INFINITE FREE RESOLUTIONS

LUCHEZAR L. AVRAMOV

Purdue University
West Lafayette, Indiana 47907, U.S.A.
avramov@math.purdue.edu

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Contents

Introduction

A free resolution of a module extends a presentation in terms of generators and relations, by choosing generators for the relations, then generators for the relations between relations, *ad infinitum*. Two questions arise:

How to write down a resolution?

How to read the data it contains?

These notes describe results and techniques for *finite* modules over a *commutative noetherian* ring R. An extra hypothesis that R is *local* (meaning that it has a unique maximal ideal) is often imposed. It allows for sharper formulations, due to the existence of unique up to isomorphism *minimal* free resolutions; in a global situation, the information can be retrieved from the local case by standard means (at least when all finite projective modules are free).

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A finite free resolution is studied 'from the end', using linear algebra over the ring(!) R. The information carried by matrices of differentials is interpreted in terms of the arithmetic of determinantal ideals. When R is a polynomial ring over a field each module has a finite free resolution; in this case, progress in computer algebra and programming has largely moved the construction of a resolution for any concrete module from algebra to hardware.

The monograph of Northcott [125] is devoted to finite free resolutions. Excellent accounts of the subject can be found in the books of Hochster [89], Roberts [136], Evans and Griffith [61], Bruns and Herzog [46], Eisenbud [58].

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Here we concentrate on modules that admit no finite free resolution.

There is no single approach to the construction of an infinite resolution and no single key to the information in it, so the exposition is built around several recurring themes. We describe them next, in the local case.

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To resolve a module of infinite projective dimension, one needs to solve an infinite string of iteratively defined systems of linear equations. The result of each step has consequences for eternity, so a measure of control on the propagation of solutions is highly desirable. This may be achieved by endowing the resolution with a *multiplicative structure* (a natural move for algebraists, accustomed to work with algebras and modules, rather than vector spaces). Such a structure can always be put in place, but its internal requirements may prevent the resolution from being minimal. Craft and design are needed to balance the diverging constraints of multiplicative structure and minimality; determination of cases when they are compatible has led to important developments.

Handling of resolutions with multiplicative structures is codified by D*ifferential* G*raded* homological algebra. Appearances notwithstanding, this theory precedes the familiar one: *Homological Algebra* [51] came after Cartan, Eilenberg, and MacLane [56], [50] developed fundamental ideas and constructions in DG categories to compute the homology of Eilenberg-MacLane spaces. An algebraist might choose to view the extra structure as an extension of the domain of rings and modules in a new, homological, dimension.

DG homological algebra is useful in *change of rings* problems. They arise in connection with a homomorphism of rings $Q \to R$ and an R–module M, when homological data on M, available over one of the rings, are needed over the other. A typical situation arises when R and M have finite free resolutions over Q, for instance when Q is a *regular ring*. It is then possible to find multiplicative resolutions of M and R over Q, that are 'not too big', and build from them resolutions of M over R. Although not minimal in general, such constructions are often useful, in part due to their functoriality and computability.

Multiplicative structures on a resolution are inherited by derived functors. It is a basic observation that the induced *higher structures in homology* do not depend of the choice of the resolution, and so are invariants of the R–module (or R–algebra) M. They suggest how to construct multiplicative structures of the resolution, or yield *obstructions* to its existence. Sometimes, they provide *criteria* for a ring- or module-theoretic property, e.g. for R to be a *complete intersection*. All known proofs that this property localizes depend on such characterizations—as all proofs that regularity localizes use the homological description of regular rings.

The behavior of resolutions at infinity gives rise to intriguing results and questions. New notions are needed to state some of them; for instance, *complexity* and *curvature* of a module are introduced to differentiate between the two known types of asymptotic growth. A striking aspect of infinite resolutions is the *asymptotic stability* that they display, both numerically (uniform patterns of Betti numbers) and structurally (high syzygies have similar module-theoretic properties). In many cases this phenomenon can be traced to a simple principle: the beginning of a resolution is strongly influenced by the defining relations of the module, that can be arbitrarily complicated; far from the source, the relations of the ring (thought of as a residue of some regular local ring) take over. In other words, the singularity of the ring dominates asymptotic behavior.

