_R(L, R)$ is called the canonical module of R/I.

Example 1.15. Let $R = \boldsymbol{k}[x, y]_{(x,y)}$ and $I = (x^2, xy, y^2)$. Then

$$P: \quad 0 \to R^2 \xrightarrow{\begin{bmatrix} -y & 0 \\ x & -y \\ 0 & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}} R$$

is a projective resolution of R/I of length 2. The grade of I is 2 and $\operatorname{pd}_R R/I = 2$. It follows that R/I is a perfect R-module and the dual of P

$$P^*: \quad 0 \to R^1 \xrightarrow{\begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} -y & x & 0 \\ 0 & -y & x \end{bmatrix}} R^2$$

is also a resolution. The module

$$\frac{R^2}{\left(\begin{bmatrix} -y \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ -y \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix} \right)}$$

is the canonical module of R/I.

January 31, 2019

Today we do many little things. Some of the highlights are:

- If R is a Noetherian local and M finitely generated R-module, then
$$\operatorname{depth} M \leq \dim M.$$
- The definition of Cohen-Macaulay module in the local setting.
- If R is a Noetherian ring, then $\operatorname{grade} I \leq \operatorname{ht} I$.
- The first hint of "unmixedness".

Observation 1.16. *Let R be a Noetherian ring, I an ideal of R, and M a finitely generated R-module with $IM \neq M$. Then*

$$\operatorname{grade}(I, M) = \operatorname{grade}(\sqrt{I}, M).$$

Proof. Notice that $\sqrt{I}M \neq M$. Indeed, there exists N with $(\sqrt{I})^N \subseteq I$. If $\sqrt{I}M = M$, then $(\sqrt{I})^2 M = \sqrt{I}M = M$ and by iterating, $(\sqrt{I})^N M = M$. In this case,

$$(\sqrt{I})^N M \subseteq IM \subseteq M \subseteq (\sqrt{I})^N M;$$

hence $IM = M$, and this is a contradiction.

Let $x_1, \dots, x_r$ be a maximal regular sequence in I on M. So, $x_1, \dots, x_r$ is a regular sequence in $\sqrt{I}$ on M and $I \subseteq \operatorname{ZeroDivisors}\left(\frac{M}{(x_1,\dots,x_r)M}\right)$. Thus,

$$I \subseteq \mathfrak{p} \in \operatorname{Ass} \frac{M}{(x_1, \dots, x_r)M}.$$

It follows that

$$\sqrt{I} \subseteq \mathfrak{p} \in \operatorname{Ass} \frac{M}{(x_1, \dots, x_r)M}.$$

Therefore, $x_1, \dots, x_r$ is a maximal regular sequence in $\sqrt{I}$ on M. □

I think that I forgot to define "system of parameters" last semester.

Definition 1.17. If $(R, \mathfrak{m})$ is a local Noetherian ring, M is a finitely generated R-module of dimension d, and $x_1, \dots, x_d$ is a set of d elements of $\mathfrak{m}$ with $\dim M/(x_1, \dots, x_d)M = 0$, then $x_1, \dots, x_d$ is a system of parameters for M.

Recall from last semester that if $(R, \mathfrak{m})$ is a local ring, M is a finitely generated R-module, and $x \in \mathfrak{m}$, then

$$\dim M - 1 \leq \dim M/(x) \leq \dim M. \tag{1.17.1}$$

The only interesting inequality is the one on the left. The proof is to take $y_1, \dots, y_s$ in M with $\bar{y}_1, \dots, \bar{y}_s$ a system of parameters for $M/(x)M$. So, $M/(x, y_1, \dots, y_s)M$ has finite length and

$$\dim M = \delta(M) \leq s + 1 = \dim M/(x)M + 1,$$

In other words, $\dim M - 1 \leq \dim M/(x)M$.

Observation 1.18. *Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and x be an element of $\mathfrak{m}$, If x is regular on M, then* $\dim M/(x)M = \dim M - 1$.

Proof. The element x is not in any associated primes of M; so, x is not in any of the prime ideals which are minimal in the support of M. It follows that $\dim M/(x)M < \dim M$. This is enough by (1.17.1). □

Corollary 1.19. *If $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module, then* $\operatorname{depth} M \leq \dim M$.

Proof. If $x_1, \dots, x_r$ is a regular sequence in $\mathfrak{m}$ on M, then

$$0 \leq \dim \left(\frac{M}{(x_1, \dots, x_r)M} \right) = \dim M - r.$$

Thus, $r \leq \dim M$. □

Corollary 1.20. *If I is an ideal in a Noetherian ring R, then* $\operatorname{grade} I \leq \operatorname{ht} I$.

Proof. Let $\mathfrak{p}$ be a prime ideal in R with $I \subseteq \mathfrak{p}$ and $\operatorname{ht} I = \operatorname{ht} p$. A regular sequence in I on R remains a regular sequence in $I_\mathfrak{p}$ on $R_\mathfrak{p}$. Thus,

$$\operatorname{grade} I \leq \operatorname{grade} I_\mathfrak{p} \leq \operatorname{depth} R_\mathfrak{p} \leq \dim R_\mathfrak{p} = \operatorname{ht} p = \operatorname{ht} I.$$ □

Definition 1.21. If $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module, then M is a Cohen-Macaulay R-module if either

(a) M is not zero and $\dim M = \operatorname{depth} M$, or
(b) $M = 0$.

If R is a Cohen-Macaulay R-module, then R is called a Cohen-Macaulay ring.

Theorem 1.22. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and M be a finitely generated Cohen-Macaulay R-module. Then the following statements hold.*

(a) *The module M does not have any embedded primes.*
(b) *If $\mathfrak{p}$ is in* $\operatorname{Ass} M$, *then* $\dim R/\mathfrak{p} = \dim M$.

Lemma 1.23. *If $(R, \mathfrak{m})$ is a local Noetherian ring and N and M are finitely generated R-modules, then*

$$\operatorname{Ext}^i_R(N, M) = 0, \quad \textit{whenever } i < \operatorname{depth} M - \dim N.$$

Remarks 1.24.

- Item 1.22.(a) is the point of this discussion. Geometers hate embedded primes! Macaulay (1916) [5] proved the unmixedness theorem for polynomial rings. Cohen (1946) [3] proved the unmixedness theorem for formal power series rings.
- An ideal I of a Noetherian ring R is called unmixed if the height of I is equal to the height of every associated prime $\mathfrak{p}$ of R/I.
- The unmixedness theorem is said to hold for the ring R if every ideal I generated by a number of elements equal to its height is unmixed.
- A Noetherian ring is Cohen-Macaulay if and only if the unmixedness theorem holds for it. (We will prove this. First we have to define Cohen-Macaulay ring for non-local rings.)
- Item 1.22.(b) gives a quick proof of 1.22.(a).
- The Lemma gives a quick proof of 1.22.(b). The Lemma is called Ischebeck's Lemma.

February 5, 2019

Last time we defined Cohen-Macaulay modules. Today we collect properties of Cohen-Macaulay modules.

Old Business:

- Last time we proved that if I is an ideal in a Noetherian ring then $\operatorname{grade} I = \operatorname{grade} \sqrt{I}$. For example, if x, y, z is a regular sequence in the Noetherian ring R, and I is an ideal of R which contains $x^n, y^n, z^n + f$, with $f \in (x, y)$, then $3 \le \operatorname{grade} I$.
- Did I say enough about embedded primes? An embedded prime of the module M is a prime ideal $\mathfrak{p}$ which is in $\operatorname{Ass} M$ but is not minimal in the support of M. A geometer feels the embedded prime because there are more zero-divisors on M than was expected, but has difficulty seeing the embedded prime because it does not correspond to an irreducible component of $\operatorname{Supp} M$.

  Algebraists are also wary of embedded primes. The "standard theorem" is $I \subseteq J$ are ideals of R. One wants to prove $I = J$. Maybe one collects evidence by showing that $I_{x_i} = J_{x_i}$ for various x_i in R. (That is, for each i one shows that there exists N_i with $x_i^{N_i} J \subseteq I$.) If $(\{x_i\})$ is not contained in any associated prime of R/I, then some x_i is regular on R/I and $J \subseteq I$. The algebraist wants some way of knowing that R/I does not have any big associated prime ideals.

  The quick example of associated primes is (x) and (x, y) are both in $\operatorname{Ass} \frac{\boldsymbol{k}[x,y]}{(x^2,xy)}$. The ideal (x) is the annihilator of y and the ideal (x, y) is the annihilator of (x).

(I think we proved this in enormous detail last semester.) Geometrically, $V(x)$ is the y-axis and $V(x, y)$ is the origin. So, we are looking at a line together with a distinguished point. A geometer would not study such a thing in the normal course of events. But, many serious calculations (like intersection multiplicity) are made by calculating the length of some module. The geometer would then be looking at a module where his or her geometric intuition is not helpful.

New Business. First Goal: Let $(R, \mathfrak{m})$ be a local Noetherian ring.

(a) If M is a finitely generated Cohen-Macaulay module, then M does not have any embedded primes.

(b) If M is a finitely generated Cohen-Macaulay module, and $\mathfrak{p}$ is in $\operatorname{Ass} M$, then

$$\dim R/\mathfrak{p} = \dim M.$$

(1.23) If M and N are finitely generated R-modules, then

$$\operatorname{Ext}^i_R(N, M) = 0, \quad \text{whenever } i < \operatorname{depth} M - \dim N.$$

We prove that 1.22.(b) implies 1.22.(a). If $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ are both in $\operatorname{Ass} M$, then

$$\dim R/\mathfrak{p}_2 < \dim R/\mathfrak{p}_1.$$

□

We prove that Lemma 1.23 implies 1.22.(b). Let $\mathfrak{p} \in \operatorname{Ass} M$. It follows that

$$\operatorname{Ext}^0(R/\mathfrak{p}, M) \text{ (which equals } \operatorname{Hom}(R/\mathfrak{p}, M)) \neq 0.$$

The hypothesis says that

$$\operatorname{Ext}^i_R(R/P, M) = 0, \quad \text{whenever } i < \operatorname{depth} M - \dim R/\mathfrak{p}.$$

So,

$$0 \text{ is NOT less than } \operatorname{depth} M - \dim R/\mathfrak{p};$$

thus

$$\operatorname{depth} M - \dim R/\mathfrak{p} \leq 0.$$

Thus

$$\operatorname{depth} M \leq \dim R/\mathfrak{p}.$$

On the other hand, M is a Cohen-Macaulay module; so,

$$\dim M = \operatorname{depth} M \leq \dim R/\mathfrak{p} \leq \dim M.$$

The last inequality holds because $\mathfrak{p}$ is in $\operatorname{Ass} M$; so, in particular, $\mathfrak{p} \in \operatorname{Supp} M$. Equality holds across the board and $\dim R/\mathfrak{p} = \dim M$. □

We prove Lemma 1.23. Induct on $\dim N$.
Base case If $\dim N = 0$, then $\operatorname{ann} N$ is $\mathfrak{m}$-primary; so,

$$\operatorname{grade}(\operatorname{ann} N, M) =_* \operatorname{grade}(\mathfrak{m}, M) = \operatorname{depth} M.$$

For (*) use the result about $\operatorname{grade}(I, M)$ and $\operatorname{grade}(\sqrt{I}, M)$.

Thus, $\operatorname{Ext}^i(N,M)=0$ for

$$i<\operatorname{grade}(\operatorname{ann} N,M)=\operatorname{depth} M=\operatorname{depth} M-\dim N.$$

The inductive step. Suppose the assertion holds for all modules N' with $\dim N'<\dim N$. We prove the assertion for N.

Special case. Assume $N=R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$. Let x be an element of $\mathfrak{m}\setminus\mathfrak{p}$. Consider the exact sequence:

$$0\to R/\mathfrak{p}\xrightarrow{x} R/\mathfrak{p}\to R/(\mathfrak{p},x)\to 0. \tag{1.24.1}$$

We have $\dim R/(\mathfrak{p},x)<\dim R/\mathfrak{p}$; so induction applies to $\dim R/(\mathfrak{p},x)$ and

$$\operatorname{depth} M-\dim R/\mathfrak{p}<\operatorname{depth} M-\dim R/(\mathfrak{p},x).$$

In particular, if $i<\operatorname{depth} M-\dim R/\mathfrak{p}$, then $i+1<\operatorname{depth} M-\dim R/(\mathfrak{p},x)$. Apply $\operatorname{Hom}_R(-,M)$ to (1.24.1) to obtain

$$\underbrace{\operatorname{Ext}^i(R/(\mathfrak{p},x),M)}_{0}\to \operatorname{Ext}^i(R/\mathfrak{p},M)\xrightarrow{x}\operatorname{Ext}^i(R/\mathfrak{p},M)\to \underbrace{\operatorname{Ext}^{i+1}(R/(\mathfrak{p},x),M)}_{0}.$$

The R-module $\operatorname{Ext}^i(R/\mathfrak{p},M)$ is finitely generated. The map

$$\operatorname{Ext}^i(R/\mathfrak{p},M)\xrightarrow{x}\operatorname{Ext}^i(R/\mathfrak{p},M)$$

is surjective. Apply Nakayama's Lemma to conclude that $\operatorname{Ext}^i(R/\mathfrak{p},M)=0$.

The general case. We have seen that there exists a filtration of N of the form

$$0=N_0\subsetneq N_1\subsetneq N_2\subsetneq\cdots\subsetneq N_s=N$$

with N_j/N_{j-1} isomorphic to $R/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j$ of R. (Here is a quick sketch of the proof: If $N\neq 0$, then N has an associated prime $\mathfrak{p}_1$. Thus, $R/\mathfrak{p}_1$ may be embedded in N. Now look at N modded out by the image of $R/\mathfrak{p}_1$. etc.) Recall, also, that

$$\dim N=\max\{\dim R/\mathfrak{p}_j\}.$$

(Here is a quick sketch of the proof: We proved that if

$$0\to N'\to N\to N''\to 0$$

is exact, then

$$\operatorname{Supp} N=\operatorname{Supp} N'\cup\operatorname{Supp} N'';$$

hence, $\dim N=\max\{\dim N',\dim N''\}$. etc.) Thus, for each j, we have an exact sequence

$$0\to N_{j-1}\to N_j\to R/\mathfrak{p}_j\to 0,$$

with

$$\operatorname{Ext}^i(N_{j-1},M)=0,\ \text{for } i<\operatorname{depth} M-\dim N$$

by induction on j and

$$\operatorname{Ext}^i(R/\mathfrak{p}_j,M)=0,\ \text{for } i<\operatorname{depth} M-\dim N$$

by the special case (or the fact that for this value of j, $\dim R/\mathfrak{p}_j$ is less than $\dim N$. Hence, $\operatorname{Ext}^i(N_j, M) = 0$ for for $i < \operatorname{depth} M - \dim N$. Conclude that $\operatorname{Ext}^i(N, M) = 0$ for $i < \operatorname{depth} M - \dim N$. □

Observation 1.25. *If $(R, \mathfrak{m})$ is a Noetherian local ring, M is a finitely generated R-module, and $x_1, \cdots, x_n$ is a regular sequence on M, then*

$$M/(x_1, \cdots, x_n)M \text{ is Cohen-Macaulay} \iff M \text{ is Cohen-Macaulay}.$$

Proof. Recall that

$$\dim M/(x_1, \cdots, x_n)M = \dim M - n$$

and

$$\operatorname{depth} M/(x_1, \cdots, x_n)M = \operatorname{depth} M - n.$$

□

Observation 1.26. *If R is a Noetherian local ring and M is a finitely generated R-module, then*

- *M Cohen-Macaulay $\implies$ $M_\mathfrak{p}$ is Cohen-Macaulay, for all prime ideals $\mathfrak{p}$, and*
- *if $M_\mathfrak{p} \neq 0$, then $\operatorname{grade}(\mathfrak{p}, M) = \operatorname{depth}_{R_P} M_\mathfrak{p}$.*

Proof. We need only deal with $M_\mathfrak{p} \neq 0$. In this case, we have

$$\operatorname{grade}(\mathfrak{p}, M) \le \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \le \dim M_\mathfrak{p}.$$

The first inequality holds because grade can only go up when you localize.

We prove $\dim M_\mathfrak{p} \le \operatorname{grade}(\mathfrak{p}, M)$.

If $\operatorname{grade}(\mathfrak{p}, M) = 0$, then $\mathfrak{p} \in \operatorname{Ass} M$. But $M_\mathfrak{p} \neq 0$; hence

$$\operatorname{ann} M \subseteq \text{ some minimal prime } \subseteq \mathfrak{p} \subseteq \text{ some associated prime}.$$

The module M is Cohen-Macaulay; so M does not have any embedded prime ideals. Thus, $\mathfrak{p}$ is minimal in the support of M; hence, $\dim M_\mathfrak{p} = 0$.

February 7, 2019. **Properties of Cohen-Macaulay modules.**

Last time we proved that if M is a finitely generated module over the local Noetherian ring $(R, \mathfrak{m})$, then the following statements hold.

- If M is Cohen-Macaulay, then M does not have any embedded primes.
- If $x_1, \dots x_n$ is a regular sequence on M in $\mathfrak{m}$, then M is Cohen-Macaulay if and only if $M/(x_1, \dots, x_n)M$ is Cohen-Macaulay.

We are in the process of proving that if M is Cohen-Macaulay and $\mathfrak{p}$ is a prime of R with $M_\mathfrak{p} \neq 0$, then $M_\mathfrak{p}$ is Cohen-Macaulay and $\operatorname{grade}(\mathfrak{p}, M) = \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$.

We saw that

$$\operatorname{grade}(\mathfrak{p}, M) \le \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \le \dim M_\mathfrak{p}$$

always holds (provided $M_\mathfrak{p} \neq 0$). We also saw that if $\operatorname{grade}(\mathfrak{p}, M) = 0$, then $\dim M_\mathfrak{p} = 0$. (This was a cute argument that took advantage of the fact that Cohen-Macaulay modules do not have embedded primes.)

We finish the proof by induction on $\operatorname{grade}(\mathfrak{p}, M)$.

If $1 \leq \operatorname{grade}(\mathfrak{p}, M)$, then there exists x in $\mathfrak{p}$ with x regular on M. Thus,

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

is exact. Apply $- \otimes_R R_\mathfrak{p}$ to get the exact sequence

$$0 \to M_\mathfrak{p} \xrightarrow{x} M_\mathfrak{p} \to (M/xM)_\mathfrak{p} \to 0.$$

The module $(M/xM)_\mathfrak{p}$ is not zero by Nakayama's Lemma. We apply induction to M/xM. We know

$$\operatorname{grade}(\mathfrak{p}, M/xM) = \operatorname{grade}(\mathfrak{p}, M) - 1.$$

So, induction gives $(M/xM)_\mathfrak{p}$ is Cohen-Macaulay with

$$\dim(M/xM)_\mathfrak{p} = \operatorname{depth}(M/xM)_\mathfrak{p} = \operatorname{grade}(\mathfrak{p}, M/xM).$$

The proof is complete because

$$(M/xM)_\mathfrak{p} \cong M_\mathfrak{p}/xM_\mathfrak{p}$$

and x is regular on $M_\mathfrak{p}$, hence

$$\dim M_\mathfrak{p}/xM_\mathfrak{p} = \dim M_\mathfrak{p} - 1 \quad \text{and} \quad \operatorname{depth} M_\mathfrak{p}/xM_\mathfrak{p} = \operatorname{depth} M_\mathfrak{p} - 1.$$

□

Theorem 1.27. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. If $a_1, \dots, a_r \in \mathfrak{m}$, then the following statements are equivalent:*

(a) *$a_1, \dots, a_r$ is a regular sequence on R;*
(b) *$\operatorname{ht}(a_1, \dots, a_i) = i$, for $1 \leq i \leq r$;*
(c) *$\operatorname{ht}(a_1, \dots, a_r) = r$;*
(d) *$a_1, \dots, a_r$ is part of a system of parameters for R.*

Remarks.

- The assertions (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) all hold without assuming that R is Cohen-Macaulay.
- The assertion (d) $\implies$ (a) requires that R be Cohen-Macaulay.
- The most important take away is that in a Cohen-Macaulay local ring "is a regular sequence" is equivalent to "is part of a system of parameters".

Proof.
(a) $\implies$ (b): On the one hand, $\operatorname{ht}(a_1, \cdots, a_i) \leq i$ by the Krull intersection Theorem. On the other hand, a_1 is not in any associated prime ideal of R; so in particular a_1 is not in any minimal prime of R. Thus, $1 \leq \operatorname{ht}(a_1)$. Similarly, a_2 is not in any minimal prime of $R/(a_1)$; hence $2 \leq \operatorname{ht}(a_1, a_2)$. etc.

(b) $\implies$ (c): This is obvious.

(c) $\implies$ (d): If $\dim R = r$, then you are finished.

Otherwise, $\mathfrak{m}$ is not a minimal prime over $(a_1, \dots, a_r)$. Pick a_{r+1} in

$$\mathfrak{m} \setminus \bigcup_{\mathfrak{p} \text{ minimal over } (a_1,\dots,a_r)} \mathfrak{p}.$$

Observe that

$$r = \operatorname{ht}(a_1, \dots, a_r) < \operatorname{ht}(a_1, \dots, a_{r+1}) \le r + 1.$$

The final inequality is due to the Krull principal ideal theorem.

Continue until finished.

(d) $\implies$ (a):

It suffices to show that every system of parameters in a Cohen-Macaulay ring is a regular sequence.

Let R be a local Cohen-Macaulay ring of dimension n and $a_1, \dots, a_n$ be a system of parameters for R.

If $\mathfrak{p} \in \operatorname{Ass} R$, then $\dim R/\mathfrak{p} = \dim R$ by Theorem 1.22.(b), which is the full version of a Cohen-Macaulay module does not have any embedded primes. It follows that $\mathfrak{p}$ is a minimal prime ideal of R. It also follows that $a_1 \notin \mathfrak{p}$. Indeed, if a_1 were in $\mathfrak{p}$, then $\dim R/(a_1) = \dim R$ and

$$1 = \dim \bar{R} - (n-1) \le \dim \bar{R}/(a_2, \dots, a_n)\bar{R} = \dim R/(a_1, \dots, a_n) = 0,$$

for $\bar{R} = R/(a_1)$, and this is nonsense.

Thus, a_1 is not in any associated prime ideals of R and a_1 is regular on R.

At this point $R/(a_1)$ is Cohen-Macaulay of dimension less than $\dim R$ and $a_2, \dots, a_r$ is a system of parameters on $R/(a_1)$. By induction we conclude that $a_2, \dots, a_n$ is a regular sequence on $R/(a_1)$; and therefore, $a_1, \dots, a_n$ is a regular sequence on R. □

Corollary 1.28. *If I is a proper ideal in the Cohen-Macaulay local ring $(R, \mathfrak{m})$, then $\operatorname{grade} I = \operatorname{ht} I$.*

Proof. We showed in Corollary 1.20 that $\operatorname{grade} I \le \operatorname{ht} I$ for every proper ideal on a Noetherian ring. When the ring is Cohen-Macaulay local, the two quantities are equal for all ideals. Let $r = \operatorname{ht} I$. Select $a_1, \dots, a_r$ in I with $\operatorname{ht}(a_1, \dots, a_r) = r$. Thus, $(a_1, \dots a_r)$ is a regular sequence in I on R by Theorem 1.27. It follows that $\operatorname{ht} I \le \operatorname{grade} I$; and therefore, the two quantities are equal. □

Proposition. *If $(R, \mathfrak{m})$ is a local Cohen-Macaulay ring, then*

$$\operatorname{ht} I + \dim R/I = \dim R.$$

Remark. Some people (and some computer packages, notably Macaulay2) use the word codimension rather than height. Or, said another way, if you say to Frank, "Consider an ideal of height 2." He will immediately ask, "What does height mean?". If you translate "height" as "codimension", he will be satisfied.

This result justifies the translation of "height" into "codimension".

February 12, 2019

Last Time: In a Cohen-Macaulay local ring, then

$$a_1, \dots, a_r \text{ is a regular sequence } \iff a_1, \dots, a_r \text{ is part of a system of parameters.}$$

Coming attractions:

- In a Cohen-Macaulay local ring, height equals codimension.
- Cohen-Macaulay local rings are catenary.
- The definition of general Cohen-Macaulay rings.
- The characterization of Cohen-Macaulay rings in terms of unmixedness.
- If R is a Cohen-Macaulay ring, then $R[x_1, \dots, x_n]$ is also Cohen-Macaulay.
- Cohen-Macaulay rings are universally catenary.
- New Chapter: regular local rings.

Proposition 1.29. *If $(R, \mathfrak{m})$ is a local Cohen-Macaulay ring, then*

$$\operatorname{ht} I + \dim R/I = \dim R.$$

Proof. It suffices to prove the claim for prime ideals because

$$\operatorname{ht} I = \min\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \text{ is minimal in } \operatorname{Supp} R/I\}, \text{ and}$$

$$\dim R/I = \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \text{ is minimal in } \operatorname{Supp} R/I\}.$$

Assume the assertion holds for prime ideals and $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are minimal in $\operatorname{Supp} R/I$ with $\operatorname{ht} I = \operatorname{ht} \mathfrak{p}_1$ and $\dim R/I = \dim R/\mathfrak{p}_2$, then

$$\operatorname{ht} \mathfrak{p}_1 \le \operatorname{ht} \mathfrak{p}_2 = \dim R - \dim R/\mathfrak{p}_2 \le \dim R - \dim R/\mathfrak{p}_1 = \operatorname{ht} \mathfrak{p}_1.$$

Thus equality holds everywhere and $\operatorname{ht} \mathfrak{p}_1 = \operatorname{ht} \mathfrak{p}_2$ and $\dim R/\mathfrak{p}_1 = \dim R/\mathfrak{p}_2$.

We prove the assertion for the prime ideal $\mathfrak{p}$. Let $r = \operatorname{ht} \mathfrak{p}$. Let $a_1, \dots, a_r$ be a regular sequence in $\mathfrak{p}$. (Keep in mind that height equals grade in a Cohen-Macaulay local ring.) The prime ideal $\mathfrak{p}$ is minimal over $a_1, \dots, a_r$; so $\mathfrak{p} \in \operatorname{Ass}(R/(a_1, \dots, a_r))$. Therefore,

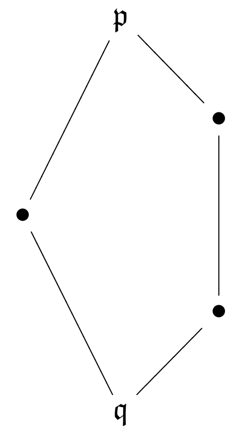
$$\dim R/\mathfrak{p} =_{\dagger} \dim R/(a_1, \dots, a_r) =_{\ddagger} \dim R - r = \dim R - \operatorname{ht} \mathfrak{p}.$$

† Use the full statement of the Theorem that Cohen-Macaulay modules do not have embedded primes.

‡ The dimension drops by one whenever one mods out by a regular element. □

Definition 1.30. A Noetherian ring R is called *catenary* if, for all pairs of prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$ in R, all saturated chains of prime ideals from $\mathfrak{q}$ to $\mathfrak{p}$ have the same length.

For example, in a catenary ring, one does not have saturated chains of prime ideals that look like:



Nagata found examples of Noetherian rings which are not catenary. There is a recent paper [2] which gives large families of Noetherian local Unique Factorization Domains which are not catenary. Furthermore, the failure of the ring to be catenary can be made to be arbitrarily bad. (Keller and I heard a lecture by Susan Loepp about this paper in April.)

Corollary 1.31. *If R is a Cohen-Macaulay local ring, then R is catenary.*

Proof. It suffices to show that

(1.31.1) $$\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q} + \operatorname{ht}\mathfrak{p}/\mathfrak{q}$$

for all prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$ in R.

Indeed, if

$$\mathfrak{q} = \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_a = \mathfrak{p}$$

and

$$\mathfrak{q} = \mathfrak{q}'_0 \subsetneq \mathfrak{q}'_1 \subsetneq \cdots \subsetneq \mathfrak{q}'_{a'} = \mathfrak{p}$$

both are saturated chains, then one applies (1.31.1) multiple times to see that $\operatorname{ht}\mathfrak{q}_i = \operatorname{ht}\mathfrak{q} + i$ and $\operatorname{ht}\mathfrak{q}'_i = \operatorname{ht}\mathfrak{q} + i$ for all i. Thus

$$\operatorname{ht}\mathfrak{q} + a = \operatorname{ht}\mathfrak{q}_a = \operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}'_{a'} = \operatorname{ht} q + a';$$

and $a = a'$.

It turns out that we know exactly enough to prove (1.31.1). Indeed, we learned in Proposition 1.29 that

$$\underbrace{\dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}}_{\operatorname{ht}\mathfrak{p}/\mathfrak{q}} + \underbrace{\operatorname{ht}\mathfrak{q}R_{\mathfrak{p}}}_{\operatorname{ht}\mathfrak{q}} = \underbrace{\dim R_{\mathfrak{p}}}_{\operatorname{ht}\mathfrak{p}}.$$

This completes the proof. □

Definition 1.32. A Noetherian ring R is *Cohen-Macaulay* if $R_{\mathfrak{m}}$ is Cohen-Macaulay for all maximal ideals $\mathfrak{m}$ of R.

Definition 1.33. If I is a proper ideal in a Noetherian ring R, then I is *unmixed* if

$$\operatorname{ht}\mathfrak{p} = \operatorname{ht} I, \quad \text{for all } \mathfrak{p} \in \operatorname{Ass} R/I.$$

Theorem 1.34. *Let R be a Noetherian ring. Then*

$$R \text{ is Cohen-Macaulay} \iff \begin{cases} \text{Every ideal of height } r \text{ generated by } r \text{ elements} \\ \text{is unmixed for all } r. \end{cases}$$

Proof.
($\Rightarrow$) We assume R is Cohen-Macaulay. Let I be a proper ideal of R of height r generated by $(a_1, \dots, a_r)$ and let $\mathfrak{p}$ be in $\operatorname{Ass} R/I$. We must show that $\operatorname{ht}\mathfrak{p} = r$. If $\mathfrak{p}$ is minimal over I, then $r = \operatorname{ht} I \le \operatorname{ht}\mathfrak{p} \le_{\dagger} r$

† This inequality is the Krull principal ideal Theorem.

Now consider $\mathfrak{p}$ to be an arbitrary element of $\operatorname{Ass} R/I$. Localize at $\mathfrak{p}$. The ideal

$$(a_1, \dots, a_r)R_{\mathfrak{p}}$$

has height r in the Cohen-Macaulay local ring $R_{\mathfrak{p}}$. It follows (from Theorem 1.27) that $a_1, \dots, a_r$ is a regular sequence in $R_{\mathfrak{p}}$. Thus,

$$(R/I)_{\mathfrak{p}} = R_{\mathfrak{p}}/(a_1, \dots a_r)R_{\mathfrak{p}}$$

is a Cohen-Macaulay local ring. Therefore, there are no embedded primes, and $\mathfrak{p}$ really is minimal over $(a_1, \dots, a_r)$ (and therefore has height r by the first part of the argument).

February 14, 2019

We are proving

Theorem. *Let R be a Noetherian ring. Then*

$$R \text{ is Cohen-Macaulay} \iff \begin{cases} \text{Every ideal of height } r \text{ generated by } r \text{ elements} \\ \text{is unmixed for all } r. \end{cases}$$

We proved ($\Rightarrow$) last time.

($\Leftarrow$) Let $\mathfrak{p}$ be a prime ideal of R. We prove that $R_{\mathfrak{p}}$ is Cohen-Macaulay.

Let $r = \operatorname{ht}\mathfrak{p}$. We choose $a_1, \dots, a_r$ in $\mathfrak{p}$ with $\operatorname{ht}(a_1, \dots, a_i) = i$, for all i. We use the usual procedure, but the hypothesis that

$$\operatorname{ht}(a_1, \dots, a_i) = i \implies (a_1, \dots, a_i) \text{ is unmixed}$$

guarantees that $a_1, \dots, a_r$ is a regular sequence in $\mathfrak{p}$ on R. (We will do this slowly.)

If $\operatorname{ht}\mathfrak{p} = 0$, then $R_{\mathfrak{p}}$ is automatically Cohen-Macaulay.

($\dagger$) The hypothesis applies to $\operatorname{ht}(0) = 0$ so every associated prime of R is a minimal prime.

If $1 \le \operatorname{ht}\mathfrak{p}$, then take $a_1 \in \mathfrak{p}$, but not in any minimal prime of R. Thus, (a_1) has height 1.

($*$) The hypothesis guarantees that every prime of $\operatorname{Ass} R/(a_1)$ has height 1.

Apply ($\dagger$) to see that a_1 is a regular element of R.

If $2 \le \operatorname{ht}\mathfrak{p}$, then take $a_2 \in \mathfrak{p}$, but not in any prime minimal over (a_1). Thus, $\operatorname{ht}(a_1, a_2) = 2$.
$(**)$ The hypothesis guarantees that every prime of $\operatorname{Ass} R/(a_1, a_2)$ has height 2.

We chose a_2 to avoid the minimal primes of $R/(a_1)$; but the hypothesis guarantees that every associated primes of $R/(a_1)$ is already a minimal prime; see $(*)$. Thus our choice of a_2 is not in any prime from $\operatorname{Ass} R/(a_1)$; that is a_1, a_2 is a regular sequence.

etc. Use $(**)$ to see that a_1, a_2, a_3 is a regular sequence.

Eventually, we have chosen $a_1, \dots, a_r$ in $\mathfrak{p}$, where $r = \operatorname{ht}\mathfrak{p}$ and every prime in

$$\operatorname{Ass} R/(a_1, \dots, a_i)$$

has height i for each i.

In particular, $a_1, \dots, a_r$ is a regular sequence in $\mathfrak{p}$ on R.

It follows that $a_1, \dots, a_r$ is a regular sequence in $\mathfrak{p}R_\mathfrak{p}$ on $R_\mathfrak{p}$.

Thus,

$$r \le \operatorname{depth} R_\mathfrak{p} \le \dim R_\mathfrak{p} = r;$$

and $R_\mathfrak{p}$ is Cohen-Macaulay. □

Theorem 1.35. *If R is a Cohen-Macaulay ring, then $R[x_1, \dots, x_n]$ is also a Cohen-Macaulay ring.*

Proof. It suffices to prove the result when $n = 1$. Let $\mathfrak{p}$ be a prime ideal of $S = R[x]$. Of course,

$$S_\mathfrak{p} = (R_{R\cap\mathfrak{p}}[x])_\mathfrak{p}.$$

It suffices to prove that if $(R, \mathfrak{m}, \boldsymbol{k})$ is a local Cohen-Macaulay ring and $\mathfrak{p}$ is a prime ideal of $S = R[x]$ with $\mathfrak{m}S \subseteq \mathfrak{p}$, then $S_\mathfrak{p}$ is a Cohen-Macaulay ring.

There are two options. Either $\mathfrak{m}S = \mathfrak{p}$ or $\mathfrak{m}S \subsetneq \mathfrak{p}$. It turns out that

$$\dim S_\mathfrak{p} = \begin{cases} \dim R & \text{if } \mathfrak{m}S = \mathfrak{p} \\ \dim R + 1 & \text{if } \mathfrak{m}S \subsetneq \mathfrak{p}. \end{cases}$$

The direction $\ge$ is clear. If $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r = \mathfrak{m}$ is a chain of primes in R that exhibit $\operatorname{ht}\mathfrak{m} = r$, then

$$\mathfrak{q}_0 S \subsetneq \cdots \subsetneq \mathfrak{q}_r S = \mathfrak{m}S$$

is a chain of prime ideals in S which exhibit $\ge$. (Keep in mind that $\mathfrak{q}_i S$ is prime because $S/\mathfrak{q}_i S$ is isomorphic to the domain $(R/\mathfrak{q}_i)[x]$. Furthermore, the inclusion $\mathfrak{q}_{i-1}S \subsetneq \mathfrak{q}_i S$ is strict because $R \cap \mathfrak{q}_i S = \mathfrak{q}_i$.)

To establish the direction $\le$, it suffices to prove

$$\mathfrak{m}S \subsetneq \mathfrak{p} \implies \dim S_\mathfrak{p} \le \dim R + 1.$$

Assume $\mathfrak{m}S \subsetneq \mathfrak{p}$. Observe that $S/\mathfrak{m}S = \boldsymbol{k}[x]$. It follows that $\mathfrak{p}/\mathfrak{m}S = (\bar{f})$ for some monic polynomial $\bar{f}$ in $\boldsymbol{k}[x]$. Lift $\bar{f}$ back to a monic polynomial f in S. So, $\mathfrak{p} = (\mathfrak{m}, f)S$. Let $a_1, \dots, a_r$ be a system of parameters for R. It follows that $a_1, \dots, a_r, f$ is a system of parameters for $S_\mathfrak{p}$.

Here is a brief explanation of the last claim. The $R/(a_1, \dots, a_r)$-module $S/(a_1, \dots, a_r, f)$ is a free $R/(a_1, \dots, a_r)$-module of finite rank because f is a monic polynomial. The ring $R/(a_1, \dots, a_r)$ has finite length; so the module $S/(a_1, \dots, a_r, f)$ has finite length. Thus $S/(a_1, \dots, a_r, f)$ is an Artinian ring. Recall from last semester that if A is an Artinian ring, then A is a direct product of local Artinian rings; in particular, $A = \prod_{\mathfrak{M} \in \operatorname{Spec} A} A_{\mathfrak{M}}$. Thus, the localization $(S/(a_1, \dots, a_r, f))_{\mathfrak{p}}$ is a summand of $S/(a_1, \dots, a_r, f)$; and this summand also has finite length.

Now it is easy to finish. Let $a_1, \dots, a_r$ be a system of parameters for R. We have seen that the following collection of elements form a system of parameters for $\mathfrak{p}$:

$$\begin{cases} a_1, \dots, a_r, & \text{if } \mathfrak{m}S = \mathfrak{p}, \text{ and} \\ a_1, \dots, a_r, f, & \text{if } \mathfrak{m}S \subsetneq \mathfrak{p}. \end{cases}$$

The elements $a_1, \dots, a_r$ form a regular sequence on R. The R-module S is flat. Thus, $a_1, \dots, a_r$ is a regular sequence on S. If $\mathfrak{m}S = \mathfrak{p}$ the proof is complete. Henceforth, $\mathfrak{m}S \subsetneq \mathfrak{p}$. The polynomial f is monic in $\frac{R}{(a_1, \dots, a_r)}[x]$. Thus, $a_1, \dots, a_r, f$ is a regular sequence on S in $\mathfrak{p}$. It follows that $a_1, \dots, a_r, f$ is a regular sequence on $S_{\mathfrak{p}}$. Some system of parameters of $S_{\mathfrak{p}}$ is a regular sequence. Thus, $S_{\mathfrak{p}}$ is Cohen-Macaulay. □

Definition 1.36. A Noetherian ring R is *universally catenary* if every finitely generated R-algebra is catenary.

Recall that

- a Noetherian ring R is called *catenary* if, for all pairs of prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$ in R, all saturated chains of prime ideals from $\mathfrak{q}$ to $\mathfrak{p}$ have the same length;
- R is a catenary if and only if $R_{\mathfrak{p}}$ is catenary for all $\mathfrak{p}$;
- if R is catenary, then R/I is catenary for all I; and
- every Cohen-Macaulay local ring is catenary.

Corollary 1.37. *If R is a Cohen-Macaulay ring, then R is universally catenary.*

Proof. If R is Cohen-Macaulay, then $R[x_1, \dots, x_n]$ is Cohen-Macaulay; hence $\left(\frac{R[x_1, \dots, x_n]}{I}\right)_{\mathfrak{p}}$ is catenary for all I and all $\mathfrak{p}$. Thus, $\frac{R[x_1, \dots, x_n]}{I}$ is catenary for all I. In particular, every finitely generated R-algebra is catenary. □

2. Regular local rings

This section loosely follows sections 19 and 20 of [6].

Definition 2.1. The Noetherian local ring $(R, \mathfrak{m}, \boldsymbol{k})$ is a *regular local ring* if $\mathfrak{m}$ can be generated by $\dim R$ elements.

Remarks 2.2.

(a) The minimal number of generators of $\mathfrak{m}$ is often denoted $\mu(\mathfrak{m})$ and is always called the *embedding dimension* of R.
(b) Nakayama's Lemma guarantees that $\mu(\mathfrak{m}) = \dim_{\boldsymbol{k}} \mathfrak{m}/\mathfrak{m}^2$.
(c) According to the Krull Principal Ideal Theorem, $\dim R \le \mu(\mathfrak{m})$. So, regular local rings satisfy an extremal condition. The Krull dimension of a regular local ring is as large as possible once its embedding dimension is known.

Examples 2.3.

(a) A local ring of Krull dimension zero is regular if and only if it is a field.
(b) A local ring of Krull dimension one is regular if and only if it is a local Principal Ideal Domain and R is not a field. See Observation 2.4. (Local Principal Ideal domains are usually called Discrete Valuation Rings (DVRs).) Some examples of DVRs are $\mathbb{Z}_{(p)}$ where p is a prime integer, $\boldsymbol{k}[x]_{(x)}$, $\boldsymbol{k}[[x]]$. If the words are meaningful to you, let D be any Dedekind domain and $\mathfrak{p}$ be any non-zero prime ideal of D, then $D_{\mathfrak{p}}$ is a DVR. In particular, if F is any finite field extension of $\mathbb{Q}$ and D is the ring of algebraic integers in F:
$$D = \{\alpha \in F \mid \alpha \text{ satisfies a monic polynomial with coefficients in } \mathbb{Z}\},$$
then D is a Dedekind domain.

Observation 2.4. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of Krull dimension one. Then R is a regular local ring if and only if R is a local Principal Ideal Domain and R is not a field.*

Proof.
($\Leftarrow$) This direction is clear.

($\Rightarrow$) Let $\mathfrak{m} = (x)$. The dimension of R is one; so $x^n \neq 0$ for any n. On the other hand, the Krull Intersection Theorem guarantees that $\cap_i \mathfrak{m}^i = 0$. If $r \in R$, then there exists i with $r \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. Thus, every non-zero element of R is equal to ux^i for some unit u of R and some non-negative integer i. It follows that every ideal of R is equal to (x^i) for some i. Of course, R is a domain because $u_1x^{i_1}$ times $u_2x^{i_2}$ is $u_1u_2x^{i_1+i_2}$, which is not zero. □

Theorem 2.5. *Every regular local ring is a domain.*

Proof. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a regular local ring. We have already seen that if $\dim R \le 1$, then R is a domain. Henceforth, we assume that $1 \le \dim R$. We know that $\mathfrak{m}$ is not contained in $\mathfrak{m}^2$ and $\mathfrak{m}$ is not contained in any minimal prime of R. Use the Prime Avoidance Lemma to select $x \in \mathfrak{m}$; but $x \notin \mathfrak{m}^2$ and x not in any minimal prime of R.

Notice that

$$\dim R - 1 \leq_{\dagger} \dim R/(x)R <_{\ddagger} \dim R$$

† If M is a finitely generated module over a local ring and x is in the maximal ideal of the local ring, then $\dim M - 1 \leq \dim M$.

‡ x is not in any minimal prime of R.

So dim $R/(x)R = \dim R - 1$. Also the embedding dimension of R/xR is equal to the embedding dimension of R minus one (because x is a minimal generator of $\mathfrak{m}$). Thus, $R/(x)R$ is a regular local ring. We learn by induction that $R/(x)R$ is a domain. Thus, $(x)R$ is a prime ideal of R. Of course,

$$0 <_{\dagger\dagger} \operatorname{ht}(x)R \leq_{\ddagger\ddagger} 1$$

†† x is not in any minimal prime of R

‡‡ Krull Principal Ideal Theorem.

Let $\mathfrak{p}$ be a minimal prime ideal inside $(x)R$. If θ is an element of $\mathfrak{p}$, then $\theta = xr$ for some $r \in R$. Recall that $\mathfrak{p}$ is a prime ideal of R, $\theta \in \mathfrak{p}$ and $x \notin \mathfrak{p}$. It follows that $r \in \mathfrak{p}$. Hence, $\mathfrak{p} = x\mathfrak{p}$. Apply Nakayama's Lemma to conclude that $\mathfrak{p} = 0$. □

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a regular local ring and $x_1, \ldots, x_n$ be a minimal generating set for $\mathfrak{m}$. Then*

(a) *$(x_1, \ldots, x_i)$ is a prime ideal of R for each i,*
(b) *$x_1, \ldots, x_n$ is a regular sequence on R, and*
(c) *R is Cohen-Macaulay.*

Proof. The element x_1 is not in $\mathfrak{m}^2$ (because x_1 is a minimal generator of R) and is not in any minimal prime of R because (according to the previous theorem) the only minimal prime of R is (0). The proof of the previous theorem now shows that x_1 generates a prime ideal. Finish by induction. □

Goal 2.7. *If R is a regular local ring and $\mathfrak{p}$ is a prime ideal of R, then $R_{\mathfrak{p}}$ is regular local ring.*

A little history. This is actually hard. Krull introduced the concept of regular local rings in 1937. In the 1940s, Zariski proved that a regular local ring corresponds to a smooth point on an algebraic variety. Let Y be an algebraic variety contained in affine n-space over a perfect field $\boldsymbol{k}$. (The field $\boldsymbol{k}$ is perfect if every algebraic extension is separable. In particular, every field of characteristic zero and every finite field is perfect. Also, if the characteristic of $\boldsymbol{k}$ is the positive prime integer p, then $\boldsymbol{k}$ is perfect if and only if $\boldsymbol{k}$ is closed under the taking of p^{th} roots.) Suppose that $Y = V(f_1, \ldots, f_m)$, where each f_i is a polynomial in $\boldsymbol{k}[x_1, \ldots, x_n]$. Then Y is nonsingular at P if Y satisfies the following Jacobian condition. If $\text{Jac} = (\partial f_i/\partial x_j)$ is the matrix of partial derivatives of the defining equations of the variety, then Y is smooth at P if and only if

$$\operatorname{rank}(\text{Jac}\,|_P) = n - \dim Y.$$

Remark 2.8. This notion is intuitively clear.

In this discussion $Y = V(f_1, \dots, f_m)$, with each f_i an element of $R = \boldsymbol{k}[x_1, \dots, x_n]$. We focus on the case when Y is a complete intersection; in other words, m is the codimension of Y; that is,

$$m = n - \dim Y.$$

Let P be a point on Y. Then Y is smooth at P if there is a linear space of dimension $\dim Y$ tangent to Y at P.