Part of the evidence comes from information gathered over specific classes of rings. At one end of the spectrum sit the complete intersections, characterized by asymptotically polynomial growth of all resolutions. The other end is occupied by the *Golod rings*, defined by an extremal growth condition on the resolution of the residue field; all resolutions over a Golod ring have asymptotically exponential growth; higher order homology operations, the *Massey products*, play a role in constructions. Results on complete intersections and Golod rings are presented in detail; generalizations are described or sketched.

A basic problem is whether some form of the polynomial/exponential dichotomy extends to all modules over local rings. No intermediate growth occurs for the residue field, a case of central importance. This result and its proof offer a glimpse at a mutually beneficial interaction between local algebra and rational homotopy theory, that has been going on for over a decade.

A major link in that connection is the *homotopy Lie algebra* of a local ring; it corresponds to the eponymous object attached to a CW complex, and has a representation in the cohomology of every R–module. Its structure affects the asymptotic patterns of resolutions, and is particularly simple when R is a complete intersection: each cohomology module is then a finite graded module over a polynomial ring, that can be investigated with all the usual tools. This brings up a strong connection with modular representations of finite groups.

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Proving a result over local rings, we imply that a corresponding statement holds for graded modules over graded rings. Results specific to the graded case are mostly excluded; that category has a life of its own: after all, Hilbert [88] introduced resolutions to study graded modules over polynomial rings.

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The notes assume a basic preparation in commutative ring theory, including a few homological routines. Modulo that, complete proofs are given for all but a couple of results used in the text. Most proofs are second or third generation, many of them are new. Constructions from DG algebra are developed from scratch. A bonus of using DG homological algebra is that spectral sequences may be eliminated from many arguments; we have kept a modest amount, for reasons of convenience and as a matter of principle.

* * *

The only earlier monographic exposition specifically devoted to infinite resolution is the influential book of Gulliksen and Levin [83], which concentrates on the residue field of a local ring. The overlap is restricted to Sections 6.1 and 6.3, with some differences in the approach. Sections 6.2, 7.2, and 8.2 contain material that has not been presented systematically before.

1. Complexes

This chapter lays the ground for the subsequent exposition. It fixes terminology and notation, and establishes some basic results.

All rings are assumed[1] commutative. No specific references are made to standard material in commutative algebra, for which the books of Matsumura [117], Bruns and Herzog [46], or Eisenbud [58], provide ample background.

For technical reasons, we choose to work throughout with algebras over a ubiquitous commutative ring $\Bbbk$, that will usually be unspecified and even unmentioned[2] (think of $\Bbbk = \mathbb{Z}$, or $\Bbbk = k$, a field).

1.1. **Basic constructions.** Let R be a ring.

A (*bounded below*) *complex* of R–modules is a sequence

$$\cdots \longrightarrow F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} F_{n-2} \longrightarrow \cdots$$

of R–linear maps with $\partial_{n-1}\partial_n = 0$ for $n \in \mathbb{Z}$ (and $F_n = 0$ for $n \ll 0$). The *underlying R–module* $\{F_n\}_{n\in\mathbb{Z}}$ is denoted $F^\natural$. Modules over R are identified with complexes concentrated in degree zero (that is, having $F_n = 0$ for $n \neq 0$); $|x|$ denotes the degree of an element x; thus, $|x| = n$ means $x \in F_n$.

Operations on complexes. Let E, F, G be complexes of R–modules.

A *degree d homomorphism* $\beta\colon F \longrightarrow G$ is simply a collection of R-linear maps $\{\beta_n\colon F_n \longrightarrow G_{n+d}\}_{n\in\mathbb{Z}}$. All degree d homomorphisms from F to G form an R–module, $\operatorname{Hom}_R(F,G)_d$. This is the degree d component of a complex of R–modules $\operatorname{Hom}_R(F,G)$, in which the boundary of β is classically defined by

$$\partial(\beta) = \partial^G\circ\beta - (-1)^{|\beta|}\beta\circ\partial^F\,.$$

The power of -1 is a manifestation of the sign that rules over homological algebra: Each transposition of homogeneous entities of degrees i, j is 'twisted' by a factor $(-1)^{ij}$, and permutations are signed accordingly.

The cycles in $\operatorname{Hom}_R(F,G)$ are those homomorphisms β that satisfy $\partial\circ\beta = (-1)^{|\beta|}\beta\circ\partial$; they are called *chain maps*. Chain maps $\beta,\beta'\colon F \longrightarrow G$ are *homotopic* if there exists a homomorphism $\sigma\colon F \longrightarrow G$ of degree $|\beta|+1$, called a *homotopy* from β to β', such that

$$\beta - \beta' = \partial\circ\sigma + (-1)^{|\beta|}\sigma\circ\partial\,;$$

equivalently, $\beta - \beta'$ is the boundary of σ in the complex $\operatorname{Hom}_R(F,G)$.