- The curve C is smooth at the point P if there is a well-defined line tangent to C at P.
  - If $C = V(f(x,y))$ is a curve in $\mathbb{A}^2$ and P is a point on C, then C is smooth at P if and only if
$$\operatorname{rank} \begin{bmatrix} \frac{\partial f}{\partial x}|_P & \frac{\partial f}{\partial y}|_P \end{bmatrix} = 1.$$
  - If $C = V(f_1(x,y,z), f_2(x,y,z))$ is a curve in $\mathbb{A}^3$ and P is a point on C, then C is smooth at P if and only if
$$\operatorname{rank} \begin{bmatrix} \frac{\partial f_1}{\partial x}|_P & \frac{\partial f_1}{\partial y}|_P & \frac{\partial f_1}{\partial z}|_P \\ \frac{\partial f_2}{\partial x}|_P & \frac{\partial f_2}{\partial y}|_P & \frac{\partial f_2}{\partial z}|_P \end{bmatrix} = 2.$$
- A surface S is smooth at the point P if S has a well-defined tangent plane at P.
  - If $S = V(f(x,y,z))$ is a surface in $\mathbb{A}^3$ and P is a point on S, then S is smooth at P if and only if
$$\operatorname{rank} \begin{bmatrix} \frac{\partial f}{\partial x}|_P & \frac{\partial f}{\partial y}|_P & \frac{\partial f}{\partial z}|_P \end{bmatrix} = 1.$$
- Here are two further observations.
  - In each example, we are using "gradients are perpendicular to level sets", which is one of the main thoughts in third semester calculus.
  - If Y is smooth at P, then the implicit function theorem may be applied to express Y near P as a function the points on the tangent plane.

Question: What does $\operatorname{rank} \operatorname{Jac}|_P$ have to do with $R_{\mathfrak{p}}$ is regular?

Answer: Everything. If P is the point $(a_1, \dots, a_n)$, then the ideal $\mathfrak{p}$ is $(x_1 - a_1, \dots, x_n - a_n)$ and each f_i is a polynomial in $x_1 - a_1, \dots, x_n - a_n$ with zero constant term. (There are at least three ways to say this. "Move P to the origin." "Use Taylor series." "Replace x_i with $(x_i - a_i) + a_i$.")

At any rate, $\frac{\partial f_i}{\partial x_j}|_P$ is the coefficient of $x_j - a_j$ in $f_i(x_1 - a_1, \dots, x_n - a_n)$. If $\operatorname{rank} \operatorname{Jac}|_P = m$, then after renumbering and taking $\boldsymbol{k}$-linear combinations, then $f_i = x_i - a_i + \text{h.o.t.}$ for $1 \le i \le m$. In other words, $f_1, \dots, f_m$ is the beginning of a minimal generating set for $\mathfrak{p}$. Thus,

$$R_{\mathfrak{p}}/(f_1, \dots, f_m)$$

is a regular local ring.

Back to the history. Zariski proved that Y is nonsingular at P if and only if the local ring of Y at P is regular. Mathematicians were not able to claim that "regular local" was exactly the right concept until Serre [9] proved that regular local localizes in 1955.

Serre used homological algebra to prove that the concept of regular local localizes. He proved the following theorem.

Theorem 2.9. [Serre] *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a Noetherian local ring, then R is regular if and only if every finitely generated R-module has finite projective dimension.*

2.10. Here is how the proof goes:

(a) If $(R, \mathfrak{m}, \boldsymbol{k})$ is a regular local ring, then $\operatorname{pd}_R \boldsymbol{k}$ is finite. (If $a_1, \dots, a_r$ is a regular sequence in a commutative Noetherian ring R, then the Koszul complex on $a_1, \dots, a_r$ is a resolution of $R/(a_1, \dots, a_r)$ by free R-modules.)

(b) If $(R, \mathfrak{m}, \boldsymbol{k})$ is a local ring, then $\operatorname{pd}_R \boldsymbol{k}$ is finite if and only if $\operatorname{pd}_R M$ is finite for all finitely generated R-modules M. (The functors $\operatorname{Tor}_i^R(M, -)$ are the left derived functors of $M \otimes_R -$; the functors $\operatorname{Tor}_i^R(-, N)$ are the left derived functors of $- \otimes_R N$; and $\operatorname{Tor}_i^R(M, N)$ may be computed using either component.)

(c) The direction $(\Leftarrow)$ from Theorem 2.9.

(d) Prove Goal 2.7, which is: If R is a regular local ring and $\mathfrak{p}$ is a prime ideal of R, then $R_\mathfrak{p}$ is regular local ring.

We start with 2.10.(d).

Assume Theorem 2.9 and assertion 2.10.(b). Let R be a regular local ring and let $\mathfrak{p}$ be a prime ideal of R. We want to show that $\operatorname{pd}_{R_\mathfrak{p}} \frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}} < \infty$. The module $R/\mathfrak{p}$ is a finitely generated R-module and R is regular local; thus, $R/\mathfrak{p}$ has a finite resolution F by projective R-modules. Apply the exact functor $- \otimes_R R_\mathfrak{p}$ to conclude that $F_\mathfrak{p}$ is a finite resolution of $\frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}}$ by projective $R_\mathfrak{p}$-modules. Assertion 2.10.(b) yields that every finitely generated $R_\mathfrak{p}$ module has finite projective dimension; hence, Theorem 2.9 guarantees that $R_\mathfrak{p}$ is a regular local ring.

February 26, 2019

Goal 1: If $(R, \mathfrak{m}, \boldsymbol{k})$ is a regular local ring and $\mathfrak{p}$ is a prime ideal of R, then $R_\mathfrak{p}$ is a regular local ring.

Goal 2: If R is a local Noetherian ring, then

$$R \text{ is regular } \iff \operatorname{pd}_R M < \infty \text{ for all finitely generated } R\text{-modules.}$$

- We already saw that Goal 2 implies Goal 1.
- To prove Goal 2 we show
  - If R is commutative Noetherian ring and $x_1, \dots, x_n$ a regular sequence on R, then $\operatorname{pd}_R R/(x_1, \dots, x_n) < \infty$. (We called this result 2.10.(a).)

  - If $(R, \mathfrak{m}, \boldsymbol{k})$ is a local Noetherian ring with $\operatorname{pd}_R \boldsymbol{k} < \infty$, then $\operatorname{pd}_R M < \infty$ for all finitely generated R-modules M. (We called this result 2.10.(b).)
  - The direction $(\Leftarrow)$ of Goal 2.

Observe that 2.10.(a) and 2.10.(b) establish the direction $(\Rightarrow)$ of Goal 2.

We attack 2.10.(b). We prove: If $(R, \mathfrak{m}, \boldsymbol{k})$ is a local ring and $\operatorname{pd}_R \boldsymbol{k}$ is finite then $\operatorname{pd}_R M$ is finite for all finitely generated R-modules M.

First we do a few warm-up comments.

- If $(R, \mathfrak{m}, \boldsymbol{k})$ is a local Noetherian ring and P is a finitely generated projective R-module, then P is a free R-module.
- If R is a ring and M and N are R-modules, then $\{\operatorname{Tor}_i^R(-, N)\}$ and $\{\operatorname{Tor}_i^R(M, -)\}$ are the left derived functors of $- \otimes_R N$ and $M \otimes_R -$, respectively; $\operatorname{Tor}_i^R(M, N)$ can be computed in either component; and each short exact sequence of modules gives rise to a long exact sequence of Tor.
- Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Noetherian ring and M be a finitely generated R-module. We want a good definition of a minimal surjection of a free R-module onto M.

I am done making lists. It is time to get to work.

Today's first job. Projective modules over local rings.

Recall that the R-module P is projective if every picture of R-module homomorphisms of the form

$$\begin{array}{ccc} & & P \\ & & \downarrow \\ A & \twoheadrightarrow & B \end{array}$$

gives rise to a commutative diagram of R-module homomorphisms of the form

$$\begin{array}{ccc} & & P \\ & \overset{\exists}{\swarrow} & \downarrow \\ A & \twoheadrightarrow & B \end{array}$$

(In a Dedekind Domain D every ideal I is a projective D-module but only the principal ideals are free R-modules.)

Remark 2.11. If $(R, \mathfrak{m}, \boldsymbol{k})$ is a local Noetherian ring and P is a (finitely generated) projective R-module, then P is a free R-module.

Proof. Let b be the minimal number of generators of P, F be a free R-module of rank b, and $\pi : F \to P$ be a surjection. (See (2.13.1), if necessary.) The defining property of projective modules says that there is a splitting R-module homomorphism $\sigma : P \to F$ with $\pi \circ \sigma = \operatorname{id}_P$. It follows that F is the internal direct sum $\operatorname{im} \sigma \oplus \ker \pi$. Look at $\frac{F}{\mathfrak{m}F} = \frac{\operatorname{im} \sigma}{\mathfrak{m}(\operatorname{im} \sigma)} \oplus \frac{\ker \pi}{\mathfrak{m}(\ker \pi)}$ to see that $\ker \pi = 0$. $\square$

Today's second job. We discuss Tor.

Recall that $M \otimes_R -$ is a right exact covariant functor. If

$$0 \to N' \to N \to N'' \to 0 \tag{2.11.1}$$

is a short exact sequence of R-modules and R-module homomorphisms, then

$$M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

is exact. The functors Tor_i are cooked up to answer the question "But what is the kernel on the left?" That is, the short exact sequence (2.11.1) gives rise to the long exact sequence

$$\cdots \to \operatorname{Tor}_2^R(M, N'') \to \operatorname{Tor}_1^R(M, N') \to \operatorname{Tor}_1^R(M, N) \to \operatorname{Tor}_1^R(M, N'') \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0.$$

Similarly, If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules and R-module homomorphisms, then

$$\cdots \to \operatorname{Tor}_2^R(M'', N) \to \operatorname{Tor}_1^R(M', N) \to \operatorname{Tor}_1^R(M, N) \to \operatorname{Tor}_1^R(M'', N) \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is exact.

To compute $\operatorname{Tor}_i^R(M, N)$, let

$$P: \quad \cdots \to P_2 \to P_1 \to P_0 \to 0$$

be a projective resolution of M. Then $\operatorname{Tor}_i^R(M, N) = \mathrm{H}_i(P \otimes_R N)$. The other way to compute $\operatorname{Tor}_i(M, N)$ is to let

$$Q: \quad \cdots \to Q_2 \to Q_1 \to Q_0 \to 0$$

be a projective resolution of N. Then $\operatorname{Tor}_i^R(M, N) = \mathrm{H}_i(M \otimes_R Q)$. The R-module

$$\operatorname{Tor}_i^R(M, N)$$

is independent of the choice of P, the choice of Q, or whether M was resolved or N was resolved. Of course, $\operatorname{Tor}_i(P, -) = \operatorname{Tor}_i(-, P) = 0$ if F is projective and i is positive. Also, $\operatorname{Tor}_0^R(M, N) = M \otimes_R N$. Of course, $\operatorname{Tor}_i^R(M, -)$ and $\operatorname{Tor}_i^R(-, N)$ both are functors.

Example 2.12. If I and J are ideals of the ring R, then $\operatorname{Tor}_1^R(R/I, R/J) = \frac{I \cap J}{IJ}$.

Proof. Start with the exact sequence $0 \to J \to R \to R/J \to 0$. Apply $R/I \otimes_R -$ to obtain the long exact sequence

$$\cdots \to \underbrace{\operatorname{Tor}_1(R/I, R)}_{0} \to \operatorname{Tor}_1(R/I, R/J) \to \underbrace{R/I \otimes_R J}_{J/IJ} \to \underbrace{R/I \otimes_R R}_{R/I} \to R/I \otimes_R R/J \to 0.$$

The kernel of the natural map $J/IJ \to R/I$ is $(I \cap J)/IJ$. $\square$

Today's third job. Minimal surjections and minimal resolutions.

Observation 2.13. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local ring, M be a finitely generated R-module minimally generated by $m_1, \dots, m_b$, F be the free R-module $F = R^b$, and $\pi : F \to M$ be the R-module homomorphism*

$$\pi\left(\begin{bmatrix} r_1 \\ \vdots \\ r_b \end{bmatrix}\right) = \sum_i r_i m_i. \tag{2.13.1}$$

then $\ker \pi \subseteq \mathfrak{m}F$.

Proof. Observe that
$$\frac{F}{\ker \pi} \cong M.$$
Thus,
$$\frac{\frac{F}{\mathfrak{m}F}}{\frac{\ker\pi+\mathfrak{m}F}{\mathfrak{m}F}} \cong \frac{F}{\ker\pi+\mathfrak{m}F} = \frac{\frac{F}{\ker\pi}}{\mathfrak{m}\frac{F}{\ker\pi}} \cong \frac{M}{\mathfrak{m}M}$$
The vector space on the left has dimension $b - \dim \frac{\ker\pi+\mathfrak{m}F}{\mathfrak{m}F}$. The vector space on the right has dimension b. Thus, $\dim \frac{\ker\pi+\mathfrak{m}F}{\mathfrak{m}F}$ is a vector space of dimension zero. In other words, $\frac{\ker\pi+\mathfrak{m}F}{\mathfrak{m}F}$ equals zero and $\ker\pi \subseteq \mathfrak{m}F$. □

Definition 2.14. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local ring, M and F be finitely generated R-modules with F free. A surjection $\pi : F \to M$ is a *minimal surjection* if $\ker\pi \subseteq \mathfrak{m}F$.

Definition 2.15. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a Noetherian local ring and M be a finitely generated R-module. A resolution
$$F: \quad \cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$
of M by free R-modules is called a *minimal resolution* if $\operatorname{im} d_i \subseteq \mathfrak{m}F_{i-1}$, for $1 \le i$.

Observation 2.16. *Every finitely generated module over a Noetherian local ring has a minimal resolution.*

Proof. Let $\pi : F_0 \to M$ be a minimal surjection. We know from Observation 2.13 that $\ker\pi \subseteq \mathfrak{m}F_0$. Let $d_1 : F_1 \to \ker\pi$ be a minimal surjection. Again, we know from Observation 2.13 that $\ker d_1 \subseteq \mathfrak{m}F_1$.

Continue in this manner. □

February 28, 2019

Goal: If R is a local Noetherian ring, then

$$R \text{ is regular} \iff \operatorname{pd}_R M < \infty \text{ for all finitely generated } R\text{-modules } M.$$

We show

(a) If R is commutative Noetherian ring and $x_1, \ldots, x_n$ a regular sequence on R, then $\operatorname{pd}_R R/(x_1, \ldots, x_n) < \infty$. (We called this result 2.10.(a).)
(b) If $(R, \mathfrak{m}, \boldsymbol{k})$ is a local Noetherian ring with $\operatorname{pd}_R \boldsymbol{k} < \infty$, then $\operatorname{pd}_R M < \infty$ for all finitely generated R-modules M. (We called this result 2.10.(b).)
(c) The direction ($\Leftarrow$).

We first attack (b).

Observation 2.17. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Noetherian ring. If $\operatorname{pd}_R \boldsymbol{k} = n$, then $\operatorname{pd}_R M \le n$ for all finitely generated R-modules M.*

Proof. Let
$$F: \quad \cdots \to R^{b_1} \to R^{b_0}$$
be a minimal resolution of M. It follows that $F \otimes_R \boldsymbol{k}$ is
$$\cdots \xrightarrow{0} \boldsymbol{k}^{b_1} \xrightarrow{0} \boldsymbol{k}^{b_0};$$
hence, $\operatorname{Tor}_i^R(M, \boldsymbol{k}) = \boldsymbol{k}^{b_i}$. If $\operatorname{pd}_R \boldsymbol{k} = n$, then $\operatorname{Tor}_i^R(M, \boldsymbol{k}) = 0$ for $n+1 \le i$. Thus, $b_i = 0$ for $n+1 \le i$ and $\operatorname{pd}_R M \le n$. □

We attack 2.10.(c) The direction ($\Leftarrow$) **from Theorem 2.9.** We prove that if $(R, \mathfrak{m}, \boldsymbol{k})$ is a Noetherian local ring and every finitely generated R-module has finite projective dimension, then R is regular.

We induct on $\operatorname{edim} R$.

If $\operatorname{edim} R = 0$, then R is a field; hence R is regular.

Henceforth, we consider $1 \le \operatorname{edim} R$.

Step 1. There exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ with x regular on R. Otherwise, $\mathfrak{m} \in \operatorname{Ass} R$ and there exists an non-zero element y with $\mathfrak{m}y = 0$. Of course, this is not possible. Look at the end of a minimal free resolution of $\boldsymbol{k}$:
$$0 \to L_t \xrightarrow{d_t} L_{t-1} \to \dots .$$
Observe that
$$\begin{bmatrix} y \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mapsto 0.$$
The map is injective and y is not zero. We have a contradiction.

So, there really does exist a minimal generator x of $\mathfrak{m}$ with x regular on R. Notice that the embedding dimension of R/xR is one less than the embedding dimension of R. Notice also that the Krull dimension of R/xR is one less than the Krull dimension of R (because x is regular on R).

It suffices to prove that every finitely generated R/xR module has finite projective dimension. (If, so, then R/xR is regular by induction and $\operatorname{edim} R/xR = \dim R/xR$. It then follows that $\operatorname{edim} R = \dim R$; that is, R is regular.)

We want to prove that $\boldsymbol{k}$ has finite projective dimension as an R/xR module.

The short exact sequence
$$0 \to \frac{\mathfrak{m}}{xR} \to \frac{R}{xR} \to \boldsymbol{k} \to 0$$
shows that it suffices to prove that $\operatorname{pd}_{R/xR} \frac{\mathfrak{m}}{xR} < \infty$.

Even $\frac{\mathfrak{m}}{xR}$ is too hard to consider directly. Instead we prove

(2.17.1) $$\operatorname{pd}_{R/xR} \frac{\mathfrak{m}}{x\mathfrak{m}} < \infty,$$

(2.17.2) $\dfrac{\mathfrak{m}}{xR}$ is a summand of $\dfrac{\mathfrak{m}}{x\mathfrak{m}}$ as R/xR modules, and

$$\operatorname{pd}_{R/xR} \frac{\mathfrak{m}}{xR} < \infty. \tag{2.17.3}$$

Assertion (2.17.3) is easy once one has (2.17.1) and (2.17.2). Indeed, if

$$\frac{\mathfrak{m}}{xR} \oplus X \cong \frac{\mathfrak{m}}{x\mathfrak{m}}$$

as R/xR modules and $\operatorname{pd}_{R/xR} \frac{\mathfrak{m}}{x\mathfrak{m}} < \infty$, then

$$\begin{aligned} &\operatorname{Tor}_i^{R/xR}\left(\frac{\mathfrak{m}}{x\mathfrak{m}}, \boldsymbol{k}\right) = 0 \text{ for } 0 \ll i \\ \Longrightarrow\ &\operatorname{Tor}_i^{R/xR}\left(\frac{\mathfrak{m}}{xR} \oplus X, \boldsymbol{k}\right) = 0 \text{ for } 0 \ll i \\ \Longrightarrow\ &\operatorname{Tor}_i^{R/xR}\left(\frac{\mathfrak{m}}{xR}, \boldsymbol{k}\right) \oplus \operatorname{Tor}_i^{R/xR}(X, \boldsymbol{k}) = 0 \text{ for } 0 \ll i \\ \Longrightarrow\ &\operatorname{Tor}_i^{R/xR}\left(\frac{\mathfrak{m}}{xR}, \boldsymbol{k}\right) = 0 \text{ for } 0 \ll i \\ \Longrightarrow\ &\operatorname{pd}_{R/xR}\left(\frac{\mathfrak{m}}{xR}\right) < \infty. \end{aligned}$$

We prove (2.17.2). We prove that if x is a minimal generator of $\mathfrak{m}$, then $\frac{\mathfrak{m}}{xR}$ is a summand of $\frac{\mathfrak{m}}{x\mathfrak{m}}$ as R/xR modules.

Observe that $x\mathfrak{m} \subseteq xR$; thus, there is a natural quotient map

$$\pi : \frac{\mathfrak{m}}{x\mathfrak{m}} \twoheadrightarrow \frac{\mathfrak{m}}{xR}.$$

Of course, π is a map of R/xR modules. On the other hand,

$$\frac{\mathfrak{m}}{xR} = \frac{xR + (x_2, \dots, x_d)R}{xR} = \frac{(x_2, \dots, x_d)R}{xR \cap (x_2, \dots, x_d)R},$$

where $x, x_2, \dots, x_d$ is a minimal generating set of $\mathfrak{m}$. Furthermore, if

$$xr \in xR \cap (x_2, \dots, x_d)R,$$

then $xr \in (x_2, \dots, x_d)R$; hence $r \in \mathfrak{m}$; since $x, x_2, \dots, x_d$ is a minimal generating set of $\mathfrak{m}$. Thus, $xr \in x\mathfrak{m}$.

Observe that

$$\frac{\mathfrak{m}}{x\mathfrak{m}} \leftarrow (x_2, \dots, x_d)R$$

is a well-defined map of R-modules $xR \cap (x_2, \dots, x_d)R$ is contained in the kernel. Thus,

$$\frac{\mathfrak{m}}{x\mathfrak{m}} \leftarrow \frac{(x_2, \dots, x_d)R}{xR \cap (x_2, \dots, x_d)R} = \frac{\mathfrak{m}}{xR}$$

is a well-defined map of R-modules; hence also, a well-defined map of R/xR-modules. Call this map σ. Observe that $\pi \circ \sigma$ is the identity. Take an element of $\frac{\mathfrak{m}}{xR}$. This element is

$$\theta + \Big(xR \cap (x_2, \dots, x_d)R\Big),$$

where $\theta \in (x_2, \dots, x_d)R$. Observe that

$$(\pi \circ \sigma)\Big(\theta + \Big(xR \cap (x_2, \dots, x_d)R\Big)\Big) = \pi(\theta + x\mathfrak{m}) = \theta + xR = \theta + \Big(xR \cap (x_2, \dots, x_d)R\Big),$$

since $\theta \in (x_2, \dots, x_d)R$.

We prove 2.17.1. Assume $(R,\mathfrak{m},\boldsymbol{k})$ is Noetherian local, $\mathrm{pd}_R\,\boldsymbol{k}<\infty$, and x is an element of $\mathfrak{m}$ which is regular on R. We prove

$$\mathrm{pd}_{R/xR}\,\frac{\mathfrak{m}}{x\mathfrak{m}}<\infty.$$

Lemma 2.18. *If R is a ring, M is an R-module, and x is an element of R which is regular on both R and M, then*

$$\mathrm{Tor}_i^R(M,N)=\mathrm{Tor}_i^{R/xR}(M/xM,N)$$

for all R/xR-modules N and for all integers i.

Assume Lemma 2.18. Prove 2.17.1. Lemma 2.18 yields that

$$\mathrm{Tor}_i^R(\mathfrak{m},\boldsymbol{k})=\mathrm{Tor}_i^{R/xR}(\mathfrak{m}/x\mathfrak{m},\boldsymbol{k})$$

for all i. The module $\mathrm{Tor}_i^R(\mathfrak{m},\boldsymbol{k})=0$ for all i with $\mathrm{pd}_R\,\boldsymbol{k}<i$. Thus $\mathrm{Tor}_i^{R/xR}(\mathfrak{m}/x\mathfrak{m},\boldsymbol{k})=0$, for all i with $\mathrm{pd}_R\,\boldsymbol{k}<i$, and $\mathrm{pd}_{R/xR}\,\mathfrak{m}/x\mathfrak{m}\le\mathrm{pd}_R\,\boldsymbol{k}$. □

Prove Lemma 2.18. Let L be a resolution of M by free R-modules. Observe that $L\otimes_R R/(x)$ is a complex with

$$\mathrm{H}_i(L\otimes_R R/(x))=\begin{cases}M/xM & \text{if } 0=i\\ \mathrm{Tor}_i^R(M,R/(x)) & \text{if } 0\le i.\end{cases}$$

On the other hand, x is regular on R and if we apply $M\otimes_R -$ to the short exact sequence

$$0\to R\xrightarrow{x}R\to R/(x)\to 0,$$

then we obtain the long exact sequence of homology

$$\begin{aligned}\cdots\to\underbrace{\mathrm{Tor}_2^R(M,R)}_{0}\to\underbrace{\mathrm{Tor}_2^R(M,R)}_{0}&\to\mathrm{Tor}_2^R(M,R/(x))\\ \to\underbrace{\mathrm{Tor}_1^R(M,R)}_{0}\to\underbrace{\mathrm{Tor}_1^R(M,R)}_{0}&\to\mathrm{Tor}_1^R(M,R/(x))\\ \to\underbrace{M\otimes_R R\to M\otimes_R R}_{M\xrightarrow{x}M}&\to M\otimes_R R/(x)\to 0\end{aligned}$$

The element x is also regular on M; so, $\mathrm{Tor}_i^R(M,R)=0$ for $1\le i$. Thus, $L\otimes_R R/(x)$ is a resolution of M/xM by free $R/(x)$ modules. We conclude that

$$\mathrm{Tor}_i^R(M,N)=\mathrm{H}_i(L\otimes_R N)\overset{\dagger}{=}\mathrm{H}_i((L\otimes_R R/(x))\otimes_{R/(x)}N)=\mathrm{Tor}_i^{R/(x)}(M/xM,N).\quad\square$$

† This equality holds because N is an $R/(x)$-module. Indeed, $R/(x)\otimes_R N=N/(x)N=N$.

2.A. **The Koszul complex.** In this subsection we finish the proof of Serre's Theorem (Theorem 2.9).

We prove the following statement.

Proposition 2.19. *If $x_1,\dots,x_n$ is a regular sequence in the commutative Noetherian ring R, then $\mathrm{pd}_R\,R/(x_1,\dots,x_n)$ is finite.*

One associates a Koszul complex K of length n to each sequence $x_1, \ldots, x_n$ of n elements in a ring R. One proves that if $x_1, \ldots, x_n$ is a regular sequence then K is a resolution.

Examples 2.20. It turns out that K amounts to a list of all of the obvious relations that one has even if one does not know anything about the elements in the sequence.

If $n = 1$, K is

$$0 \to R \xrightarrow{x_1} R.$$

If $n = 2$, K is

$$0 \to R \xrightarrow{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} R$$

If $n = 3$, K is

$$0 \to R \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}} R$$

If $n = 4$, K is

$$0 \to R \xrightarrow{\begin{bmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & -x_2 & -x_3 & 0 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{bmatrix}} R^6 \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 & -x_4 & 0 & 0 \\ x_1 & 0 & -x_3 & 0 & -x_4 & 0 \\ 0 & x_1 & x_2 & 0 & 0 & -x_4 \\ 0 & 0 & 0 & x_1 & x_2 & x_3 \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}} R$$

Definition 2.21. FIRST DEFINITION. The Koszul complex on x_1 is $0 \to R \xrightarrow{x_1} R$. The Koszul complex on $x_1, \ldots, x_n$ is $K' \otimes K''$, where K' is the Koszul complex on $x_1, \ldots, x_{n-1}$ and K'' is the Koszul complex on x_n.

Recall that if (A, d_A) and (B, d_B) are complexes of R-modules then the tensor product of these two complexes is the complex $(A \otimes_R B, d_{A \otimes B})$, where $(A \otimes_R B)_n = \sum_{p+q=n} A_p \otimes B_q$ and

$$d_{A \otimes B}(a_p \otimes b_q) = d_A(a_p) \otimes b_q + (-1)^p a_p \otimes d_B(b_q),$$

for $a_p \in A_p$ and $b_q \in B_q$.

Definition 2.22. SECOND DEFINITION. The Koszul complex on x_1 is $0 \to R \xrightarrow{x_1} R$. The Koszul complex on $x_1, \ldots, x_n$ is the total complex of

$$\begin{array}{c} K' \\ \big\downarrow {\scriptstyle x_n} \\ K' \end{array},$$

where K' is the Koszul complex on $x_1, \ldots, x_{n-1}$. Recall that the total complex of

$$\begin{array}{ccccccc} \cdots & \xrightarrow{t_3} & T_2 & \xrightarrow{t_2} & T_1 & \xrightarrow{t_1} & T_0 \\ & & \big\downarrow {\scriptstyle \alpha_2} & & \big\downarrow {\scriptstyle \alpha_1} & & \big\downarrow {\scriptstyle \alpha_0} \\ \cdots & \xrightarrow{b_3} & B_2 & \xrightarrow{b_2} & B_1 & \xrightarrow{b_1} & B_0 \end{array}$$

is

$$(2.22.1)\qquad \cdots \to \begin{matrix} T_2\\ \oplus \\ B_3\end{matrix} \xrightarrow{\begin{bmatrix} -t_2 & 0\\ \alpha_2 & b_3\end{bmatrix}} \begin{matrix} T_1\\ \oplus \\ B_2\end{matrix} \xrightarrow{\begin{bmatrix} -t_1 & 0\\ \alpha_1 & b_2\end{bmatrix}} \begin{matrix} T_0\\ \oplus \\ B_1\end{matrix} \xrightarrow{\begin{bmatrix} \alpha_0 & b_1\end{bmatrix}} B_0.$$

The total complex (2.22.1) is also called the mapping cone of the map of complexes

$$\alpha : T \to B.$$

A little preparation is needed before I can state the third definition of the Koszul complex. In this third definition, one says that K is the exterior algebra on a free R-module of rank n, $\dots$. I think that the right way to do this is to define the tensor algebra, the symmetric algebra, and the exterior algebra of a module.

Definition 2.23. Let R be a commutative ring and M be an R-module. The tensor algebra $T_\bullet(M)$ of the R-module M is the (non-commutative) graded R-algebra

$$T_\bullet(M) = R \oplus M \oplus (M \otimes_R M) \oplus \underbrace{(M \otimes_R M \otimes_R M)}_{T_3(M)} \oplus \cdots .$$

One multiplies in $T_\bullet M$ as follows:

$$\underbrace{(m_1 \otimes \cdots \otimes m_r)}_{\in T_r(M)} \text{ times } \underbrace{(m_1' \otimes \cdots \otimes m_s')}_{\in T_s(M)} = \underbrace{m_1 \otimes \cdots \otimes m_r \otimes m_1' \otimes \cdots \otimes m_s'}_{\in T_{r+s}(M)}.$$

The symmetric algebra $\operatorname{Sym}(M)$ of the R-module M is the (commutative) R-algebra

$$\operatorname{Sym}_\bullet(M) = \frac{T_\bullet(M)}{\text{the two sided ideal generated by } \{m_1 \otimes m_2 - m_2 \otimes m_1 \mid m_i \in M\}}.$$

The exterior algebra $\bigwedge^\bullet(M)$ of the R-module M is the (anti-commutative) R-algebra

$$\textstyle\bigwedge^\bullet(M) = \frac{T_\bullet(M)}{\text{the two sided ideal generated by } \{m \otimes m \mid m \in M\}}.$$

These algebras satisfy the following Universal mapping properties.

Proposition 2.24.

- *Let M be an R-module and A a (not necessarily commutative) R-algebra. Then for any R-module homomorphism $\phi : M \to A$ there exists unique R-algebra homomorphism $\Phi : T(M) \to A$ such that $\Phi|M = \phi$. That is, the picture*

$$\begin{array}{ccc} M & \longrightarrow & T(M) \\ & \overset{\phi}{\searrow} & \Big\downarrow {\scriptstyle \exists!\Phi} \\ & & A \end{array}$$

  commutes.
- *Let M be an R-module and A a commutative R-algebra. Then for any R-module homomorphism $\phi : M \to A$ there exists unique R-algebra homomorphism $\Phi : \mathrm{Sym}(M) \to A$ such that $\Phi|M = \phi$. That is, the picture*

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{Sym}(M) \\ & \overset{\phi}{\searrow} & \Big\downarrow {\scriptstyle \exists!\Phi} \\ & & A \end{array}$$

  commutes.
- *Let M be an R-module and A a (not necessarily commutative) R-algebra. Then for any R-module homomorphism $\phi : M \to A$ with the property that $\phi(m)^2 = 0$, for all $m \in M$, there exists unique R-algebra homomorphism $\Phi : \bigwedge^\bullet M \to A$ such that $\Phi|M = \phi$. That is, the picture*

$$\begin{array}{ccc} M & \longrightarrow & \bigwedge^\bullet M \\ & \overset{\phi}{\searrow} & \Big\downarrow {\scriptstyle \exists!\Phi} \\ & & A \end{array}$$

  commutes.

Example 2.25. Let F be a free R-module of rank n. Then $\bigwedge^\bullet F = \bigoplus_{i=0}^n \bigwedge^i F$, where $\bigwedge^i F$ is free of rank $\binom{n}{i}$. Indeed, if $e_1, \dots, e_n$ is a basis for F, then

$$\{e_{j_1} \wedge \ldots \wedge e_{j_i} \mid 1 \le j_1 < \cdots < j_i \le n\}$$

is a basis for $\bigwedge^i F$. (One can prove this using the Universal mapping property for exterior algebras.)

If F is free of finite rank, then $\bigwedge^\bullet F$ is a $\bigwedge^\bullet F^*$-module. The R-algebra $\bigwedge^\bullet F^*$ is generated by $\bigwedge^1 F^*$. It suffices to tell how $\bigwedge^1 F^*$ acts on $\bigwedge^\bullet F$. We know how $\bigwedge^1 F^*$ acts on $\bigwedge^1 F$. We need only decide how $\bigwedge^1 F^*$ on homogeneous products. If $\phi_1 \in F^*$, $u_i \in \bigwedge^i F$, and $u_j \in \bigwedge^j F$, then

$$\phi_1(u_i \wedge u_j) = \phi_1(u_i) \wedge u_j + (-1)^i u_i \wedge \phi_1(u_j).$$

If F is free of finite rank and $\phi : F \to R$ is a homomorphism, then we define the Koszul complex $(\bigwedge^\bullet F, \phi)$. If F is a free R-module of finite rank and $\phi : F \to R$ is a homomorphism, then $(\bigwedge^\bullet F, \phi)$ is the Koszul complex associated to ϕ.

Example 2.26. If $F = \bigoplus_{i=1}^n Re_i$ and $\phi : F \to R$ sends $\sum r_i e_i$ to $\sum r_i a_i$, then the Koszul complex associated to ϕ is

$$0 \to \bigwedge^n F \xrightarrow{\phi} \bigwedge^{n-1} F \xrightarrow{\phi} \cdots \xrightarrow{\phi} \bigwedge^2 F \xrightarrow{\phi} \bigwedge^1 F \xrightarrow{\phi} \bigwedge^0 F,$$

where

$$\phi(e_{j_1} \wedge \cdots \wedge e_{j_t}) = \sum_{i=1}^{t} (-1)^{i+1} a_{j_i} e_{j_1} \wedge \ldots \wedge \widehat{e_{j_i}} \wedge \ldots e_{j_n}.$$

We prove Proposition 2.19.

Proposition. [2.19] *If $x_1, \ldots, x_n$ is a regular sequence in the commutative Noetherian ring R, then $\operatorname{pd}_R R/(x_1, \ldots, x_n)$ is finite.*

The Koszul complex on $x_1, \ldots, x_n$ is the total complex (or mapping cone) of

$$\begin{array}{c} K' \\ \big\downarrow{\scriptstyle x_n} \\ K' \end{array},$$

where K' is the Koszul complex on $x_1, \ldots, x_{n-1}$. Recall that the total complex (or mapping cone) of

$$\begin{array}{ccccccc} \cdots \xrightarrow{t_3} & T_2 & \xrightarrow{t_2} & T_1 & \xrightarrow{t_1} & T_0 \\ & \big\downarrow{\scriptstyle \alpha_2} & & \big\downarrow{\scriptstyle \alpha_1} & & \big\downarrow{\scriptstyle \alpha_0} \\ \cdots \xrightarrow{b_3} & B_2 & \xrightarrow{b_2} & B_1 & \xrightarrow{b_1} & B_0 \end{array}$$

is

$$\cdots \to \begin{matrix} T_2 \\ \oplus \\ B_3 \end{matrix} \xrightarrow{\begin{bmatrix} -t_2 & 0 \\ \alpha_2 & b_3 \end{bmatrix}} \begin{matrix} T_1 \\ \oplus \\ B_2 \end{matrix} \xrightarrow{\begin{bmatrix} -t_1 & 0 \\ \alpha_1 & b_2 \end{bmatrix}} \begin{matrix} T_0 \\ \oplus \\ B_1 \end{matrix} \xrightarrow{\begin{bmatrix} \alpha_0 & b_1 \end{bmatrix}} B_0.$$

Recall also that there is a long exact sequence of homology associated to a mapping cone:

(2.26.1) $$\cdots \to \mathrm{H}_1(T) \to \mathrm{H}_1(B) \to \mathrm{H}_1(M) \to \mathrm{H}_0(T) \to \mathrm{H}_0(B) \to \mathrm{H}_0(M) \to 0$$

For us, the long exact sequence becomes

$$\cdots \to 0 \to 0 \to \mathrm{H}_2(M) \to 0 \to 0 \to \mathrm{H}_1(M) \to R/(x_1, \ldots, x_{n-1}) \xrightarrow{x_{n-1}} R/(x_1, \ldots, x_{n-1}) \to \mathrm{H}_0(M) \to 0.$$

Thus,

$$\mathrm{H}_i(M) = \begin{cases} R/(x_1, \ldots, x_n), & \text{if } i = 0, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

One could prove the long exact sequence of homology for mapping cones from scratch; but it would be more clever to observe that a mapping cone gives rise to a short exact sequence of complexes:

(2.26.2) $$0 \to B \to M \to T[-1] \to 0,$$

as described below:

$$\begin{array}{ccccccc}
\cdots \longrightarrow B_3 & \xrightarrow{\quad b_3 \quad} & B_2 & \xrightarrow{\quad b_2 \quad} & B_1 & \xrightarrow{\quad b_1 \quad} & B_0 \\
\Big\downarrow \begin{bmatrix}0\\1\end{bmatrix} & & \Big\downarrow \begin{bmatrix}0\\1\end{bmatrix} & & \Big\downarrow \begin{bmatrix}0\\1\end{bmatrix} & & \Big\downarrow 1 \\
\cdots \longrightarrow \begin{matrix}T_2\\ \oplus \\ B_3\end{matrix} & \xrightarrow{\begin{bmatrix}-t_2 & 0\\ \alpha_2 & b_3\end{bmatrix}} & \begin{matrix}T_1\\ \oplus \\ B_2\end{matrix} & \xrightarrow{\begin{bmatrix}-t_1 & 0\\ \alpha_1 & b_2\end{bmatrix}} & \begin{matrix}T_0\\ \oplus \\ B_1\end{matrix} & \xrightarrow{\begin{bmatrix}\alpha_0 & b_1\end{bmatrix}} & B_0 \\
\Big\downarrow \begin{bmatrix}1 & 0\end{bmatrix} & & \Big\downarrow \begin{bmatrix}1 & 0\end{bmatrix} & & \Big\downarrow \begin{bmatrix}1 & 0\end{bmatrix} & & \\
\cdots \longrightarrow T_2 & \xrightarrow{\quad t_2 \quad} & T_1 & \xrightarrow{\quad t_1 \quad} & T_0 & &
\end{array}$$

The long exact sequence of homology that corresponds to the short exact sequence of complexes (2.26.2) is (2.26.1).

2.B. **Regular local implies UFD.**

Theorem 2.27. [Auslander-Buchsbaum, Serre] *If R is a regular local ring, then R is a Unique Factorization Domain.*

Recall that an element r of the domain R is <u>irreducible</u> if r is not a unit and whenever $r = r_1 r_2$ in R, then either r_1 or r_2 is a unit.

The domain R is a Unique Factorization Domain (UFD) if the following two properties hold.

(a) Every non-unit can be factored into a finite product of irreducible elements.
(b) Whenever $r_1 \cdots r_n = r'_1 \cdots r'_{n'}$ are products of irreducible elements, then $n = n'$ and after re-numbering, then $r_i = u_i r'_i$ where u_i is a unit in R.

Observation 2.28. *Let R be a Noetherian domain. Then*

$$R \text{ is a UFD } \iff \text{ every irreducible element of } R \text{ generates a prime ideal.}$$

Proof.
($\Rightarrow$) This direction is clear. If r is an irreducible element in R and r_1 and r_2 are elements of R with $r_1 r_2$ is in the ideal (r), then r is one of the irreducible factors of $r_1 r_2$; hence r is one of the irreducible factors of r_1 or r_2; hence r_1 or r_2 is in (r). Thus, (r) is a prime ideal of R.

($\Leftarrow$) We must show that every non-unit of R can be written as a finite product of irreducible elements. The hypothesis "every irreducible element of R generates a prime ideal" together with the hypothesis that R is a domain guarantees that every factorization into irreducible elements is necessarily unique.

Claim. *If $r \in R$ and r is not a unit, then r has an irreducible factor.*

If r is not irreducible, then $r = r_1 r_1'$, where neither r_1 nor r_1' is a unit. Thus

$$(r) \subsetneq (r_1).^1$$

If r_1 is not irreducible, then $r_1 = r_2 r_2'$, where neither r_2 nor r_2' is a unit. Thus,

$$(r) \subsetneq (r_1) \subsetneq (r_2).$$

The ring R is Noetherian, so the chain eventually stops and an irreducible factor of r has been identified. This concludes the proof of the claim.

Claim. *Every non-unit in R is equal to a finite product of irreducible elements.*

Suppose r is a counterexample. Then r is not irreducible. Thus r has an irreducible factor r_1 and the complementary factor r_1' is not a unit and is not equal to a finite product of irreducible elements. Thus, $r_1' = r_2 r_2'$, where r_2 is irreducible and r_2' is not a unit and is not equal to a finite product of irreducible elements. Observe that

$$(r) \subsetneq (r_1') \subsetneq (r_2') \subsetneq \dots \, .$$

This claim is also established.

The proof is complete. □

Observation 2.29. *Let R be a Noetherian domain. Then*

$$R \text{ is a UFD} \iff \text{every height one prime ideal of } R \text{ is principal}.$$

Proof.
($\Rightarrow$) Assume R is a UFD. Let $\mathfrak{p}$ be a height one prime of R. Let r be a non-zero element of $\mathfrak{p}$. Write $r = \prod r_i$, where each r_i is an irreducible element of R. It follows that least one of the r_i is in $\mathfrak{p}$. This irreducible element r_i generates a prime ideal of R. (See Observation 2.28, if necessary.) Thus, $(0) \subsetneq (r_i) \subseteq \mathfrak{p}$ are prime ideals of R with $\operatorname{ht} \mathfrak{p} = 1$. It follows that $(r_i) = \mathfrak{p}$; hence, $\mathfrak{p}$ is principal.

($\Leftarrow$) Assume every height one prime of R is principal. We prove that R is a UFD. We apply Observation 2.28 and show that every irreducible element of R generates a prime ideal.

Let r be an irreducible element of R. Let $\mathfrak{p}$ be a prime ideal of R which is minimal over (r). Thus,

$\operatorname{ht} \mathfrak{p} \le 1,$	by the Krull Principal Ideal Theorem, and
$0 < \operatorname{ht} \mathfrak{p},$	because R is a domain.

Thus, $\mathfrak{p}$ is a height one prime ideal of R. The hypothesis ensures that $\mathfrak{p}$ is principal. Thus, there exists $\pi \in R$ with $\mathfrak{p} = (\pi)$. It follows that $r \in (\pi)$ and $r = \pi r'$ for some $r' \in R$. The

[1] If $(r_1) = (r)$, then $r_1 = r r_1'' = r_1 r_1' r_1''$. The ring R is a domain; so $1 = r_1' r_1''$ which is not possible because r_1' is not a unit.

element r of R is irreducible; therefore, one of the factors π or r' of r is a unit. The element π generates a proper ideal of R; so π is not a unit. Thus, r' is a unit and r generates the prime ideal $(\pi) = \mathfrak{p}$. □

Observation 2.30. *Let R be a Noetherian domain and S be a multiplicatively closed subset of R which is generated by prime elements of R. (In other words, the generators of S generate prime ideals in R.) If $S^{-1}R$ is a UFD, then R is a UFD.*

Proof. Let $\mathfrak{p}$ be a height one prime of R. According to Observation 2.29, it suffices to prove that $\mathfrak{p}$ is a principal ideal of R. There are two cases.

Either $\mathfrak{p}S^{-1}R$ is equal to all of $S^{-1}R$ or $\mathfrak{p}S^{-1}R$ is a proper ideal of $S^{-1}R$.

In the first case, there is an element $s \in S \cap \mathfrak{p}$. The element s is equal to a finite product of elements of R each of which generates a prime ideal. Thus, there is an element $s' \in S \cap \mathfrak{p}$, with (s') a prime ideal of R. Observe that

$$(0) \subsetneq (s') \subseteq \mathfrak{p}$$

are prime ideals of R and $\mathfrak{p}$ has height one. We conclude that $(s') = \mathfrak{p}$ and therefore $\mathfrak{p}$ is principal.

In the second case, $\mathfrak{p}S^{-1}R$ is a height one prime ideal in the UFD $S^{-1}R$. Thus, $\mathfrak{p}S^{-1}R$ is a principal ideal of $S^{-1}R$ and there exists elements r of $\mathfrak{p}$ with $\mathfrak{p}S^{-1}R = (r)S^{-1}R$. Consider the set of ideals of R of the form

$$\{(r)R \mid r \in \mathfrak{p} \text{ and } (r)S^{-1}R = \mathfrak{p}S^{-1}R\}. \tag{2.30.1}$$

The ring R is Noetherian; thus, the above set has a maximal element $(r)R$. We claim that $\mathfrak{p} = (r)R$.

$\supseteq$ This direction is clear.

$\subseteq$. Let π be an element of $\mathfrak{p}$. The hypothesis that $\mathfrak{p}S^{-1}R = (r)S^{-1}R$ guarantees that there is an element of S with $s\pi \in (r)R$. Write

$$\prod s_i \pi = rr',$$

where each s_i is an element of S which generates a prime ideal in R and r' is an element of R. Our choice of r ensures that none of the s_i can divide r in R. (If $r = s_i x$, then $(r) \subsetneq (x)$ and both ideals are in (2.30.1).) So, each s_i, one at a time, divides the complement to r. Ultimately, we obtain $\pi = rr''$, where $r'' \prod s_i = r'$. In other words, $\pi \in (r)R$; and the proof is complete. □

Lemma 2.31. *Let R be a ring and M be a finitely presented R-module, then M is a projective R-module if and only if $R_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.*

Proof.
($\Rightarrow$) We already did this direction. If M is a projective R-module, then $M_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module. We proved that every finitely generated projective module over a local ring is free. Thus $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module.