A chain map β induces a natural homomorphism $\operatorname{H}(\beta)\colon \operatorname{H}(F) \longrightarrow \operatorname{H}(G)$ of degree $|\beta|$; homotopic chain maps induce the same homomorphism. Chain maps of degree zero are the *morphisms* of the category of complexes. A *quasi-isomorphism* is a morphism that induces an isomorphism in homology; the symbol $\simeq$ next to an arrow identifies a quasi[3]-isomorphism.

The *tensor product* $E\otimes_R F$ has $(E\otimes_R F)_n = \sum_{i+j=n} E_i \otimes_R F_j$ and

$$\partial(e\otimes f) = \partial^E(e)\otimes f + (-1)^{|e|}e\otimes\partial^F(f)\,.$$

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[1]Following a grand tradition of making blanket statements, to be violated down the road.

[2]Thus, the unqualified word 'module' stands for '$\Bbbk$–module'; homomorphisms are $\Bbbk$-linear; writing 'ring' or 'homomorphism of rings', we really mean '$\Bbbk$–algebra' or 'homomorphism of $\Bbbk$–algebras'. The convention is only limited by your imagination, and my forgetfulness.

[3]The symbol $\cong$ is reserved for the real thing.

The *transposition map* $\tau(e\otimes f)=(-1)^{|e||f|}f\otimes e$ is an isomorphism of complexes $\tau\colon E\otimes_R F\to F\otimes_R E$.

A degree d homomorphism $\beta\colon F\to G$ induces degree d homomorphisms

$$\begin{aligned}
\operatorname{Hom}_R(E,\beta)\colon \operatorname{Hom}_R(E,F) &\to \operatorname{Hom}_R(E,G)\ ,\\
\operatorname{Hom}_R(E,\beta)(\alpha) &= \beta\circ\alpha\,;\\
\operatorname{Hom}_R(\beta,E)\colon \operatorname{Hom}_R(G,E) &\to \operatorname{Hom}_R(F,E)\ ,\\
\operatorname{Hom}_R(\beta,E)(\gamma) &= (-1)^{|\beta||\gamma|}\gamma\circ\beta\,;\\
\beta\otimes_R E\colon F\otimes_R E\to G\otimes_R E\ ,&\quad (\beta\otimes_R E)(f\otimes e)=\beta(f)\otimes e\,;\\
E\otimes_R \beta\colon E\otimes_R F\to E\otimes_R G\ ,&\quad (E\otimes_R \beta)(e\otimes f)=(-1)^{|\beta||e|}e\otimes\beta(f)\,,
\end{aligned}$$

with signs determined by the Second Commandment[4]. All maps are natural in both arguments. If β is a chain map, then so are the induced maps.

The *shift* ΣF of a complex F has $(\Sigma F)_n=F_{n-1}$ for each n. In order for the degree 1 bijection $\Sigma^F\colon F\to\Sigma F$, sending $f\in F_n$ to $f\in(\Sigma F)_{n+1}$, to be a chain map the differential on ΣF is defined by $\partial^{\Sigma F}(\Sigma^F(f))=-\Sigma^F(\partial^F(f))$.

The *mapping cone* of a morphism $\beta\colon F\to G$ is the complex $\mathrm{C}(\beta)$ with underlying module $G^\natural\oplus(\Sigma F)^\natural$, and differential $\begin{pmatrix}\partial^G & (\Sigma^G)^{-1}\Sigma(\beta)\\ 0 & \partial^{\Sigma F}\end{pmatrix}$. The connecting homomorphism defined by the exact *mapping cone sequence*

$$0\to G\to\mathrm{C}(\beta)\to\Sigma F\to 0$$

is equal to $\mathrm{H}(\beta)$. Thus, β is a quasi-isomorphism if and only if $\mathrm{H}(\mathrm{C}(\beta))=0$.

A *projective* (respectively, *free*) *resolution* of an R–module M is a quasi-isomorphism $\epsilon^F\colon F\to M$ from a complex F of projective (respectively, free) modules with $F_n=0$ for $n<0$. The *projective dimension* of M is defined by $\operatorname{pd}_R M=\inf\{p\,|\text{M a projective resolution with } F_n=0 \text{ for } n>p\}$.