($\Leftarrow$) Let $\pi : A \to B$ be a surjection of R-modules. We must prove that R-module homomorphisms of the form

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ A & \xrightarrow{\pi} & B \end{array}$$

gives rise to a commutative diagram of R-module homomorphisms of the form

$$\begin{array}{ccc} & & M \\ & \exists \swarrow & \downarrow \\ A & \xrightarrow{\pi} & B. \end{array}$$

In other words, we must prove that

$$\operatorname{Hom}_R(M, A) \xrightarrow{\pi_*} \operatorname{Hom}_R(M, B)$$

is onto. Let C be the cokernel:

$$\operatorname{Hom}_R(M, A) \xrightarrow{\pi_*} \operatorname{Hom}_R(M, B) \to C \to 0 \tag{2.31.1}$$

is exact. We prove that $C_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$. Localize (2.31.1) at $\mathfrak{m}$:

$$\operatorname{Hom}_R(M, A)_\mathfrak{m} \xrightarrow{\pi_*} \operatorname{Hom}_R(M, B)_\mathfrak{m} \to C_\mathfrak{m} \to 0$$

Recall from last semester that

$$\operatorname{Hom}_R(M, X)_\mathfrak{m} = \operatorname{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, X_\mathfrak{m}),$$

for all R-modules X, because M is finitely presented. Thus,

$$\operatorname{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, A_\mathfrak{m}) \xrightarrow{\pi_*} \operatorname{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m}) \to C_\mathfrak{m} \to 0$$

is exact. The hypothesis that M_m is free over $R_\mathfrak{m}$ guarantees that

$$\operatorname{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, A_\mathfrak{m}) \xrightarrow{\pi_*} \operatorname{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m})$$

is onto. Thus, $C_\mathfrak{m} = 0$ for all $\mathfrak{m}$ and the proof is complete. □

Observation 2.32. *If*

$$0 \xrightarrow{d_{r+1}} P_r \xrightarrow{d_r} P_{r-1} \xrightarrow{d_{r-1}} \dots P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0$$

is a finite exact sequence of projective modules over some ring T, then there is an isomorphism of T-modules

$$\bigoplus_{i \text{ odd}} P_i \cong \bigoplus_{i \text{ even}} P_i.$$

Proof. Let $K_i = \ker d_i$ for each integer i. Notice that $K_i = 0$ for $r \le i$ and for $i \le -1$. Observe that

$$0 \to K_i \to P_i \to K_{i-1} \to 0$$

is a short exact sequence of projective modules for each i. Thus $P_i \cong K_i \oplus K_{i-1}$ for each i. Notice that

$$\bigoplus_{i \text{ even}} P_i = (\underbrace{K_{-1}}_{0} \oplus K_0) \oplus (K_1 \oplus K_2) \oplus \cdots$$

and

$$\bigoplus_{i \text{ odd}} P_i = (K_0 \oplus K_1) \oplus (K_2 \oplus K_3) \oplus \cdots .$$

□

Theorem. 2.27 [Auslander-Buchsbaum, Serre] *If R is a regular local ring, then R is a Unique Factorization Domain.*

Proof. Induct on $\dim R$. If $\dim R \leq 1$, then the assertion holds. Assume $2 \leq \dim R$. Take $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. We saw in the proof of Theorem 2.5 (and we stated in Corollary 2.6) that (x) is a prime ideal of R. In light of Observation 2.30, it suffices to show that R_x is a UFD. In light of Observation 2.29, it suffices to show that every height one prime ideal of R_x is principal.

Let P be a height one prime ideal of R_x. The ring R_x has less dimension than the ring R (because, for example, $\mathfrak{m}R_x = R_x$). The ideal P is a height one prime of $(R_x)_\mathfrak{q}$ for each prime ideal $\mathfrak{q}$ of R_x. Each $(R_x)_\mathfrak{q}$ is a regular local ring of less dimension than $\dim R$; hence, by induction the regular local ring $(R_x)_\mathfrak{q}$ is a UFD. Thus, $P_\mathfrak{q}$ is principal for all prime ideals $\mathfrak{q}$ of R_x. Thus, P is a projective R_x-module by Lemma 2.31.

Let $\mathfrak{p} = P \cap R$. The R-module $\mathfrak{p}$ is finitely generated and the ring R is regular local; so there is a finite resolution F of $\mathfrak{p}$ by free R-modules. The functor $- \otimes_R R_x$ is flat; so F_x is a finite resolution of $\mathfrak{p}_x = P$ by free R_x-modules. Apply Observation 2.32 to see that there are finitely generated free R_x-modules F and G with

$$F \cong G \oplus P.$$

The rank of a finitely generated projective module X over a domain T is defined to $\dim_K(X \otimes_T K)$, where K is the field of fractions $K = T_{(0)}$ of T.

Observe that the rank of P is one; hence $\operatorname{rank} G + 1 = \operatorname{rank} F$. The following really cool trick is due to Kaplansky. Observe that

$$R_x \cong \textstyle\bigwedge_{R_x}^{\operatorname{rank} F} F \cong \bigwedge^{\operatorname{rank} F}(G \oplus P)$$

$$= \underbrace{\left(\textstyle\bigwedge^{\operatorname{rank} F} G \otimes_{R_x} \bigwedge^0 P\right)}_{0} \oplus \underbrace{\left(\textstyle\bigwedge^{\operatorname{rank} F-1} G \otimes_{R_x} \bigwedge^1 P\right)}_{R_x \otimes_{R_x} P = P} \oplus \underbrace{\left(\textstyle\bigwedge^{\operatorname{rank} F-2} G \otimes_{R_x} \bigwedge^2 P\right) \oplus \cdots \oplus \left(\textstyle\bigwedge^0 G \otimes_{R_x} \bigwedge^{\operatorname{rank} F} P\right)}_{0}.$$

We conclude that $P \cong R_x$. Thus P is a principal ideal and the proof is complete. □

3. The relationship between transcendence degree and Krull dimension.

This section closely follows section 5 of [6]. There are two goals in front of us.

Goal. 6.1 *Let $\boldsymbol{k}$ be a perfect field, $P = \boldsymbol{k}[x_1, \dots, x_n]$, $I = (f_1, \dots, f_m)$ be an ideal of P, $\mathfrak{p}$ be a prime ideal of P with $I \subseteq \mathfrak{p}$, and L be the field $\frac{P_\mathfrak{p}}{\mathfrak{p}P_\mathfrak{p}}$. Let* Jac *be the matrix*

$$\mathrm{Jac} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

Then

$$\mathrm{rank}_L \mathrm{Jac}|_L \le \dim P_\mathfrak{p} - \dim(P/I)_\mathfrak{p}$$

and

$$(P/I)_\mathfrak{p} \textit{ is regular } \iff \mathrm{rank}_L \mathrm{Jac}|_L = \dim P_\mathfrak{p} - \dim(P/I)_\mathfrak{p}.$$

Remarks. 6.2

(a) The entries in Jac are polynomials in the polynomial ring $\boldsymbol{k}[x_1, \dots, x_n]$.
(b) We have used the symbol "$|_{\frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}}}$" to mean: take the image in $\frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}}$.
(c) Recall that the field $\boldsymbol{k}$ is perfect if every algebraic extension is separable. In particular, every field of characteristic zero and every finite field is perfect. Also, if the characteristic of $\boldsymbol{k}$ is the positive prime integer p, then $\boldsymbol{k}$ is perfect if and only if $\boldsymbol{k}$ is closed under the taking of p^{th} roots.

Goal 3.1. *If R is a commutative domain which is finitely generated as an algebra over the field $\boldsymbol{k}$, then the Krull dimension of R is equal to* $\mathrm{trdeg}_{\boldsymbol{k}} R$.

Remarks 3.2. Let $\boldsymbol{k} \subseteq L$ be fields.

(a) A set S in L is a *transcendence basis* for L over $\boldsymbol{k}$ if S is algebraically independent and L is an algebraic extension of $\boldsymbol{k}(S)$.
(b) Any two transcendence bases for L over $\boldsymbol{k}$ have the same cardinality. The proof uses the same replacement argument as one uses to prove that vector space dimension makes sense. (In linear algebra the result is called the Steinitz Replacement Theorem.) See, for example, [7, Chapt. 9].
(c) The transcendence degree of L over $\boldsymbol{k}$ (which is written $\mathrm{trdeg}_{\boldsymbol{k}} L$) is the cardinality of a transcendence basis of L over $\boldsymbol{k}$.
(d) If the domain R is a $\boldsymbol{k}$-algebra, then $\mathrm{trdeg}_{\boldsymbol{k}} R$ is the transcendence degree of the quotient field of R over $\boldsymbol{k}$.

March 26, 2019.

Goal. 3.1 *If R is a commutative domain which is finitely generated as an algebra over the field $\boldsymbol{k}$, then the Krull dimension of R is equal to $\operatorname{trdeg}_{\boldsymbol{k}} R$.*

One small piece of Goal 3.1 is: If R is a commutative domain which is finitely generated as an algebra over the field $\boldsymbol{k}$, then

$$1 \le \operatorname{trdeg}_{\boldsymbol{k}} R \implies R \text{ is not a field.}$$

Lets prove this first. (In fact, this is the main thing we have to prove.)

Our thought process goes something like this.

- $1 \le \operatorname{trdeg}_{\boldsymbol{k}} R \implies$ there is a polynomial ring inside R. Certainly polynomial rings have lots of non-units. We have to show that some of these non-units of the polynomial ring remain non-units in R
- It would be useful to understand the structure of (some approximation of) R over (some approximation of) the polynomial ring.
- Here is the plan:
  - Lets record the structure of $\operatorname{Qf}(\text{polynomial ring}) \subseteq \operatorname{Qf}(R)$. (We know this because it is a finite dimensional field extension.)
  - Then use this information to learn about

$$\text{Polynomial ring} \subseteq R.$$

(I write $\operatorname{Qf}(D)$ to be the quotient field of the domain D.)

Project 3.3. *Suppose $K \subseteq \mathbb{L}$ are fields, $\alpha_1, \dots, \alpha_n$ are elements of $\mathbb{L}$ which are algebraic over K, and L is the ring $K[\alpha_1, \dots, \alpha_n]$.*

(a) (*Of course, L is a field with $\dim_K L < \infty$.*) *What is a good basis for L over K?*

(b) *Let $\Phi : K[x_1, \dots, x_n] \to K[\alpha_1, \dots, \alpha_n]$ be the homomorphism which is defined by*

$$\Phi(f(x_1, \dots, x_n)) = f(\alpha_1, \dots, \alpha_n).$$

What is the kernel of Φ?

Get to work.

(a) Define

$$\begin{aligned} \dim_K K(\alpha_1) &= d_1 \\ \dim_{K(\alpha_1)} K(\alpha_1, \alpha_2) &= d_2 \\ &\vdots \\ \dim_{K(\alpha_1, \dots, \alpha_{n-1})} K(\alpha_1, \dots, \alpha_n) &= d_n \end{aligned}$$

Observe that

$$\left\{ \alpha_1^{e_1} \cdots \alpha_n^{e_n} \,\middle|\, \begin{matrix} 0 \le e_1 \le d_1 - 1, \\ \vdots \\ 0 \le e_n \le d_n - 1 \end{matrix} \right\} \tag{3.3.1}$$

is a basis for $K[\alpha_1, \dots, \alpha_n]$ over K.

(b)

Step (i) Identify $f_1(x_1) \in K[x_1]$ monic of degree d_1 with $f_1(\alpha_1) = 0$.
Identify $f_2(x_1, x_2) \in (K[x_1])[x_2]$ monic (in x_2) of degree d_2 in x_2 with $f_2(\alpha_1, \alpha_2) = 0$.
Identify $f_3(x_1, x_2, x_3) \in (K[x_1, x_2])[x_3]$ monic (in x_3) of degree d_3 in x_3 with $f_3(\alpha_1, \alpha_2, \alpha_3) = 0$.
$\vdots$
Identify $f_n(x_1, x_2, \dots, x_n) \in (K[x_1, \dots x_{n-1}])[x_n]$ monic (in x_n) of degree d_n in x_n with $f_n(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$.

Step (ii) Let p be an arbitrary element of $K[x_1, \dots, x_n]$. Apply the division algorithm many times to see that

$$p = \text{ an element of } (f_1, \dots, f_n) \ + p_0$$

where $\deg_{x_i} p_0 \le d_i - 1$, for all i.

Explanation. Apply the division algorithm to f_n and p to write

$$p = q_n f_n + \sum_{i_n=0}^{d_n-1} r_{i_n} x_n^{i_n},$$

with $q_n \in K[x_1, \dots, x_n]$ and $r_{i_n} \in K[x_1, \dots, x_{n-1}]$.

Apply the division algorithm to f_{n-1} and each r_{i_n} to write

$$p = q_n f_n + q_{n-1} f_{n-1} + \sum_{i_n=0}^{d_n-1} \sum_{i_{n-1}=0}^{d_{n-1}-1} r_{i_{n-1}, i_n} x_{n-1}^{i_{n-1}} x_n^{i_n},$$

with $q_{n-1} \in K[x_1, \dots, x_{n-1}]$ and $r_{i_{n-1}, i_n} \in K[x_1, \dots, x_{n-2}]$.

Continue in this manner until (ii) is obtained.

Step (iii) If $p \in \ker \Phi$, then $p_0 \in \ker \Phi$. Degree considerations show that p_0 is the zero polynomial. It follows that $p \in (f_1, \dots, f_n)$.

We conclude that $\ker \Phi = (f_1, \dots, f_n)$. □

Lemma 3.4. *Let $D \subseteq R$ be domains, with R finitely generated as an algebra over D, and R an algebraic extension of D. Then there is a non-zero element δ of D with R_δ a finitely generated free D_δ-module.*

Remark. The hypothesis "R is an algebraic extension of D" means that, for each element r of R, there is a nonzero polynomial $p \in D[x]$ with $p(r) = 0$.

Proof. There are elements $\alpha_1, \dots, \alpha_n$ in R with $R = D[\alpha_1, \dots, \alpha_n]$. Everything is taking place inside the quotient field $\text{Qf}(R)$ of R. Let $K = \text{Qf}(D)$ and $L = K[\alpha_1, \dots, \alpha_n]$. We may apply everything we learned in Project 3.3 to $K \subseteq L$. In particular, define the dimensions d_i as was done in Project 3.3.(a) and the polynomials f_i as was done in Project 3.3.(b). A basis for L over K is given in (3.3.1). This basis consists of elements of R. The polynomials $f_i \in K[x_1, \dots, x_n]$ are used to show that each element of L can be written in terms of the basis. Recall that $K = \operatorname{Qf}(D)$. Let δ in D be a common denominator for $f_1, \dots, f_n$. In other words, $f_1, \dots, f_n$ all are elements of $D_\delta[x_1, \dots, x_n]$. The Division Algorithm may be applied in $D_\delta[x_1, \dots, x_n]$ to write every element of $D_\delta[x_1, \dots, x_n]$ as an element of $(f_1, \dots, f_n)D_\delta[x_1, \dots, x_n]$ plus an element of degree less than d_i in x_i for all i. Thus,

$$D_\delta[\alpha_1, \dots, \alpha_n] \cong \frac{D_\delta[x_1, \dots, x_n]}{(f_1, \dots, f_n)D_\delta[x_1, \dots, x_n]}$$

is a free D_δ-module with basis (3.3.1). □

Lemma 3.5. *Let R be a commutative domain which is finitely generated as an algebra over the field $\boldsymbol{k}$. If any element of R is transcendental over $\boldsymbol{k}$, then R is not a field.*

Proof. Identify a generating set $\gamma_1, \dots, \gamma_r, \alpha_1, \dots, \alpha_n$ for R over $\boldsymbol{k}$ with $\gamma_1, \dots, \gamma_r$ algebraically independent over $\boldsymbol{k}$ and R algebraic over $\boldsymbol{k}[\gamma_1, \dots, \gamma_r]$. The hypothesis ensures that $1 \le r$. Let $D = \boldsymbol{k}[\gamma_1, \dots, \gamma_r]$. Apply Lemma 3.4 and identify a non-zero element δ in D with the property that R_δ is a free D_δ-module and 1 is one of the basis elements. We will carry out the following steps.

(a) Observe D_δ is not a field.
(b) Conclude that R_δ is not a field.
(c) Conclude that R is not a field.

(a) This is obvious. The polynomial ring $D = \boldsymbol{k}[\gamma_1, \dots, \gamma_r]$ is a UFD with infinitely many irreducible elements. (If there were only a finite number of irreducible elements, then the product of these irreducible elements plus one is not a unit (for degree reasons) and is not divisible by any irreducible element.) On the other hand, g has only a finite number of irreducible factors. Pick $h \in \boldsymbol{k}[\gamma_1, \dots, \gamma_r]$ with h irreducible and h not a factor of δ. Observe that h is not a unit in D_δ. Indeed, if $\frac{1}{h} \in \boldsymbol{k}[\gamma_1, \dots, \gamma_r]_\delta$, then

$$\frac{1}{h} = \frac{\text{polynomial}}{\delta^N}$$

Thus,

$$\delta^N = \text{polynomial } h,$$

which contradicts that the fact that h is an irreducible element of of UFD which does not divide δ.

(b) Now, this is obvious. We already identified an element b in D_δ which is not a unit in D_δ. The ring R_δ is a free D_δ-module and 1 is one of the generators; say

$$R_\delta = D_\delta \cdot 1 \oplus D_\delta \cdot \beta_2 \oplus \cdots.$$

We show that b is not a unit in R_δ. A typical element of R_δ is equal to

$$b_0 \cdot 1 \oplus b_1 \cdot \beta_2 \oplus \dots,$$

with $b_i \in D_\delta$. Observe that b times the typical element is

$$bb_0 \cdot 1 \oplus bb_1 \cdot \beta_2 \oplus \dots,$$

and this is equal to 1 only if $bb_0 = 1$ and $bb_i = 0$ for $1 \le i$. Of course, $bb_0 = 1$ is not possible since b is not a unit of D_δ.

(c) This is also obvious now. We argue by contradiction. Assume R is a field. It follows that δ, which is a non-zero element of R, is a unit; hence, $\frac{1}{\delta}$ is already in R. Thus, $R_\delta = R$ is a field. On the other hand, we saw in (b) that R_δ is not a field. This contradiction guarantees that R is not a field. □

March 28, 2019

Theorem. 3.1. *If R is a commutative domain which is finitely generated as an algebra over the field $\boldsymbol{k}$, then the Krull dimension of R is equal to $\operatorname{trdeg}_{\boldsymbol{k}} R$.*

Lemma. 3.5. *Let R be a commutative domain which is finitely generated as an algebra over the field $\boldsymbol{k}$. If any element of R is transcendental over $\boldsymbol{k}$, then R is not a field.*

Proof. Identify $\alpha_1, \dots, \alpha_r \in R$ so that

(a) $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\boldsymbol{k}$, and
(b) R is algebraic over $\boldsymbol{k}[\alpha_1, \dots, \alpha_r]$.

Let $D = \boldsymbol{k}[\alpha_1, \dots, \alpha_r]$.

- Observe that there exists δ in D with the property that R_δ is a finitely generated free D_δ module.
- Observe that D_δ is not a field.
- Observe that R_δ is not a field.
- Conclude that R is not a field. (Indeed, if R were a field, then δ, which is in $D \subseteq R$, is a unit in R; hence $R_\delta = R$. This contradicts the fact that R_δ is not a field.) □

Proof of Theorem 3.1. Let $P = \boldsymbol{k}[x_1, \dots, x_n]$ and $R = \frac{P}{\mathfrak{p}}$.

Step 1. $\dim R \le \operatorname{trdeg}_{\boldsymbol{k}} R$.
Let

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$$

be prime ideals in P. We will show that

$$\operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_{i+1} < \operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_i.$$

In this case,

$$0 \le \operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_d < \operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_{d-1}, \cdots < \operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_0 = \operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}$$

and

$$\dim P/\mathfrak{p} \le \operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}.$$

The inclusion

$$\mathfrak{p}_i \subseteq \mathfrak{p}_{i+1}$$

induces a surjection

$$P/\mathfrak{p}_i \twoheadrightarrow P/\mathfrak{p}_{i+1}.$$

A set of algebraically elements of $P/\mathfrak{p}_{i+1}$ pulls back to set of algebraically independent elements in $P/\mathfrak{p}_i$. Thus,

$$\operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_{i+1} \le \operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_i.$$

We prove $<$.

Assume $<$ fails (that is, assume $\operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_{i+1} = \operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{p}_i$). We will obtain a contradiction.

Renumber the variables so that $\operatorname{im} x_1, \dots, \operatorname{im} x_r$ is a transcendence basis for $P/\mathfrak{p}_{i+1}$ over $\boldsymbol{k}$.

Let $\beta_j = \operatorname{im} x_j$ in $P/\mathfrak{p}_{i+1}$ for all j.

Let $\alpha_j = \operatorname{im} x_j$ in $P/\mathfrak{p}_i$ for all j.

It follows that $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\boldsymbol{k}$. We have assumed that $P/\mathfrak{p}_i$ and $P/\mathfrak{p}_{i+1}$ have the same transcendence degree over $\boldsymbol{k}$. It follows that $\alpha_1, \dots, \alpha_r$ are a transcendence basis for $R/\mathfrak{p}_i$.

Let $S = \boldsymbol{k}[x_1, \dots, x_r] \setminus \{0\}$ and $K = S^{-1}(\boldsymbol{k}[x_1, \dots, x_r])$. Notice that $\mathfrak{p}_i \cap S$ and $\mathfrak{p}_{i+1} \cap S$ are both empty because $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_r$ are both algebraically independent sets over $\boldsymbol{k}$.

Notice that $S^{-1}P = K[x_{r+1}, \dots, x_n]$. Thus,

$$\frac{S^{-1}P}{\mathfrak{p}_i S^{-1}P} = S^{-1}\left(\frac{P}{\mathfrak{p}_i}\right) = \boldsymbol{k}(\alpha_1, \dots, \alpha_r)[\alpha_{r+1}, \dots, \alpha_n]$$

and **this is a field!**

Thus, $\mathfrak{p}_i S^{-1}P$ is a maximal ideal of $S^{-1}P$; but this makes no sense because,

$$\mathfrak{p}_i S^{-1}P \subsetneq \mathfrak{p}_{i+1} S^{-1}P$$

are proper ideals of $S^{-1}P$. We have reached our contradiction.

We have established $\dim R \le \operatorname{trdeg}_{\boldsymbol{k}} R$.

Step 2. Now we prove that $\operatorname{trdeg}_{\boldsymbol{k}} R \le \dim R$.

The proof is by induction on $\operatorname{trdeg}_{\boldsymbol{k}} R$. If $\operatorname{trdeg}_{\boldsymbol{k}} R = 0$, then there is nothing to prove.

Henceforth, $1 \le \operatorname{trdeg}_{\boldsymbol{k}} R = r$. Continue to take $P = \boldsymbol{k}[x_1, \dots, x_n]$ and $R = P/\mathfrak{p}$. Let α_i be the image of x_i in R. Assume α_1 is transcendental over $\boldsymbol{k}$. Let $S = \boldsymbol{k}[x_1] \setminus \{0\}$. Observe that $S \cap \mathfrak{p} = \emptyset$ and that $S^{-1}R = \boldsymbol{k}(\alpha_1)[\alpha_2, \dots, \alpha_r]$. Observe that

$$\operatorname{trdeg}_{\boldsymbol{k}(\alpha_1)} \boldsymbol{k}(\alpha_1)[\alpha_2, \dots, \alpha_r] = r - 1.$$

Thus, by induction, there are prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{r-1}$$

in P which miss S. It is clear that

$$\mathfrak{p}R = \mathfrak{p}_0 R \subsetneq \mathfrak{p}_1 R \subsetneq \cdots \subsetneq \mathfrak{p}_{r-1} R$$

are prime ideals of R. Notice that the class of x_1 in

$$\frac{P}{\mathfrak{p}_{r-1}}$$

is transcendental over $\boldsymbol{k}$ because $\mathfrak{p}_{r-1} \cap S$ is the empty set. Apply Lemma 3.5 to see that

$$\frac{P}{\mathfrak{p}_{r-1}}$$

is not a field. It follows that $\mathfrak{p}_{r-1}$ is not a maximal ideal of P and $r \le \dim R$. □

Theorem 3.6. *Let $\boldsymbol{k}$ be a field and $\mathfrak{m}$ be a maximal ideal of the polynomial ring $P = \boldsymbol{k}[x_1, \dots, x_n]$. Then the following statements hold*:

(a) *$P/\mathfrak{m}$ is algebraic over $\boldsymbol{k}$;*
(b) *$\mathfrak{m}$ can be generated by n elements; and*
(c) *if $\boldsymbol{k}$ is algebraically closed, then $\mathfrak{m} = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$ for some $\alpha_1, \dots, \alpha_n$ in $\mathfrak{m}$.*

Proof. **(a)** Let $K = P/\mathfrak{m}$. Observe that K is finitely generated as an algebra over $\boldsymbol{k}$. We know from Goal 3.1 that the transcendence degree of K over $\boldsymbol{k}$ is equal to the Krull dimension of K, which, of course, is zero. We conclude that K is algebraic over $\boldsymbol{k}$.

(b) We proved in Project 3.3.(b) that the kernel of the natural quotient map $P \to P/\mathfrak{m}$ can be generated by n elements.

April 2, 2019.

Last time we proved the following result.

Theorem. 3.1. *If R is a commutative domain which is finitely generated as an algebra over the field $\boldsymbol{k}$, then the Krull dimension of R is equal to $\operatorname{trdeg}_{\boldsymbol{k}} R$.*

Corollary. 3.6.(c) *If $\boldsymbol{k}$ is an algebraically closed field and $\mathfrak{m}$ is a maximal ideal of the polynomial ring $P = \boldsymbol{k}[x_1, \dots, x_n]$, then $\mathfrak{m} = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$ for some $\alpha_1, \dots, \alpha_n$ in $\mathfrak{m}$.*

(c) The field $K = P/\mathfrak{m}$ is algebraic over $\boldsymbol{k}$; (since $\operatorname{trdeg}_{\boldsymbol{k}} P/\mathfrak{m} = \dim P/\mathfrak{m} = 0$), thus there is a $\boldsymbol{k}$-algebra embedding of K into the algebraic closure, $\bar{\boldsymbol{k}}$, of $\boldsymbol{k}$. The hypothesis $\boldsymbol{k} = \bar{\boldsymbol{k}}$ ensures that for each i there exists $\alpha_i \in \boldsymbol{k}$ with $x_i \equiv \alpha_i \mod \mathfrak{m}$. In other words, $x_i - \alpha_i \in \mathfrak{m}$. Thus, $(x_1 - \alpha_1, \dots, x_n - \alpha_n) \subseteq \mathfrak{m}$. Of course, $(x_1 - \alpha_1, \dots, x_n - \alpha_n)$ is already a maximal ideal; thus $(x_1 - \alpha_1, \dots, x_n - \alpha_n) = \mathfrak{m}$. □

Definition 3.7. Let $\boldsymbol{k} \subseteq K$ be fields.

(a) If J is an ideal of $\boldsymbol{k}[x_1, \dots, x_n]$, then

$$V_K(J) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{A}_K^n \mid f(\alpha_1, \dots, \alpha_n) = 0, \text{ for all } f \in J\}.$$

(b) If V is a subset of $\mathbb{A}_K^n$, then

$$I_{\boldsymbol{k}}(V) = \{f \in \boldsymbol{k}[x_1, \dots, x_n] \mid f(\alpha_1, \dots, \alpha_n) = 0, \text{ for all } (\alpha_1, \dots, \alpha_n) \text{ in } V\}.$$

Theorem 3.8. [Hilbert's Nullstellensatz] *Let $\boldsymbol{k}$ be a field and $\bar{\boldsymbol{k}}$ be the algebraic closure of $\boldsymbol{k}$.*

(a) *If J is an ideal of $\boldsymbol{k}[x_1, \dots, x_n]$, with $V_{\bar{\boldsymbol{k}}}(J)$ empty, then $J = \boldsymbol{k}[x_1, \dots, x_n]$.*
(b) *If J is a proper ideal of $\boldsymbol{k}[x_1, \dots, x_n]$, then $I_{\boldsymbol{k}}(V_{\bar{\boldsymbol{k}}}(J)) = \sqrt{J}$.*

Proof. **(a)** We prove that if J is a proper ideal of $\boldsymbol{k}[x_1, \dots, x_n]$, then $V_{\bar{\boldsymbol{k}}}(J)$ is not empty.

Assume J is a proper ideal of $\boldsymbol{k}[x_1, \dots, x_n]$. Then J is contained in a maximal ideal $\mathfrak{m}$ of $\boldsymbol{k}[x_1, \dots, x_n]$. We proved in 3.6.(a) that $\boldsymbol{k}[x_1, \dots, x_n]/\mathfrak{m}$ is algebraic over $\boldsymbol{k}$. Let

$$\theta : \boldsymbol{k}[x_1, \dots, x_n]/\mathfrak{m} \to \bar{\boldsymbol{k}}$$

be a $\boldsymbol{k}$-algebra homomorphism and let α_i in $\bar{\boldsymbol{k}}$ be θ of the class of x_i. If $g(x_1, \dots, x_n) \in \mathfrak{m}$, then

$$0 = \theta g(x_1, \dots, x_n) = g(\alpha_1, \dots, \alpha_n).$$

In particular, each element of J annihilates $(\alpha_1, \dots, \alpha_n)$ and $(\alpha_1, \dots, \alpha_n)$is in $V_{\bar{\boldsymbol{k}}}(J)$.

(b) The inclusion $\supseteq$ is clear. We prove $\subseteq$. Suppose $f \in I_{\boldsymbol{k}}(V_{\bar{\boldsymbol{k}}}(J))$. We must show that $f^N \in J$ for some N. Consider the ideal $(J, 1 - yf)$ in the polynomial ring $\boldsymbol{k}[x_1, \dots, x_n, y]$. Observe that $V_{\bar{\boldsymbol{k}}}(J, 1 - yf)$ is empty. Thus, by (a), $(J, 1 - yf) = \boldsymbol{k}[x_1, \dots, x_n, y]$ and there are elements $j_i \in J$ and p_i and q in $\boldsymbol{k}[x_1, \dots, x_n, y]$ such that

$$1 = \sum_i j_i p_i + q(1 - yf).$$

Apply the $\boldsymbol{k}[x_1, \dots, x_n]$-algebra homomorphism

$$\boldsymbol{k}[x_1, \dots, x_n, y] \to \boldsymbol{k}[x_1, \dots, x_n]_f,$$

which sends y to $\frac{1}{f}$, to obtain

$$1 = \sum_i j_i (\text{an element of } \boldsymbol{k}[x_1, \dots, x_n]_f).$$

Multiply both sides of the most recent equation by a large power of f to finish the proof. □

I would next like to prove the Jacobian criterion for regularity.

Goal. 6.1 *Let $\boldsymbol{k}$ be a perfect field, $P = \boldsymbol{k}[x_1, \dots, x_n]$, $I = (f_1, \dots, f_m)$ be an ideal of P, $\mathfrak{p}$ be a prime ideal of P with $I \subseteq \mathfrak{p}$, and L be the field $\frac{P_\mathfrak{p}}{\mathfrak{p}P_\mathfrak{p}}$. Let* Jac *be the matrix*

$$\operatorname{Jac} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

Then

$$\operatorname{rank}_L \operatorname{Jac}|_L \le \dim P_\mathfrak{p} - \dim(P/I)_\mathfrak{p}$$

and

$$(P/I)_{\mathfrak{p}} \textit{ is regular} \iff \operatorname{rank}_L \operatorname{Jac}|_L = \dim P_{\mathfrak{p}} - \dim(P/I)_{\mathfrak{p}}.$$

Remarks. 6.2

(a) The entries in Jac are polynomials in the polynomial ring $\boldsymbol{k}[x_1, \ldots, x_n]$.
(b) We have used the symbol "$|_{\frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}}}$" to mean: take the image in $\frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}}$.
(c) Recall that the field $\boldsymbol{k}$ is perfect if every algebraic extension is separable. In particular, every field of characteristic zero and every finite field is perfect. Also, if the characteristic of $\boldsymbol{k}$ is the positive prime integer p, then $\boldsymbol{k}$ is perfect if and only if $\boldsymbol{k}$ is closed under the taking of p^{th} roots.

We proved that if R is a domain which is finitely generated as an algebra over the field $\boldsymbol{k}$, then

$$\operatorname{trdeg}_{\boldsymbol{k}} R = \dim R$$

as a step in proving the Jacobian criterion.

To complete the proof, we also need to know that if

(3.8.1) R is a domain which is finitely generated as an algebra over the field $\boldsymbol{k}$,

then

(3.8.2) $$\operatorname{ht} \mathfrak{p} + \dim \tfrac{R}{\mathfrak{p}} = \dim R.$$

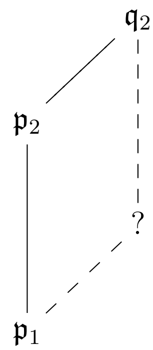
(We proved (3.8.2) if R is local and Cohen-Macaulay in Proposition 1.29.)

Example 3.9. Here is a quick example to show that hypotheses are needed in order to get (3.8.2). Let $R = \frac{\boldsymbol{k}[x,y,z]}{(x)(y,z)}$. Geometrically, R is the coordinate ring of the union of the y, z plane and the x-axis.

- If $\mathfrak{p} = (y,z)R$, then $\operatorname{ht} \mathfrak{p} = 0$, $\dim R/\mathfrak{p} = \dim \boldsymbol{k}[x,y,z]/(y,z) = 1$ and $\dim R = 2$. So
$$\operatorname{ht} \mathfrak{p} + \dim R/\mathfrak{p} = 0 + 1 \neq 2 = \dim R.$$
- If $\mathfrak{p} = (x)R$, then $\operatorname{ht} \mathfrak{p} = 0$, $\dim R/\mathfrak{p} = \dim \boldsymbol{k}[x,y,z]/(x) = 2$ and $\dim R = 2$. In this case,
$$\operatorname{ht} \mathfrak{p} + \dim R/\mathfrak{p} = 0 + 2 = 2 = \dim R.$$

The problem with this ring (or geometric object) is that the irreducible components of the geometric object have different dimensions.

It will take some work to prove (3.8.2) under hypothesis (3.8.1). In section 4 we will prove that if $A \subseteq B$ is an integral extension of domains and A is integrally closed in its quotient field, then the extension $A \subseteq B$ has the Going Down property, Theorem 4.9. In other words, if $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ are prime ideals of A and $\mathfrak{q}_2$ is a prime ideal of B with $\mathfrak{p}_2 = \mathfrak{q}_2 \cap A$. Then there is a prime ideal $\mathfrak{q}_1$ in B with $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\mathfrak{p}_1 = \mathfrak{q}_1 \cap A$. Here is a picture:



Then, in Section 5 we will prove that under hypothesis (3.8.1), if I is any ideal R, then there exists a polynomial ring P with $P \subseteq R$ an integral extension and such that $I \cap P$ is generated by variables (A polynomial ring is easily seen to be integrally closed in its quotient field.) The proof of (3.8.2) is given in 5.5.(d).

4. INTEGRAL EXTENSIONS

Definition 4.1. Let $A \subseteq B$ be rings.

(a) If $b \in B$, then b is integral over A if there exists a monic polynomial $p(x) \in A[x]$ such that $p(b) = 0$.

(b) If every element of B is integral over A, then $A \subseteq B$ is called an integral extension.

Examples 4.2.

(a) If A is a UFD, then A is integrally closed in $\text{Qf}(A)$. Indeed, if $\theta = \alpha_1/\alpha_2$ is a non-zero element of $\text{Qf}(A)$, with α_1, α_2 relatively prime elements of A, and θ is integral over A, then there exist elements a_i in A with

$$(\alpha_1/\alpha_2)^n + a_1(\alpha_1/\alpha_2)^{n-1} + \cdots + a_n(\alpha_1/\alpha_2)^0 = 0.$$

Multiply both sides by α_2^n to obtain and

$$\alpha_1^n + a_1\alpha_2\alpha_1^{n-1} + \cdots + a_n\alpha_2^n = 0.$$

Thus $\alpha_1^n \in (\alpha_2)$. The elements α_1 and α_2 of A are relatively prime so α_2 is a unit of A and θ is an element of A.

(b) The ring $\mathbb{Z}[\sqrt{5}]$ is not integrally closed in its quotient field. Indeed $\frac{1+\sqrt{5}}{2}$ is a solution of

$$x^2 - x - 1 = 0.$$

Lemma 4.3. *If $A \subseteq B$ are rings and B is finitely generated as an A-module, then $A \subseteq B$ is an integral extension.*

Proof. Suppose B is generated by $b_1, \dots, b_n$ over A. Let b be an element of B. Write bb_i as $\sum_j a_{i,j}b_j$. Thus,

$$b\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Let M be the name of the $n \times n$ matrix. Thus,

$$(bI - M)\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = 0.$$

Multiply both sides of the equation by the classical adjoint of $bI - M$ to conclude $\det(bI - M)$ annihilates B. But $1 \in B$; so, $\det(bI - M) = 0$. Thus b satisfies the n-th degree monic polynomial $\det(xI - M) = 0$. Observe that $\det(xI - M) \in A[x]$. □

Lemma 4.4. [This result called Incomparable.] *If $A \subseteq B$ is an integral extension of rings and $P_1 \subsetneq P_2$ are prime ideals of B, then $P_1 \cap A \subsetneq P_2 \cap B$.*

Proof. The ring extension $A/(P_1 \cap A) \subseteq B/P_1$ is also an integral extension.

It suffices to show that if $A \subseteq B$ is an integral extension of domains, then (0) is the only ideal of B contracting to (0). If $b \in B$ and $b \neq 0$, then $b^n + a_1b^{n-1} + \cdots + a_n = 0$ for some n and for some $a_i \in A$ with $a_n \neq 0$. Thus, $a_n \in A \cap (b)B$. □

Observation 4.5. *If $A \subseteq B$ are rings and $\mathfrak{p}$ is a prime ideal of A with $A \cap (\mathfrak{p}B) \subseteq \mathfrak{p}$, then there is a prime ideal $\mathfrak{q}$ of B with $\mathfrak{q} \cap A = \mathfrak{p}$.*

Proof. Let $S = A \setminus \mathfrak{p}$. Think of S as a multiplicatively closed subset of B. The hypothesis ensures that $\mathfrak{p}B$ misses S. Consider the set

$$\{I \text{ is an ideal of } B \mid \mathfrak{p}B \subseteq I \quad \text{and} \quad I \cap S = \emptyset\}.$$

The set is non-empty because $\mathfrak{p}B$ is in it! Take an ideal $\mathfrak{q}$ of B maximal in the set. Such an ideal $\mathfrak{q}$ exists (either by Zorn's Lemma or because B is Noetherian), is a prime ideal of B, and $\mathfrak{q} \cap A = \mathfrak{p}$. □

The rest of the section closely follows [1, 5.14, 5.15, 5.16].

Definition 4.6. Let $A \subseteq B$ be rings, and I be an ideal of A. If the element b of B satisfies a polynomial equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

with all $a_i \in I$, then b is integral over I.

Observation 4.7. *If $A \subseteq B$ are rings, I is an ideal of A, and C is the integral closure of A in B, then*

$$\text{the integral closure of } I \text{ in } B = \sqrt{IC}.$$

Remark. The most important consequence of this observation is the fact that the integral closure of I in B is closed under addition and multiplication.

Proof. ($\subseteq$) If $b \in B$ is in the integral closure of I in B, then there exists $a_i \in I$ with

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Thus, $b \in C$ and $b^n \in IC$. It follows that $b \in \sqrt{IC}$.

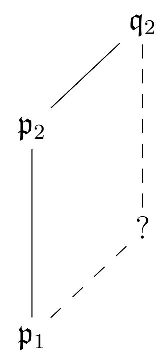
($\supseteq$). If the element b of B is in $\sqrt{IC}$, then $b^n \in IC$, for some n, and $b^n = \sum_{j=1}^{N} i_j c_j$ with $i_j \in I$ and $c_j \in C$. Let R be the ring $A[c_1, \dots, c_N]$. The ring R is finitely generated as an A-module. Observe that $b^n R \subseteq IR$. The usual determinant trick argument shows that

$$\det\Big(b^n \text{ identity matrix} - (\text{a matrix with entries from } I)\Big) R = 0.$$

The ring R contains the element 1 of A. Thus the determinant is zero. The equation $\det = 0$ demonstrates that b^n is integral over I; hence b is integral over I. □

April 9, 2019

Goal. 4.9 *Let $A \subseteq B$ be an integral extension of domains with A integrally closed in $\mathrm{Qf}(A)$.*



Observation. 4.5 *If $A \subseteq B$ are rings and $\mathfrak{p}$ is a prime ideal of A with $A \cap (\mathfrak{p}B) \subseteq \mathfrak{p}$, then there is a prime ideal $\mathfrak{q}$ of B with $\mathfrak{q} \cap A = \mathfrak{p}$.*

Observation. 4.7 *If $A \subseteq B$ are rings and I is an ideal of A, then*

$$\text{the integral closure of } I \text{ in } B = \sqrt{I(\text{the integral closure of } A \text{ in } B)}.$$

Lemma 4.8. *Let $A \subseteq B$ be domains with A integrally closed in $\mathrm{Qf}(A)$. Let I be an ideal of A and b be an element of B with b integral over I. (In particular b is algebraic over $\mathrm{Qf}(A)$.) Then the minimal polynomial of b over $\mathrm{Qf}(A)$ has the form*

$$x^n + a_1x^{n-1} + \cdots + a_nx^n,$$

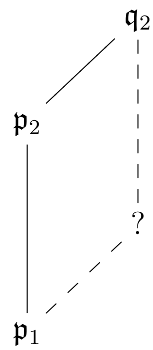
with $a_i \in \sqrt{I}$.

Proof. The hypothesis guarantees that there exists a polynomial $g(x) = x^m + i_1x^{m-1} + \cdots + i_m$ in $A[x]$ with $i_j \in I$ and $g(b) = 0$. Let $f(x) = \mathrm{Qf}(A)[x]$ be the minimal polynomial of b over $\mathrm{Qf}(A)$ and let L be an extension field of $\mathrm{Qf}(A)$ in which $f(x)$ factors into linear factors $f(x) = \prod_j (x - b_j)$ in $L[x]$. The polynomial $f(x)$ divides $g(x)$ in $[\mathrm{Qf}(A)][x]$; so $g(b_j) = 0$ (in L) for each j. In other words, each b_j is integral over I. The coefficients of f are the elementary symmetric polynomials evaluated at the b_j's. Thus each coefficient of f is integral over A. (We use the Remark following Observation 4.7 to conclude the sums and products of elements in L which are integral over I are also integral over I.) Now we use Observation 4.7 again. The coefficients of f are elements of $\mathrm{Qf}(A)$ which are integral over I. Thus these coefficients are in

$$\sqrt{I(\text{the integral closure of } A \text{ in } \mathrm{Qf}(A))} = \sqrt{IA} = \sqrt{I}.$$

Thus each coefficient of f is in $\sqrt{I}$. $\square$

Theorem 4.9. [This result is called Going Down.] *Let $A \subseteq B$ be an integral extension of domains with A integrally closed in $\mathrm{Qf}(A)$. Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ be prime ideals of A and $\mathfrak{q}_2$ be a prime ideal of B with $\mathfrak{p}_2 = \mathfrak{q}_2 \cap A$. Then there is a prime ideal $\mathfrak{q}_1$ in B with $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\mathfrak{p}_1 = \mathfrak{q}_1 \cap A$.*



Proof. According to Observation 4.5, it suffices to show that

$$A \cap \mathfrak{p}_1 B_{\mathfrak{q}_2} \subseteq \mathfrak{p}_1.$$

Let x be an arbitrary element of $\mathfrak{p}_1 B_{\mathfrak{q}_2}$. So, $x = \frac{y}{s}$, with $y \in \mathfrak{p}_1 B$ and $s \in B \setminus \mathfrak{q}_2$. Apply Observation 4.7:

$$y \in \mathfrak{p}_1 B \subseteq \sqrt{\mathfrak{p}_1 B} = \sqrt{\mathfrak{p}_1 (\text{the integral closure of } A \text{ in } B)}.$$

Therefore, y is in the integral closure of $\mathfrak{p}_1$ in B. The present hypothesis that A is integrally closed in $\mathrm{Qf}(A)$ allows us to apply Lemma 4.8 in order to conclude that the minimal polynomial of y over $\mathrm{Qf}(A)$ is

$$m(t) = t^r + u_1 t^{r-1} + \cdots + u_r \in \mathrm{Qf}(A)[t],$$

for some $u_i \in \mathfrak{p}_1$.

Now assume that x is in $A \cap \mathfrak{p}_1 B_{\mathfrak{q}_2}$. Of course, $s = yx^{-1}$ and $x^{-1} \in \mathrm{Qf}(A)$. Observe that

$$0 = m(y) = (xs)^r + u_1 (xs)^{r-1} + \cdots + u_r.$$

Multiply by x^{-r} to obtain

$$0 = s^r + x^{-1} u_1 s^{r-1} + \cdots + (x^{-1})^r u_r. \tag{4.9.1}$$

Thus, s satisfies the polynomial

$$m_1(t) = t^r + x^{-1} u_1 t^{r-1} + \cdots + (x^{-1})^r u_r$$

and $m_1(t)$ is an irreducible polynomial in $\mathrm{Qf}(A)[t]$. It follows that $m_1(t)$ is the minimal polynomial of s over $\mathrm{Qf}(A)$.

The element s of B is integral over A (because every element of B is integral over A). Apply Lemma 4.8 again. This time take $I = A$. Conclude $(x^{-1})^i u_i \in A$ for all i with $0 \le i \le r-1$.

The crucial equation is

$$x^i \underbrace{(x^{-1})^i u_i}_{\in A} = \underbrace{u_i}_{\in \mathfrak{p}_1}.$$

If $x \notin \mathfrak{p}_1$, then $(x^{-1})^i u_i \in \mathfrak{p}_1$, for $1 \le i \le r$. In this case, (4.9.1) exhibits

$$s^r \in \mathfrak{p}_1 B \subseteq \mathfrak{p}_2 B \subseteq \mathfrak{q}_2.$$

In this case, $s \in \mathfrak{q}_2$, which is a contradiction.