Example 1.1.1. In the *Koszul complex* $K=K(\boldsymbol{g};R)$ on a sequence $\boldsymbol{g}=g_1,\dots,g_r$ of elements of R the module K_1 is free with basis $x_1,\dots,x_r$, the module K_n is equal to $\bigwedge_R^n K_1$ for all n, and the differential is defined by

$$\partial(x_{i_1}\wedge\cdots\wedge x_{i_n})=\sum_{j=1}^{n}(-1)^{j-1}g_{i_j}\,x_{i_1}\wedge\cdots\wedge x_{i_{j-1}}\wedge x_{i_{j+1}}\wedge\cdots\wedge x_{i_n}\,.$$

For each R–module M, set $K(\boldsymbol{g};M)=K(\boldsymbol{g};R)\otimes_R M$, and note that $\mathrm{H}_0(K(\boldsymbol{g};M))=M/(\boldsymbol{g})M$. A crucial property of Koszul complexes is their *depth*[5]-*sensitivity*: If M is a finite module over a noetherian ring R, then

$$\sup\{i\mid \mathrm{H}_i(K(\boldsymbol{g};M))\neq 0\}=r-\operatorname{depth}_R((\boldsymbol{g}),M)\,.$$

In particular, if $g_1,\dots,g_r$ is an R–regular sequence, then $K(\boldsymbol{g};R)$ is a free resolution of $R'=R/(\boldsymbol{g})$ over R, and $\mathrm{H}(K(\boldsymbol{g};M))\cong\operatorname{Tor}^R(R',M)$.

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[4]Obey the Sign! While orthodox compliance is a nuisance, transgression may have consequences ranging anywhere from mild embarrassment (confusion of an element and its opposite) to major disaster (creation of complexes with $\partial^2\neq 0$).

[5]Recall that the *depth* of an ideal $I\subseteq R$ on M, denoted $\operatorname{depth}_R(I,M)$, is the maximal length of an M-regular sequence contained in I.

Minimal complexes. A *local ring* $(R,\mathfrak{m},k)$ is a noetherian ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k=R/\mathfrak{m}$. A complex of R–modules F, such that $\partial_n(F_n)\subseteq \mathfrak{m}F_{n-1}$ for each n, is said to be *minimal*. Here is the reason:

Proposition 1.1.2. *If F is a bounded below complex of finite free modules over a local ring $(R,\mathfrak{m},k)$, then the following conditions are equivalent.*

(i) *F is minimal.*
(ii) *Each quasi-isomorphism $\alpha\colon F\to F$ is an isomorphism.*
(iii) *Each quasi-isomorphism $\beta\colon F\to F'$ to a bounded below minimal complex of free modules is an isomorphism.*
(iv) *Each quasi-isomorphism $\beta\colon F\to G$ to a bounded below complex of free modules is injective, and $G=\operatorname{Im}\beta\oplus E$ for a split-exact subcomplex E.*

Proof. (i) $\implies$ (iii). The mapping cone $\mathrm{C}(\beta)$ is a bounded below complex of free modules with $\mathrm{H}(\mathrm{C}(\beta))=0$. Such a complex is split-exact, hence so is $\mathrm{C}(\beta)\otimes_R k\cong \mathrm{C}(\beta\otimes_R k)$. Thus, $\mathrm{H}(\beta\otimes_R k)$ is an isomorphism. Since both $F\otimes_R k$ and $F'\otimes_R k$ have trivial differentials, each $\beta_n\otimes_R k$ is an isomorphism. As F_n and F'_n are free modules, β_n is itself an isomorphism by Nakayama.

(iii) $\implies$ (iv). Choose a subset $Y\subseteq G$ such that $\partial(Y)\otimes_R 1$ is a basis of the vector space $\partial(G\otimes_R k)$. As $Y\cup\partial(Y)$ is linearly independent modulo $\mathfrak{m}G$, it extends to a basis of the R–module $G^\natural$. Thus, the complex

$$E\colon\quad \bigoplus_{y\in Y}\big(0\to Ry\to R\partial(y)\to 0\big) \qquad (*)$$

is split-exact and a direct summand of G, hence $G\to G/E=G'$ is a quasi-isomorphism onto a bounded below free complex, that is minimal by construction. As the composition $\beta'\colon F\to G\to G'$ is a quasi-isomorphism, it is an isomorphism by our hypothesis. The assertions of (iv) follow.

(iv) $\implies$ (ii): the split monomorphisms $\alpha_n\colon F_n\to F_n$ are bijective.

(ii) $\implies$ (i). Assume that (i) fails, and form a surjective quasi-isomorphism $\beta\colon F\to F/F'=G$ onto a bounded below free complex as in the argument above. There is then a morphism $\gamma\colon G\to F$ with $\beta\gamma=\mathrm{id}^G$, cf. Proposition 1.3.1. Such a γ is necessarily a quasi-isomorphism, hence $\gamma\beta=\alpha$ is a quasi-isomorphism $F\to F$ with $\operatorname{Ker}\alpha\supseteq F'\neq 0$, a contradiction. □

1.2. **Syzygies.** In this section $(R,\mathfrak{m},k)$ is a local ring. Over such rings, projectives are free; for finite modules, this follows easily from Nakayama.

A *minimal resolution* F of an R–module M is a free resolution, that is also a minimal complex. If $m_1,\dots,m_r$ minimally generate M, then the Third Commandment[6] prescribes to start the construction of a resolution by a map $F_0=R^r\to M$ with $(a_1,\dots,a_r)\mapsto a_1m_1+\cdots+a_rm_r$. This is a surjection with kernel in $\mathfrak{m}R^r$; iterating the procedure, one sees that M has a minimal resolution. As any two resolutions of M are linked by a morphism that induces the identity of M, Proposition 1.1.2 completes the proof of the following result of Eilenberg from [55], where minimal resolutions are introduced.

Proposition 1.2.1. *Each finite R–module M has a minimal resolution, that is unique up to isomorphism of complexes. A minimal resolution F is isomorphic to*

[6]Resolve minimally!

a direct summand of any resolution of M, with complementary summand a split-exact free complex. In particular, $\operatorname{pd}_R M = \sup\{n \mid F_n \neq 0\}$. ☐

The 'uniqueness' of a minimal resolution F implies that each R-module $\operatorname{Syz}_n^R(M) = \operatorname{Coker}(\partial_{n+1}\colon F_{n+1} \to F_n) \cong \partial_n(F_n)$ is defined uniquely up to a (non-canonical) isomorphism; it is called the n'th *syzygy* of M; note that $\operatorname{Syz}_0^R(M) = M$, and $\operatorname{Syz}_n^R(M) = 0$ for $n < 0$.

The number $\beta_n^R(M) = \operatorname{rank}_R F_n$ is called the n'th *Betti number* of M (over R). The complexes $F \otimes_R k$ and $\operatorname{Hom}_R(F,k)$ have zero differentials, so $\operatorname{Tor}_n^R(M,k) \cong F_n \otimes_R k$ and $\operatorname{Ext}_R^n(M,k) \cong \operatorname{Hom}_R(F_n,k)$; in other words:

Proposition 1.2.2. *If M is a finite R–module, then*

$$\beta_n^R(M) = \nu_R\big(\operatorname{Syz}_n^R(M)\big) = \dim_k \operatorname{Tor}_n^R(M,k) = \dim_k \operatorname{Ext}_R^n(M,k)$$

and $\operatorname{pd}_R M = \sup\{n \mid \beta_n^R(M) \neq 0\}$. ☐

Syzygies behave well under certain base changes by *local homomorphisms*, that is, homomorphisms of local rings $\varphi(R,\mathfrak{m}) \to (R',\mathfrak{m}')$ with $\varphi(\mathfrak{m}) \subseteq \mathfrak{m}'$.

Proposition 1.2.3. *If M is a finite R–module and $\varphi\colon R \to R'$ is a local homomorphism such that $\operatorname{Tor}_n^R(R',M) = 0$ for $n > p$, then*

$$\operatorname{Syz}_{n-p}^{R'}\big(R' \otimes_R \operatorname{Syz}_p^R(M)\big) \cong R' \otimes_R \operatorname{Syz}_n^R(M) \qquad \textit{for} \quad n \geq p\,.$$

Proof. Let F be a minimal free resolution of M over R.