Thus, $x \in \mathfrak{p}_1$, and the proof is complete. $\square$

5. Noether Normalization.

This section closely follows [4, Chapt. II, sect. 3]. We continue to move in the direction of proving the Jacobian criteria for regular local rings. One critical part of the proof is the fact (see Corollary 5.5.(d)) that if $\mathfrak{p}$ is a prime ideal in a domain R, which is finitely generated as an algebra over a field, then

$$\operatorname{ht}\mathfrak{p} + \dim R/\mathfrak{p} = \dim R.$$

We already know this conclusion when R is a Cohen-Macaulay local ring. I think that Noether Normalization provides the best way to get the conclusion when R is a domain which is finitely generated as an algebra over a field.

Corollary 5.6 is another consequence of Noether Normalization which is used in the proof of the Jacobian criteria. This result states that if P is a polynomial ring over a field, then $P_{\mathfrak{p}}$ is a regular local ring for all prime ideals $\mathfrak{p}$ of P.

Theorem 5.1. [Noether Normalization] *If R is a ring which is finitely generated as an algebra over a field $\boldsymbol{k}$ and I is a non-zero proper ideal of R, then there exist elements $y_1, \dots, y_d$ in R such that*

(a) *$y_1, \dots, y_d$ are algebraically independent over $\boldsymbol{k}$,*
(b) *R is finitely generated as a module over $\boldsymbol{k}[y_1, \dots, y_d]$, and*
(c) *$I \cap \boldsymbol{k}[y_1, \dots, y_d] = (y_\delta, \dots, y_d)$, for some δ with $\delta \le d$.*

Definition 5.2. If R is a ring which is finite generated as an algebra over a field $\boldsymbol{k}$ and $y_1, \dots, y_d$ are elements of R which are algebraically independent over $\boldsymbol{k}$ with R finitely generated as a module over $\boldsymbol{k}[y_1, \dots, y_d]$, then $\boldsymbol{k}[y_1, \dots, y_d]$ is called a <u>Noether normalization</u> of R.

Remark 5.3. If R is a domain which is finite generated as an algebra over a field $\boldsymbol{k}$ and $N = \boldsymbol{k}[y_1, \dots, y_d]$ is a Noether normalization of R, then $\operatorname{trdeg}_{\boldsymbol{k}} R = \operatorname{trdeg}_{\boldsymbol{k}} N = d$; hence $\dim R = \dim N = d$ by Theorem 3.1.

We use Lemma 5.4 in our proof of Noether Normalization.

Lemma 5.4. *Let $f \in \boldsymbol{k}[x_1, \dots, x_n]$ be a non-constant polynomial. Then there exist non-negative integers $r_1, \dots, r_{n-1}$ with*

$$f = a_m x_n^m + \rho_1 x_n^{m-1} + \dots + \rho_m,$$

with $a_m \neq 0$ in $\boldsymbol{k}$ and $\rho_i \in \boldsymbol{k}[y_1, \dots, y_{r-1}]$, for $y_i = x_i - x_n^{r_i}$, with $1 \le i \le r-1$.

Proof. Write $f = \sum_{\vec{v}} a_{\vec{v}} x_1^{v_1} \dots x_n^{v_n}$. Replace x_i with $y_i + x_n^{r_i}$ for $1 \le i \le n-1$. Observe that

$$\begin{aligned} f &= \sum_{\vec{v}} a_{\vec{v}} (y_1 + x_n^{r_1})^{v_1} \dots (y_{n-1} + x_n^{r_{n-1}})^{v_{r-1}} x_n^{v_n} \\ &= \sum_{\vec{v}} a_{\vec{v}} \left(x_n^{\vec{v}\cdot(r_1,\dots,r_{n-1},1)} + \text{l.o.t. in } x_n \right) \end{aligned}$$

Pick $\overrightarrow{r}$ so that all $\overrightarrow{v}\cdot\overrightarrow{r}$ are distinct. In particular, if k is an integer which is larger than all v_i, and $\overrightarrow{r}=(k^{n-1},\dots,k^2,k,1)$, then each $\overrightarrow{v}\cdot\overrightarrow{r}$ is the k-adic expansion of

$$v_1k^{n-1}+\cdots+v_{n-1}k+v_n.$$

Each integer has a unique k-adic expansion. □

The proof of Theorem 5.1. We consider three cases.

Case 1. Suppose $I=(f)$ for some non-zero $f\in R=\boldsymbol{k}[x_1,\dots,x_n]$.

We find $y_1,\dots,y_{n-1}$ in R so that

- $y_1,\dots,y_{n-1},f$ are algebraically independent over $\boldsymbol{k}$ and
- R is a finitely generated $\boldsymbol{k}[y_1,\dots,y_{n-1},f]$-module.

Then we let y_n denote f.

Apply Lemma 5.4 to find r_i so that $f=ax_n^m+\rho_1x_n^{m-1}+\cdots+\rho_m$ for some $\rho_i\in\boldsymbol{k}[y_1,\dots,y_{n-1}]$, where $y_i=x_i-x_n^{r_i}$ (for $1\le i\le n-1$) and a is a non-zero element of $\boldsymbol{k}$. Observe that

$$R=\Big(\boldsymbol{k}[y_1,\dots,y_{n-1},f]\Big)[x_n]$$

and x_n satisfies $f(X)-f=0$, and

$$f(X)-f=aX^m+\rho_1X^{m-1}+\cdots+\rho_m-(ax_n^m+\rho_1x_n^{m-1}+\cdots+\rho_m)$$

is a polynomial in X with coefficients in $\boldsymbol{k}[y_1,\dots,y_n]$. Thus, R is finitely generated as a module over $\boldsymbol{k}[y_1,\dots,y_{n-1},f]$.

It is clear that $(I\cap\boldsymbol{k}[y_1,\dots,y_n])\supseteq(y_n)\boldsymbol{k}[y_1,\dots,y_n]$.

It remains to show that $\subseteq$. Take $\theta\in I\cap\boldsymbol{k}[y_1,\dots,y_n]$. It follows that

$$\theta=fg \text{ for some } g \text{ in } R \text{ and } \theta\in\boldsymbol{k}[y_1,\dots,y_n].$$

The hypothesis that R is a finitely generated module over $\boldsymbol{k}[y_1,\dots,y_n]$ guarantees that g satisfies a monic polynomial with coefficients in $\boldsymbol{k}[y_1,\dots,y_n]$ (see Lemma 4.3):

$$g^s+a_1g^{s-1}+\cdots+a_s=0$$

for some $a_i\in\boldsymbol{k}[y_1,\dots,y_n]$. Multiply both sides by f^s to obtain

$$\underbrace{(fg)^s}_{\theta^s}+\underbrace{a_1f(fg)^{s-1}+\cdots+a_sf^s=0}_{\in(f)\boldsymbol{k}[y_1,\dots,y_n]}.$$

So θ is in the polynomial ring $\boldsymbol{k}[y_1,\dots,y_n]$ and θ^s is in the prime ideal $(y_n)\boldsymbol{k}[y_1,\dots,y_n]$. (Remember, $f=y_n$.) Thus, $\theta\in(f)\boldsymbol{k}[y_1,\dots,y_n]$.

Case 2. Suppose I is a non-zero proper ideal in $R=\boldsymbol{k}[x_1,\dots,x_n]$.

Induct on n. The case $n=1$ is trivial (and is covered in case 1). Henceforth, assume $2\le n$. Let f be a non-zero element of I. Apply case 1 and identify $y_1,\dots,y_{n-1}$ so that when $y_n=f$, then

- $y_1,\dots,y_n$ are algebraically independent over $\boldsymbol{k}$,
- R is a finitely generated module over $\boldsymbol{k}[y_1,\dots,y_n]$, and

- $I \cap \boldsymbol{k}[y_1, \dots, y_n] = (y_n)\boldsymbol{k}[y_1, \dots, y_n]$.

Apply induction to the ideal $I \cap \boldsymbol{k}[y_1, \dots, y_{n-1}]$ in the ring $\boldsymbol{k}[y_1, \dots, y_{n-1}]$. There are elements $t_1, \dots, t_d$ in $\boldsymbol{k}[y_1, \dots, y_{n-1}]$ which are algebraically independent over $\boldsymbol{k}$ with

- $\boldsymbol{k}[y_1, \dots, y_{n-1}]$ finitely generated as a module over $\boldsymbol{k}[t_1, \dots, t_d]$ and
- $I \cap \boldsymbol{k}[t_1, \dots, t_d] = (t_\delta, \dots, t_d)\boldsymbol{k}[t_1, \dots, t_d]$, for some δ.

Notice that

$$d = \operatorname{trdeg}_{\boldsymbol{k}} \boldsymbol{k}[t_1, \dots, t_d] = \operatorname{trdeg}_{\boldsymbol{k}} \boldsymbol{k}[y_1, \dots, y_{n-1}] = n - 1.$$

Notice also that $\boldsymbol{k}[x_1, \dots, x_n]$ is finitely generated as a module over $\boldsymbol{k}[t_1, \dots, t_{n-1}, y_n]$.

Here is a quick proof of the most recent claim. We started with

$$\boldsymbol{k}[y_1, \dots, y_n] \subseteq \boldsymbol{k}[x_1, \dots, x_n] \text{ is a module-finite extension} \quad \text{and} \qquad (*)$$

$$\boldsymbol{k}[t_1, \dots, \underbrace{t_d}_{t_{n-1}}] \subseteq \boldsymbol{k}[y_1, \dots, y_{n-1}] \text{ is a module-finite extension}$$

hence,

$$\boldsymbol{k}[t_1, \dots, t_d, y_n] \subseteq \boldsymbol{k}[y_1, \dots, y_{n-1}, y_n] \text{ is a module-finite extension.} \qquad (**)$$

Now combine (*) and (**). *This completes this small proof.*

We must show that

$$I \cap \boldsymbol{k}[t_1, \dots, t_{n-1}, y_n] = (t_\delta, \dots, t_{n-1}, y_n)\boldsymbol{k}[t_1, \dots, t_{n-1}, y_n]. \tag{5.4.1}$$

The inclusion $\supseteq$ is obvious.

We prove $\subseteq$. Every element of $\boldsymbol{k}[t_1, \dots, t_{n-1}, y_n]$ is

$$A + y_n B,$$

where $A \in \boldsymbol{k}[t_1, \dots, t_{n-1}]$ and $B \in \boldsymbol{k}[t_1, \dots, t_{n-1}, y_n]$. Observe that $y_n B$ is in

$$(t_\delta, \dots, t_{n-1}, y_n)\boldsymbol{k}[t_1, \dots, t_{n-1}, y_n] \subseteq I \cap \boldsymbol{k}[t_1, \dots, t_{n-1}, y_n].$$

Thus $A + y_n B$ is in $I \cap \boldsymbol{k}[t_1, \dots, t_{n-1}, y_n]$ if and only if

$$A \in I \cap \boldsymbol{k}[t_1, \dots, t_{n-1}] = (t_\delta, \dots, t_{n-1})\boldsymbol{k}[t_1, \dots, t_{n-1}].$$

We have established (5.4.1).

April 16, 2019

We are proving

Theorem. 5.1 [Noether Normalization] *If R is a ring which is finitely generated as an algebra over a field $\boldsymbol{k}$ and I is a non-zero proper ideal of R, then there exist elements $y_1, \dots, y_d$ in R such that*

(a) *$y_1, \dots, y_d$ are algebraically independent over $\boldsymbol{k}$,*
(b) *R is finitely generated as a module over $\boldsymbol{k}[y_1, \dots, y_d]$, and*
(c) *$I \cap \boldsymbol{k}[y_1, \dots, y_d] = (y_\delta, \dots, y_d)$, for some δ with $\delta \le d$.*

So far we have established the theorem completely when R is a polynomial ring over $\boldsymbol{k}$. In this case each Noether Normalization of R is a polynomial ring in $\dim R$ variables.

Case 3. Suppose I is a non-zero proper ideal in $R = \boldsymbol{k}[x_1, \dots, x_n]/J$, where J is an ideal of $\boldsymbol{k}[x_1, \dots, x_n]$.

Let $P = \boldsymbol{k}[x_1, \dots, x_n]$. Apply case 2 to identify a polynomial ring $\boldsymbol{k}[y_1, \dots, y_n]$ which is contained in P with P finitely generated as as a module over $\boldsymbol{k}[y_1, \dots, y_n]$ and

$$J \cap \boldsymbol{k}[y_1, \dots, y_n] = (y_{d+1}, \dots, y_n)$$

for some d. Thus, $R = P/J$ is finitely generated as a module over

$$\frac{\boldsymbol{k}[y_1, \dots, y_n]}{J \cap \boldsymbol{k}[y_1, \dots, y_n]} = \frac{\boldsymbol{k}[y_1, \dots, y_n]}{(y_{d+1}, \dots, y_n)} = \boldsymbol{k}[y_1, \dots, y_d].$$

Now apply case 2 to the ideal $I \cap \boldsymbol{k}[y_1, \dots, y_d]$ to identify $t_1, \dots, t_d$ in $\boldsymbol{k}[y_1, \dots, y_d]$ with

- $t_1, \dots, t_d$ algebraically independent over $\boldsymbol{k}$,
- $\boldsymbol{k}[y_1, \dots, y_d]$ finitely generated as a module over $\boldsymbol{k}[t_1, \dots, t_d]$,
- $(I \cap \boldsymbol{k}[y_1, \dots, y_d]) \cap \boldsymbol{k}[t_1, \dots, t_d] = (t_\delta, \dots, t_d)$, for some δ.

Observe that

$$\boldsymbol{k}[t_1, \dots, t_d] \subseteq \boldsymbol{k}[y_1, \dots, y_d] = \frac{\boldsymbol{k}[y_1, \dots, y_n]}{J \cap \boldsymbol{k}[y_1, \dots, y_n]} \subseteq \frac{P}{J} = R$$

exhibits $t_1, \dots, t_d$ as elements of R. These elements are algebraically independent. The original ring R is finitely generated as a $\boldsymbol{k}[t_1, \dots, t_d]$-module. The intersection

$$I \cap \boldsymbol{k}[t_1, \dots, t_d] = (t_\delta, \dots, t_d).$$

The proof is complete. □

Corollary 5.5. *Let R be a domain which is finitely generated as an algebra over the field $\boldsymbol{k}$. The following statements hold.*

(a) *If $\mathfrak{m}$ is a maximal ideal of R, then $\operatorname{ht} \mathfrak{m} = \dim R$.*
(b) *Every saturated chain of prime ideals from (0) to a maximal ideal of R has length equal to $\dim R$.*
(c) *If $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals of R, then every saturated chain of prime ideals from $\mathfrak{p}$ to $\mathfrak{q}$ has length equal to $\dim \frac{R}{\mathfrak{p}} - \dim \frac{R}{\mathfrak{q}}$. (In particular, R is a catenary ring.)*

(d) *If I is an ideal of R, then*

$$\dim R = \dim \tfrac{R}{I} + \operatorname{ht} I.$$

Remark. We saw in Proposition 1.29 and Corollary 1.31 that (d) and (c) hold if R is a Cohen-Macaulay local ring.

5.5.1. The first step in the proof of Corollary 5.5. Let N be a Noether Normalization of R and

$$(0) = \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m \tag{5.5.2}$$

be a saturated chain of prime ideals in R with $\mathfrak{q}_m$ a maximal ideal of R. Then

$$(0) = N \cap \mathfrak{q}_0 \subsetneq \cdots \subsetneq N \cap \mathfrak{q}_m \tag{5.5.3}$$

is a saturated chain of prime ideals in N and $N \cap \mathfrak{q}_m$ is a maximal ideal of N.

Proof. We know from INC, Lemma 4.4, that

$$N \cap \mathfrak{q}_0 \subsetneq \cdots \subsetneq N \cap q_m.$$

It is clear that $N \cap \mathfrak{q}_0 = (0)$. It is not hard to see that $N \cap q_m$ is a maximal ideal of N. Indeed, $\frac{N}{N \cap q_m} \to \frac{R}{q_m}$ is an integral extension and if $D \subset L$ is an integral extension where L is a field, then D is also a field. Indeed, if d is a non-zero element of D, then $\frac{1}{d}$ is in L and therefore is integral over D. So,

$$(\tfrac{1}{d})^n + d_1(\tfrac{1}{d})^{n-1} + \cdots + d_n = 0 \tag{5.5.4}$$

for some d_i in D. It follows that

$$1 + \underbrace{d_1 d + \cdots + d_n d^n}_{\text{in } (d)D} = 0$$

and d is a unit in D.

It will take some effort to show that (5.5.3) is saturated. Suppose a prime $\mathfrak{p}$ can be inserted between $N \cap \mathfrak{q}_i$ and $N \cap \mathfrak{q}_{i+1}$; that is, suppose that

$$N \cap \mathfrak{q}_i \subsetneq \mathfrak{p} \subsetneq N \cap \mathfrak{q}_{i+1}$$

are prime ideals in N.

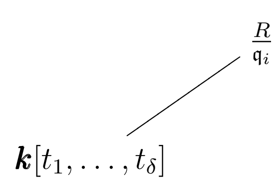
Find a Noether normalization $T = \boldsymbol{k}[t_1, \ldots, t_d]$ of N with

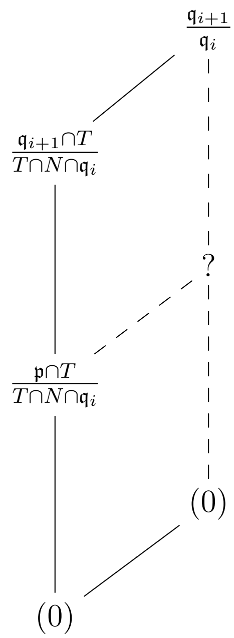
$$T \cap N \cap \mathfrak{q}_i = (t_{\delta+1}, \ldots, t_d)$$

for some δ. At this point,

$$\boldsymbol{k}[t_1, \ldots, t_\delta] = \frac{T}{(t_{\delta+1}, \ldots, t_d)} \subseteq \frac{R}{\mathfrak{q}_i}$$

is a Noether normalization and one has the following diagram of prime ideals in





The polynomial ring $\boldsymbol{k}[t_1, \dots, t_\delta]$ is a UFD. Every UFD is integrally closed in its quotient field. (This is easy to see. Use an argument like (5.5.4).) We may apply the Going Down Theorem, Theorem 4.9, to see that a prime ideal in R may be inserted between $\frac{\mathfrak{q}_i}{\mathfrak{q}_i}$ and $\frac{\mathfrak{q}_{i+1}}{\mathfrak{q}_i}$ in $R/\mathfrak{q}_i$. Thus, there is a prime ideal $\mathfrak{q}$ of R with $\mathfrak{q}_i \subsetneq \mathfrak{q} \subsetneq \mathfrak{q}_{i+1}$. Of course this is not possible because (5.5.2) is a saturated chain of prime ideals. This completes the proof of (5.5.3). ☐

5.5.5. The second step in the proof of Corollary 5.5. Let

$$(0) = \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m \tag{5.5.6}$$

be a saturated chain of prime ideals in R with $\mathfrak{q}_m$ a maximal ideal of R. We prove that $m = \dim R$.

Proof. Induct on m. If $m = 0$, then (0) is a maximal ideal of R, which is a field. (Of course, every saturated chain of prime ideals in a field has length equal to the Krull dimension of R.)

Henceforth $1 \le m$. Consider a Noether Normalization $N = \boldsymbol{k}[y_1, \dots, y_d]$ of R with $N \cap \mathfrak{q}_1 = (y_{\delta+1}, \dots, y_d)$ for some δ. The prime ideal $\mathfrak{q}_1$ of R height 1 because it properly contains the prime ideal (0) and the chain $(0) \subsetneq \mathfrak{q}_1$ is saturated. The chain $(0) \subsetneq \mathfrak{q}_1 \cap N$ of prime ideals in N is also saturated (by (5.5.3)); hence, $N \cap \mathfrak{q}_1 = (y_{\delta+1}, \dots, y_d)$ has height one. It follows that $N \cap \mathfrak{q}_1 = (y_d)$ and

$$\boldsymbol{k}[y_1, \dots, y_{d-1}] = \frac{N}{N \cap \mathfrak{q}_1} \to \frac{R}{\mathfrak{q}_1}$$

is a Noether Normalization. The saturated chain of prime ideals

$$\frac{\mathfrak{q}_1}{\mathfrak{q}_1} \subsetneq \cdots \subsetneq \frac{\mathfrak{q}_m}{\mathfrak{q}_1},$$

from (0) to a maximal ideal in $R/\mathfrak{q}_1$ contracts to give a saturated chain of prime ideals

$$\frac{\mathfrak{q}_1}{\mathfrak{q}_1} \cap \boldsymbol{k}[y_1, \dots, y_{d-1}] \subsetneq \cdots \subsetneq \frac{\mathfrak{q}_m}{\mathfrak{q}_1} \cap \boldsymbol{k}[y_1, \dots, y_{d-1}], \tag{5.5.7}$$

from (0) to a maximal ideal in $\boldsymbol{k}[y_1, \dots, y_{d-1}]$. The saturated chain of prime ideals (5.5.7) has length $m-1$; so by induction

$$m - 1 = \dim \boldsymbol{k}[y_1, \dots, y_{d-1}] = d - 1.$$

Thus, $m = d$ and the proof of 5.5.5 is complete. □

Remark. Be sure to notice WHY we went to such effort. It is clear that

$$\dim \boldsymbol{k}[y_1, \dots, y_{d-1}] = d - 1.$$

It is not clear (until we prove it) that $\dim R/\mathfrak{q}_1 = d - 1$. We proved the Going Down theorem (Theorem 4.9) and the Noether Normalization theorem (Theorem 5.1) to get that apparently small fact.

April 18, 2019

Corollary. 5.5 *Let R be a domain which is finitely generated as an algebra over the field $\boldsymbol{k}$. The following statements hold.*

(a) *If $\mathfrak{m}$ is a maximal ideal of R, then $\operatorname{ht} \mathfrak{m} = \dim R$.*
(b) *Every saturated chain of prime ideals from (0) to a maximal ideal of R has length equal to $\dim R$.*
(c) *If $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals of R, then every saturated chain of prime ideals from $\mathfrak{p}$ to $\mathfrak{q}$ has length equal to $\dim \frac{R}{\mathfrak{p}} - \dim \frac{R}{\mathfrak{q}}$. (In particular, R is a catenary ring.)*
(d) *If I is an ideal of R, then*
$$\dim R = \dim \tfrac{R}{I} + \operatorname{ht} I.$$

We proved Step 1: If N is a Noether Normalization of R, then every saturated chain of prime ideals in R from (0) to a maximal ideal contracts to become a saturated chain of prime ideals in N from (0) to a maximal ideal.

The proof used the Going Down Theorem.

We proved Step 2: Every saturated chain of prime ideals in R from (0) to a maximal ideal has length equal to $\dim R$.

The proof is by induction. Of course, we want to look at $R/\mathfrak{q}_1$ where $\mathfrak{q}_1$ is a prime ideal of R of height 1. At the beginning of the process, it is not obvious what the dimension of $R/\mathfrak{q}_1$ is. We used Noether Normalization and step 1 to learn that $\dim R/\mathfrak{q}_1$ is equal to $\dim N/(\mathfrak{q}_1 \cap N)$ where $N = \boldsymbol{k}[y_1, \dots, y_d]$ is a Noether Normalization of R (so $\dim R = \dim N$) and $\mathfrak{q}_1 \cap N = (y_d)$. Now it is obvious that $\dim N/(\mathfrak{q}_1 \cap N) = \dim N - 1$ and hence $\dim R/\mathfrak{q}_1 = \dim R - 1$.