Since $\mathrm{H}_n(R' \otimes_R F) \cong \operatorname{Tor}_n^R(R',M) = 0$ for $n > p$, the complex of R'–modules $(R' \otimes_R F)_{\geqslant p}$ is a free (and obviously minimal) resolution of the module $\operatorname{Coker}(R' \otimes_R \partial_{p+1}) \cong R' \otimes_R \operatorname{Coker}\partial_{p+1} \cong R' \otimes_R \operatorname{Syz}_p^R(M)$. ☐

Corollary 1.2.4. *Let M be a finite R–module.*

(1) *If $R \to R'$ is a faithfully flat homomorphism of local rings, then*

$$\operatorname{Syz}_n^{R'}(R' \otimes_R M) \cong R' \otimes_R \operatorname{Syz}_n^R(M) \qquad \textit{for} \quad n \geq 0\,.$$

(2) *If a sequence $g_1,\dots,g_r \in R$ is both R–regular and M–regular, then*

$$\operatorname{Syz}_n^{R'}(M') \cong R' \otimes_R \operatorname{Syz}_n^R(M) \qquad \textit{for} \quad n \geq 0$$

with $R' = R/(g_1,\dots,g_r)R$, and $M' = M/(g_1,\dots,g_r)M$. ☐

Proof. The proposition applies with $p = 0$, twice: by definition in case (1), and by Example 1.1.1 in case (2). ☐

When R is local, $\operatorname{depth}_R M = \operatorname{depth}_R(\mathfrak{m},M)$ and $\operatorname{depth} R = \operatorname{depth}_R R$.

Corollary 1.2.5. *If R is a direct summand of $\operatorname{Syz}_n^R(M)$, then $n \leq m$, where $m = \max\{0, \operatorname{depth} R - \operatorname{depth} M\}$.*

Proof. Let $n > m$, and assume that $\operatorname{Syz}_n^R(M)$ has R as a direct summand. Choose a maximal $(R \oplus M)$–regular sequence, and complete it (if necessary) to a maximal R–regular sequence $\boldsymbol{g}$. By Example 1.1.1 and Proposition 1.2.3, $R' = R/(\boldsymbol{g})$ is a direct summand of a syzygy of the R'–module $R' \otimes_R \operatorname{Syz}_m^R(M)$, hence sits in $\mathfrak{m}F'$, where F' is a free R'–module. As $\operatorname{depth} R' = 0$, the ideal $(0 :_{R'} \mathfrak{m}) \neq 0$ annihilates R'; this is absurd. ☐

Depth can be computed cohomologically, by the formula

$$\operatorname{depth}_R M = \inf\{n \mid \operatorname{Ext}^n_R(k,M) \neq 0\}\,.$$

The following well known fact is recorded for ease of reference.

Lemma 1.2.6. *If M is a finite R–module, then*

$$\operatorname{depth}_R \operatorname{Syz}^R_1(M) = \begin{cases} \operatorname{depth}_R M + 1 & \text{when} \quad \operatorname{depth}_R M < \operatorname{depth} R\,; \\ g \geq \operatorname{depth} R & \text{when} \quad \operatorname{depth}_R M = \operatorname{depth} R\,; \\ \operatorname{depth} R & \text{when} \quad \operatorname{depth}_R M > \operatorname{depth} R\,. \end{cases}$$

Proof. Track the vanishing of $\operatorname{Ext}^n_R(k,-)$ through the long exact cohomology sequence induced by the exact sequence $0 \to \operatorname{Syz}^R_1(M) \to F_0 \to M \to 0$. □

Proposition 1.2.7. *If M is a finite R–module with $\operatorname{pd}_R M < \infty$, then*

(1) $\operatorname{pd}_R M + \operatorname{depth}_R M = \operatorname{depth} R$.
(2) $(0 :_R M) = 0$, *or* $(0 :_R M)$ *contains a non-zero-divisor on* R.
(3) $\sum_{n \geqslant 0} (-1)^n \beta^R_n(M) \geq 0$, *with equality if and only if* $(0 :_R M) \neq 0$.

Proof. Set $m = \operatorname{pd}_R M$, $g = \operatorname{depth}_R M$, and $d = \operatorname{depth} R$.

(1) As $\operatorname{Syz}^R_m(M) \neq 0$ is free, Corollary 1.2.5 yields $m + g \leq d$. Thus, $g \leq d$, and if $g = d$, then M is free and (1) holds. If $g < d$, then assume by descending induction that (1) holds for modules of depth $> g$, and use the lemma:

$$m + g = (\operatorname{pd}_R \operatorname{Syz}^R_1(M) + 1) + (\operatorname{depth}_R \operatorname{Syz}^R_1(M) - 1) = d\,.$$

(2) and (3). If F is a minimal resolution of M and $\mathfrak{p} \in \operatorname{Spec} R$, then

$$0 \to (F_m)_{\mathfrak{p}} \to (F_{m-1})_{\mathfrak{p}} \to \dots \to (F_1)_{\mathfrak{p}} \to (F_0)_{\mathfrak{p}} \to M_{\mathfrak{p}} \to 0$$

is exact. If $\mathfrak{p} \in \operatorname{Ass} R$, then $\operatorname{depth} R_{\mathfrak{p}} = 0$, hence $M_{\mathfrak{p}}$ is free by (1).