• At this point we have established parts (a) and (b) of Corollary 5.5. That is, if R is a domain which is finitely generated as an algebra over a field, then every maximal ideal of R has height equal to $\dim R$ and every saturated chain of prime ideals in R, from (0) to a maximal ideal, has length equal to $\dim R$.

We prove (c). Let R be a domain which is finitely generated as an algebra over the field $\boldsymbol{k}$. If $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals of R, then every saturated chain of prime ideals from $\mathfrak{p}$ to $\mathfrak{q}$ has length equal to $\dim \frac{R}{\mathfrak{p}} - \dim \frac{R}{\mathfrak{q}}$.

Let

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_s = \mathfrak{q}$$

be a saturated chain of prime ideals in R from $\mathfrak{p}$ to $\mathfrak{q}$. Let

$$\mathfrak{q} = \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_t$$

be a saturated of prime ideals in R from $\mathfrak{q}$ to a maximal ideal of R. Observe that

$$\frac{\mathfrak{p}}{\mathfrak{p}} = \frac{\mathfrak{p}_0}{\mathfrak{p}} \subsetneq \frac{\mathfrak{p}_1}{\mathfrak{p}} \subsetneq \cdots \subsetneq \frac{\mathfrak{p}_s}{\mathfrak{p}} = \frac{\mathfrak{q}}{\mathfrak{p}} = \frac{\mathfrak{q}_0}{\mathfrak{p}} \subsetneq \frac{\mathfrak{q}_1}{\mathfrak{p}} \subsetneq \cdots \subsetneq \frac{\mathfrak{q}_t}{\mathfrak{p}}$$

is a saturated chain of prime ideals in $R/\mathfrak{p}$ from the zero ideal to a maximal ideal and

$$\frac{\mathfrak{q}}{\mathfrak{q}} = \frac{\mathfrak{q}_0}{\mathfrak{q}} \subsetneq \frac{\mathfrak{q}_1}{\mathfrak{q}} \subsetneq \cdots \subsetneq \frac{\mathfrak{q}_t}{\mathfrak{q}}$$

is a saturated chain of prime ideals in $R/\mathfrak{q}$ from the zero ideal to a maximal ideal. The domains $R/\mathfrak{p}$ and $R/\mathfrak{q}$ both are finitely generated as algebras over $\boldsymbol{k}$; thus (b) applies to both of these rings; so

$$\dim R/\mathfrak{p} = s + t = s + \dim R/\mathfrak{q}$$

and $s = \dim R/\mathfrak{p} - \dim R/\mathfrak{q}$, as claimed.

We prove (d). Let R be a domain which is finitely generated as an algebra over the field $\boldsymbol{k}$. If I is an ideal of R, then

$$\dim R = \dim \tfrac{R}{I} + \operatorname{ht} I.$$

The argument of Proposition 1.29 continues to show that it suffices to prove the result when I is a prime ideal because

$$\operatorname{ht} I = \min\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \text{ is minimal in } \operatorname{Supp} R/I\}, \text{ and}$$

$$\dim R/I = \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \text{ is minimal in } \operatorname{Supp} R/I\}.$$

Assume the assertion holds for prime ideals and $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are minimal in $\operatorname{Supp} R/I$ with $\operatorname{ht} I = \operatorname{ht} \mathfrak{p}_1$ and $\dim R/I = \dim R/\mathfrak{p}_2$, then

$$\operatorname{ht} \mathfrak{p}_1 \le \operatorname{ht} \mathfrak{p}_2 = \dim R - \dim R/\mathfrak{p}_2 \le \dim R - \dim R/\mathfrak{p}_1 = \operatorname{ht} \mathfrak{p}_1.$$

Thus equality holds everywhere and $\operatorname{ht} \mathfrak{p}_1 = \operatorname{ht} \mathfrak{p}_2$ and $\dim R/\mathfrak{p}_1 = \dim R/\mathfrak{p}_2$.

So, we prove the result when $I = \mathfrak{p}$ is a prime ideal. Apply (c) to the prime ideals $(0) \subseteq \mathfrak{p}$ in R Every saturated chain of prime ideals from (0) to $\mathfrak{p}$ has length equal to $\dim R/(0) - \dim R/\mathfrak{p}$. Thus,

$$\operatorname{ht} \mathfrak{p} = \dim R - \dim R/\mathfrak{p}.$$

We have finished the proof of Corollary 5.5.

Corollary 5.6. *If $P = \boldsymbol{k}[x_1, \dots, x_n]$ is a polynomial ring over the field $\boldsymbol{k}$ and $\mathfrak{p}$ is a prime ideal of P, then $P_\mathfrak{p}$ is a regular local ring.*

Proof. Apply Goal 2.7, which was established in 2.10.(d) on page 29. It suffices to prove that $P_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of P. Fix a maximal ideal $\mathfrak{m}$. We know from Corollary 5.5.(a) that $\operatorname{ht}\mathfrak{m} = \dim P$. Of course, $\dim P = n$ from Theorem 3.1. We know from Theorem 3.6.(b) that $\mathfrak{m}$ can be generated by n elements. The Krull Principal Ideal Theorem guarantees that neither $\mathfrak{m}$ nor $\mathfrak{m}P_\mathfrak{m}$ can be generated by fewer than n elements. Thus $\mathfrak{m}P_\mathfrak{m}$ can be generated by $\dim P_\mathfrak{m}$ elements and $P_\mathfrak{m}$ is a regular local ring. □

6. THE JACOBIAN CRITERION FOR REGULAR LOCAL RINGS.

This section closely follows [4, Chapt. 4, Thm. 1.15].

Goal 6.1. *Let $\boldsymbol{k}$ be a perfect field, $P=\boldsymbol{k}[x_1,\dots,x_n]$, $I=(f_1,\dots,f_m)$ be an ideal of P, $\mathfrak{p}$ be a prime ideal of P with $I\subseteq\mathfrak{p}$, and L be the field $\frac{P_\mathfrak{p}}{\mathfrak{p}P_\mathfrak{p}}$. Let Jac be the matrix*

$$\mathrm{Jac}=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \vdots & & \vdots\\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{bmatrix}.$$

Then

$$\operatorname{rank}_L \mathrm{Jac}|_L \le \dim P_\mathfrak{p}-\dim(P/I)_\mathfrak{p}$$

and

$$(P/I)_\mathfrak{p} \text{ is regular} \iff \operatorname{rank}_L \mathrm{Jac}|_L = \dim P_\mathfrak{p}-\dim(P/I)_\mathfrak{p}.$$

Remarks 6.2.

(a) The entries in Jac are polynomials in the polynomial ring $\boldsymbol{k}[x_1,\dots,x_n]$.

(b) We have used the symbol "$|_{\frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}}}$" to mean: take the image in $\frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}}$.

(c) Recall that the field $\boldsymbol{k}$ is perfect if every algebraic extension is separable. In particular, every field of characteristic zero and every finite field is perfect. Also, if the characteristic of $\boldsymbol{k}$ is the positive prime integer p, then $\boldsymbol{k}$ is perfect if and only if $\boldsymbol{k}$ is closed under the taking of p^{th} roots.

Proof. It suffices to prove

$$\operatorname{rank}_L \mathrm{Jac}|_L = \dim P_\mathfrak{p}-\operatorname{edim}(P/I)_\mathfrak{p}. \tag{6.2.1}$$

Indeed, once we prove (6.2.1), then

$$\begin{aligned}&\operatorname{rank}_L \mathrm{Jac}|_L\\ =\ &\dim P_\mathfrak{p}-\operatorname{edim}(P/I)_\mathfrak{p}\\ =\ &(\dim P_\mathfrak{p}-\dim(P/I)_\mathfrak{p})-\underbrace{(\operatorname{edim}(P/I)_\mathfrak{p}-\dim(P/I)_\mathfrak{p})}_{\text{I am non-negative}},\end{aligned}$$

which is at most $\dim P_\mathfrak{p}-\dim(P/I)_\mathfrak{p}$; furthermore,

$$\begin{aligned}\operatorname{rank}_L \mathrm{Jac}|_L=\dim P_\mathfrak{p}-\dim(P/I)_\mathfrak{p} &\iff \operatorname{edim}(P/I)_\mathfrak{p}-\dim(P/I)_\mathfrak{p}=0\\ &\iff (P/I)_\mathfrak{p} \text{ is a regular local ring.}\end{aligned}$$

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The situation: $\boldsymbol{k}$ is a perfect field, $P=\boldsymbol{k}[x_1,\dots,x_n]$ is a polynomial ring, $I=(f_1,\dots,f_m)$ is an ideal of P, $\mathfrak{p}$ is a prime ideal of P with $I\subseteq\mathfrak{p}$, $L=\operatorname{Qf}(P/\mathfrak{p})$, and $\operatorname{Jac}=(\partial f_i/\partial x_j)$.

The Goal:

$$(P/I)_{\mathfrak{p}} \text{ is a regular local ring } \iff \dim P_{\mathfrak{p}}-\dim(P/I)_{\mathfrak{p}}\le \operatorname{rank}\operatorname{Jac}|_L.$$

It suffices to show:

$$\operatorname{rank}_L \operatorname{Jac}|_L=\dim P_{\mathfrak{p}}-\operatorname{edim}(P/I)_{\mathfrak{p}}. \tag{6.2.1}$$

Consider the following exact sequence of L-modules:

$$0\to\frac{\mathfrak{p}^2P_{\mathfrak{p}}+IP_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}}\to\frac{\mathfrak{p}P_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}}\to\frac{\mathfrak{p}P_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}+IP_{\mathfrak{p}}}\to 0.$$

Observe that

$$\dim_L\frac{\mathfrak{p}P_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}}=\operatorname{edim}P_{\mathfrak{p}}.$$

Recall from Corollary 5.6 that $P_{\mathfrak{p}}$ is a regular local ring. It follows that

$$\dim_L\frac{\mathfrak{p}P_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}}=\dim P_{\mathfrak{p}}.$$

Observe that

$$\frac{\mathfrak{p}\left(\frac{P}{I}\right)_{\mathfrak{p}}}{\mathfrak{p}^2\left(\frac{P}{I}\right)_{\mathfrak{p}}}=\frac{\frac{\mathfrak{p}P_{\mathfrak{p}}}{IP_{\mathfrak{p}}}}{\frac{\mathfrak{p}^2P_{\mathfrak{p}}+IP_{\mathfrak{p}}}{IP_{\mathfrak{p}}}}=\frac{\mathfrak{p}P_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}+IP_{\mathfrak{p}}};$$

thus,

$$\dim_L\frac{\mathfrak{p}P_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}+IP_{\mathfrak{p}}}=\operatorname{edim}(P/I)_{\mathfrak{p}}.$$

Let $\Lambda=\frac{\mathfrak{p}^2P_{\mathfrak{p}}+IP_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}}$. We have shown that

$$\dim_L\Lambda=\dim P_{\mathfrak{p}}-\operatorname{edim}(P/I)_{\mathfrak{p}}.$$

Combine the most recent equation with (6.2.1). It suffices to prove that

$$\operatorname{rank}_L\operatorname{Jac}|_L=\dim_L\Lambda. \tag{6.2.2}$$

Let $G_1,\dots,G_r$ be elements of $\mathfrak{p}P_{\mathfrak{p}}$ with $\bar{G}_1,\dots,\bar{G}_r$ a basis for $\frac{\mathfrak{p}P_{\mathfrak{p}}}{\mathfrak{p}^2P_{\mathfrak{p}}}$. (We will make a special choice of $\{G_1,\dots,G_r\}$ later.) Observe that $\bar{f}_1,\dots,\bar{f}_m$ generates Λ. Identify $\sigma_{i,j}$ in $P_{\mathfrak{p}}$ with

$$f_i=\sum_{k=1}^{r}\sigma_{i,k}G_k\quad\text{in } P_{\mathfrak{p}}. \tag{6.2.3}$$

In other words,

$$\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = \begin{bmatrix} \sigma_{1,1} & \dots & \sigma_{1,r} \\ \vdots & & \vdots \\ \sigma_{m,1} & \dots & \sigma_{m,r} \end{bmatrix} \begin{bmatrix} G_1 \\ \vdots \\ G_r \end{bmatrix} \tag{6.2.4}$$

(The f's and the G's are in $\mathfrak{p}P_\mathfrak{p}$; the σ's are in $P_\mathfrak{p}$.) The equation (6.2.4) continues to hold when the f's and G's are viewed as elements of $\frac{\mathfrak{p}P_\mathfrak{p}}{\mathfrak{p}^2 P_\mathfrak{p}}$ and the σ's continue to be viewed as elements of $P_\mathfrak{p}$:

$$\begin{bmatrix} \bar{f}_1 \\ \vdots \\ \bar{f}_m \end{bmatrix} = \begin{bmatrix} \sigma_{1,1} & \dots & \sigma_{1,r} \\ \vdots & & \vdots \\ \sigma_{m,1} & \dots & \sigma_{m,r} \end{bmatrix} \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_r \end{bmatrix}.$$

At this point, it does no harm to view the σ's as elements of $P_\mathfrak{p}/\mathfrak{p}P_\mathfrak{p} = L$:

$$\begin{bmatrix} \bar{f}_1 \\ \vdots \\ \bar{f}_m \end{bmatrix} = \begin{bmatrix} \sigma_{1,1} & \dots & \sigma_{1,r} \\ \vdots & & \vdots \\ \sigma_{m,1} & \dots & \sigma_{m,r} \end{bmatrix}\Bigg|_L \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_r \end{bmatrix}.$$

The $\bar{G}$'s are a basis for $\mathfrak{p}P_\mathfrak{p}/\mathfrak{p}^2P_\mathfrak{p}$; the $\bar{f}$'s are a generating set for Λ. It follows that

$$\dim_L \Lambda = \operatorname{rank}[\sigma_{i,j}]|_L.$$

Combine the most recent equation with (6.2.2). It suffices to prove that G's can be chosen so that

$$\operatorname{rank}_L \operatorname{Jac}|_L = \operatorname{rank}[\sigma_{i,j}]|_L. \tag{6.2.5}$$

Take $\frac{\partial}{\partial x_j}$ of equation (6.2.3):

$$\frac{\partial f_i}{\partial x_j} = \sum_{k=1}^{r} \left(\sigma_{i,k} \frac{\partial G_k}{\partial x_j} + \frac{\partial \sigma_{i,k}}{\partial x_j} G_k \right).$$

When we look at this equation in $L = \frac{P_\mathfrak{p}}{\mathfrak{p}P_\mathfrak{p}}$, the G's are zero; hence,

$$\frac{\partial f_i}{\partial x_j}\Bigg|_L = \sum_{k=1}^{r} \sigma_{i,k}|_L \frac{\partial G_k}{\partial x_j}\Bigg|_L.$$

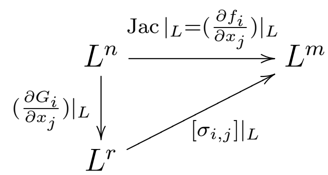
In other words,

$$\operatorname{Jac}|_L = [\sigma_{i,j}]|_L \left(\frac{\partial G_i}{\partial x_j} \right)\Bigg|_L.$$

In Lemma 6.3 we prove that there exist G's so that

$$\left(\frac{\partial G_i}{\partial x_j} \right)\Bigg|_L : L^n \to L^r \tag{6.2.6}$$

is surjective. Once we do this, then (6.2.5) happens, (that is $\operatorname{rank}_L \operatorname{Jac}|_L = \operatorname{rank}[\sigma_{i,j}]|_L$). Indeed, if (6.2.6) is surjective, then a quick look at



shows that the image of $\operatorname{Jac}|_L$ is then **equal to** the image of $[\sigma_{i,j}]|_L$. The proof will then be complete. □

Lemma 6.3. *Let $\boldsymbol{k}$ be a perfect field, P be the polynomial ring $\boldsymbol{k}[x_1, \dots, x_n]$, $\mathfrak{p}$ be a prime ideal of P, and L be the field $P_\mathfrak{p}/\mathfrak{p}P_\mathfrak{p}$. Then there exist $G_1, \dots G_r \in \mathfrak{p}P_\mathfrak{p}$ such that $\bar{G}_1, \dots \bar{G}_r$ is a basis for $\mathfrak{p}P_\mathfrak{p}/\mathfrak{p}^2P_\mathfrak{p}$ and*

$$\left.\begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \dots & \frac{\partial G_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial G_r}{\partial x_1} & \dots & \frac{\partial G_r}{\partial x_n} \end{bmatrix}\right|_L$$

has rank r.

Proof. Let ξ_i be the image of x_i in L. The field $\boldsymbol{k}$ is perfect and $L = \boldsymbol{k}(\xi_1, \dots, \xi_n)$. Apply Lemma 6.4 to conclude that some subset Γ of $\{\xi_1, \dots, \xi_n\}$ has the property that Γ is a transcendence basis for L over $\boldsymbol{k}$ and L is a separable algebraic extension of $\boldsymbol{k}(\Gamma)$. We renumber the variables, if necessary, in order to have $\Gamma = \{\xi_1, \dots, \xi_t\}$, where $t = \operatorname{trdeg}_{\boldsymbol{k}} L$. According to Goal 3.1 and Corollary 5.5.(d),

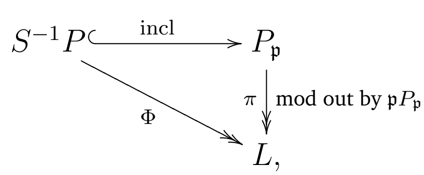
$$t = \dim P/\mathfrak{p} = \dim P - \operatorname{ht}\mathfrak{p}.$$

Let $S = \boldsymbol{k}[x_1, \dots, x_t] \backslash \{0\}$ and $K = S^{-1}\boldsymbol{k}[x_1, \dots, x_t]$. Of course, K is the field $\boldsymbol{k}(x_1, \dots, x_t)$; $S^{-1}P$ is the polynomial ring $S^{-1}P = K[x_{t+1}, \dots, x_n]$; and $K \subseteq L$ is a finite dimensional, separable, algebraic field extension.

Observe that the kernel of the composition

$$\boldsymbol{k}[x_1, \dots, x_t] \subseteq P \to L$$

is zero since $\xi_1, \dots, \xi_t$ are algebraically independent over $\boldsymbol{k}$; hence S is disjoint from $\mathfrak{p}$ and $P_\mathfrak{p}$ is obtained from $S^{-1}P = K[x_{t+1}, \dots, x_n]$ by further localization. (If it is helpful $P_\mathfrak{p} = T^{-1}P$, where $T = P \setminus \mathfrak{p}$ and $S \subseteq T$.) Consider the following picture:



where Φ is the K-algebra homomorphism with $\Phi(x_i) = \xi_i$, for $t+1 \le i \le n$. Notice that $(\ker\Phi)P_\mathfrak{p} = \ker\pi$. (The containment $\subseteq$ is obvious. We look at $\supseteq$. If $\theta \in \ker\pi$, then

$\tau\theta \in S^{-1}P$ for some $\tau \in P \setminus \mathfrak{p}$. Thus $\tau\theta$ is in $\ker \Phi$ and $\theta \in (\ker \Phi)P_{\mathfrak{p}}$.) We carefully worked out

$$\ker \Big(\Phi : K[x_{t+1}, \dots, x_n] \to K(\xi_{t+1}, \dots, \xi_n)\Big)$$

in Project 3.3. Indeed,

$$\ker \Phi = \Big(F_1(x_{t+1}), F_2(x_{t+1}, x_{t+2}), \dots, F_{n-t}(x_{t+1}, x_{t+2}, \dots, x_n)\Big),$$

where

$$F_i(\xi_{t+1}, \dots, \xi_{t+i-1}, x_{t+i})$$

is the minimal polynomial of ξ_{t+i} over $K(\xi_{t+1}, \dots, \xi_{t+i-1})$. Call this minimal polynomial $g_i(x_{t+i})$. Notice that $\frac{\partial F_i}{\partial x_{t+i}}|_L = g_i'(\xi_{t+i}) \neq 0$. (The extension $K \subseteq L$ is algebraic and separable. It follows that minimal polynomials do not have repeated roots.)

The ideal $\ker \Phi$ is generated by $(F_1, \dots, F_{n-t})$. Thus,

$$(F_1, \dots, F_{n-t})P_{\mathfrak{p}} = \ker \pi = \mathfrak{p}P_{\mathfrak{p}}.$$

The ring $P_{\mathfrak{p}}$ is regular local (by Corollary 5.6) of dimension equal to

$$\begin{aligned} \operatorname{ht} \mathfrak{p} &= \dim P - \dim P/\mathfrak{p}, && \text{by Corollary 5.5.(d)},\\ &= n - t, && \text{by Goal 3.1}. \end{aligned}$$

So, $F_1, \dots, F_{n-t}$ is a minimal generating set for $\mathfrak{p}P_{\mathfrak{p}}$. Now consider

$$\left(\frac{\partial F_i}{\partial x_j}\right)\Bigg|_L = \begin{bmatrix} * & \dots & * & \frac{\partial F_1}{\partial x_{t+1}}|_L & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ * & & & & & * & \frac{\partial F_{n-t}}{\partial x_n}|_L \end{bmatrix}.$$

This matrix clearly has rank $n - t$. Multiplication by this matrix is a **surjective** map from L^n to L^{n-t}. This completes the proof. □

Lemma 6.4. *If* $\boldsymbol{k}$ *is a perfect field and* $L = \boldsymbol{k}(\xi_1, \dots, \xi_n)$ *is a field extension* $\boldsymbol{k}$*, then some subset* Γ *of* $\{\xi_1, \dots, \xi_n\}$ *is a transcendence basis for* L *over* $\boldsymbol{k}$ *and the field extension* $\boldsymbol{k}(\Gamma) \subset L$ *is algebraic and separable.*

Proof. See Theorems 30 and 31 in [11]. These theorems are in section 13 of Chapter 2 of Volume 1. The chapter is called “Elements of Field Theory”. □

CONTENTS

REFERENCES

[1] M. AtiyahAtiyah and I. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, MA, 1969.
[2] C. Avery, C. Booms, T. Kostolansky, S. Loepp, and A. Semendinger, *Characterization of completions of noncatenary local domains and noncatenary local UFDs*, J. Algebra **524** (2019), 1–18.
[3] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc., **59** (1946), 54–106.
[4] E. Kunz, Introduction to commutative algebra and algebraic geometry, Birkhäuser Boston, Inc., Boston, MA, 1985.
[5] F. S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge University Press, 1916.
[6] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1989.
[7] J. S. Milne, Fields and Galois Theory `http://www.jmilne.org/math/CourseNotes/FT.pdf`
[8] J. Rotman, *An introduction to homological algebra*, Springer, New York, 2009.
[9] J.-P. Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proceedings of the international symposium on algebraic number theory, Tokyo and Nikko, 1955, pp. 175-189. Science Council of Japan, Tokyo, 1956.
[10] C. Weibel, *An introduction to homological algebra* Cambridge University Press, Cambridge, 1994.
[11] O. Zariski and P. Samuel, *Commutative algebra, Volume I.* The University Series in Higher Mathematics. D. Van Nostrand Company, Inc., Princeton, New Jersey, 1958.