Counting ranks, we get $\sum_n (-1)^n \beta^R_n(M) = \operatorname{rank}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq 0$.

If $\sum_n (-1)^n \beta^R_n(M) = 0$, then $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Ass} R$, so $(0 :_R M) \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass} R} \mathfrak{p}$, that is, $(0 :_R M)$ contains a non-zero-divisor.

If $(0 :_R M) \neq 0$, then $(0 :_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = (0 :_R M)_{\mathfrak{p}} \neq 0$ for $\mathfrak{p} \in \operatorname{Ass}(0 :_R M) \subseteq \operatorname{Ass} R$; as $M_{\mathfrak{p}}$ is free, this implies $M_{\mathfrak{p}} = 0$, and so $\sum_n (-1)^n \beta^R_n(M) = 0$. □

The arguments for (2) and (3) are from Auslander and Buchsbaum's paper [17]; there is a new twist in the proof of their famous equality (1). It computes depths of syzygies when $\operatorname{pd}_R M < \infty$; otherwise, Okiyama [126] proves:

Proposition 1.2.8. *If M is a finite R–module with $\operatorname{pd}_R M = \infty$, then*

$$\operatorname{depth} \operatorname{Syz}^R_n(M) \geq \operatorname{depth} R \qquad \text{for} \quad n \geq \max\{0, \operatorname{depth} R - \operatorname{depth}_R M\}$$

with at most one strict inequality, at $n = 0$ or at $n = \operatorname{depth} R - \operatorname{depth}_R M + 1$.

Proof. Set $M_n = \operatorname{Syz}^R_n(M)$ and $d = \operatorname{depth} R$. Iterated use of Lemma 1.2.6 yields the desired inequality, and reduces the last assertion to proving that inequalities for n and $n+1$ imply equality for $n+2$.

Break down a minimal free resolution F of M into short exact sequences $E^i : 0 \to M_{i+1} \xrightarrow{\iota_{i+1}} F_i \xrightarrow{\pi_i} M_i \to 0$. If $\operatorname{depth}_R M_{n+2} > d$, then the cohomology exact sequence of E^{n+1} implies that the homomorphism

$$\operatorname{Ext}^d_R(k, \pi_{n+1}) : \operatorname{Ext}^d_R(k, F_{n+1}) \to \operatorname{Ext}^d_R(k, M_{n+1})$$

is injective. As $\operatorname{depth}_R M_n \geq d$, the cohomology sequence of E^n shows that

$$\operatorname{Ext}_R^d(k, \iota_{n+1})\colon \operatorname{Ext}_R^d(k, M_{n+1}RM) \to \operatorname{Ext}_R^d(k, F_n)$$

is injective. Since $\iota_n \circ \pi_{n+1} = \partial_{n+1}\colon F_{n+1} \to F_n$, we see that the map

$$\operatorname{Ext}_R^d(k, \partial_{n+1}) = \operatorname{Ext}_R^d(k, \iota_n) \circ \operatorname{Ext}_R^d(k, \pi_{n+1})$$

is injective as well. But $\partial_{n+1}\colon F_{n+1} \to F_n$ is a matrix with elements in $\mathfrak{m}$, so $\operatorname{Ext}_R^d(k, \partial_{n+1}) = 0$, hence $\operatorname{Ext}_R^d(k, F_{n+1}) = 0$. Since $\operatorname{depth}_R F_{n+1} = d$ this is impossible, so $\operatorname{depth} M_{n+2} \leq d$, as desired. □

Remark 1.2.9. By the last two results, there exists an R–regular sequence $\boldsymbol{g}$ of length $d = \operatorname{depth} R$, that is also regular on $N = \operatorname{Syz}_m^R(M)$, where $m = \max\{0, d - \operatorname{depth} M\}$. Proposition 1.2.4.2 yields $\beta_{n+m}^R(M) = \beta_n^R(N) = \beta_n^{R/(\boldsymbol{g})}(N/(\boldsymbol{g})N)$ for $n \geq 0$. When k is infinite, a sequence $\boldsymbol{g}$ may be found that also preserves multiplicity: $\operatorname{mult}(R) = \operatorname{mult}(R/(\boldsymbol{g}))$; if k is finite, then $R' = R[t]_{\mathfrak{m}[t]}$ has $\operatorname{mult}(R') = \operatorname{mult}(R)$, and $\beta_n^R(N) = \beta_n^{R'}(N \otimes_R R')$, for $n \geq 0$.

Remark 1.2.10. The name *graded ring* is reserved[1] for rings equipped with a direct sum decomposition $R = \bigoplus_{i \geqslant 0} R_i$, and having $R_0 = k$, a field. For such a ring we denote $\mathfrak{m}$ the *irrelevant maximal ideal* $\bigoplus_{i > 0} R_i$. An R–module$M$ is *graded* if $M = \bigoplus_{j \in \mathbb{Z}} M_j$ and $R_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. To minimize confusion with gradings arising from complexes, we say that $a \in M_i$ has *internal degrees* i, and write $\deg(a) = i$. The d'th *translate* of M is the graded R–module $M(d)$ with $M(d)_j = M_{j+d}$. A degree zero homomorphism $\alpha\colon M \to N$ of graded R–modules is an R–linear map such that $\alpha(M_j) \subseteq N_j$ for all j.

The free objects in the category of graded modules and degree zero homomorphisms are isomorphic to direct sums of modules of the form $R(d)$. Each graded R–module M has a *graded resolution* by free graded modules with differentials that are homomorphisms of degree zero. If $M_j = 0$ for $j \ll 0$ (in particular, if M is finitely generated), then such a resolution F exists with $\partial(F_n) \subseteq \mathfrak{m}F_n$ for all n. This *minimal graded resolution* is unique up to isomorphism of complexes of graded R–modules, so the numbers β_{nj} appearing in isomorphisms $F_n \cong \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{nj}}$ are uniquely defined, and finite if M is a finite R–module; these *graded Betti numbers* of M over R are denoted $\beta_{nj}^R(M)$.

1.3. **Differential graded algebra.** The term refers to a hybrid of homological algebra and ring theory. When describing the progeny[7], we systematically replace the compound 'differential graded' by the abbreviation DG.

DG algebras. A *DG algebra* A is a complex (A, ∂), with an element $1 \in A_0$ (the *unit*), and a morphism of complexes (the *product*)

$$A \otimes_{\Bbbk} A \to A\,, \qquad a \otimes b \mapsto ab\,,$$

that is unitary: $1a = a = a1$, and associative: $a(bc) = (ab)c$. In addition, we assume the A is (*graded*) *commutative*:

$$ab = (-1)^{|a||b|}ba \quad \text{for } a, b \in A \qquad \text{and} \qquad a^2 = 0 \quad \text{when } |a| \quad \text{is odd}\,,$$

and that $A_i = 0$ for $i < 0$; without them, we speak of *associative* DG algebras.

[7]It is not that exotic: a commutative ring is precisely a DG algebra concentrated in degree zero, and a DG module over it is simply a complex. A prime example of a 'genuine' DG algebra is a Koszul complex, with multiplication given by wedge product.

The fact that the product is a chain map is expressed by the *Leibniz rule*:

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b) \qquad \text{for} \quad a, b \in A\,,$$

Its importance comes from a simple observation: The cycles $\mathrm{Z}(A)$ are a graded subalgebra of A, the boundaries $\partial(A)$ are an ideal in $\mathrm{Z}(A)$, hence the canonical projection $\mathrm{Z}(A) \to \mathrm{H}(A)$ makes the homology $\mathrm{H}(A)$ into a graded algebra. In particular, each $\mathrm{H}_n(A)$ is a module over the ring $\mathrm{H}_0(A)$.

A *morphism of DG algebras* is a morphism of complexes $\phi\colon A \to A'$, such that $\phi(1) = 1$ and $\phi(ab) = \phi(a)\phi(b)$; we say that A' is a *DG algebra over* A.

If A and A' are DG algebras, then the tensor products of complexes $A \otimes_{\Bbbk} A'$ is a DG algebra with multiplication $(a \otimes a')(b \otimes b') = (-1)^{|a'||b|}(ab \otimes a'b')$.

A *graded algebra* is a DG algebra with zero differential, that is, a *family*[8] $\{A_n\}$, rather than a direct sum $\bigoplus_n A_n$.

A *DG module* U over the DG algebra A is a complex together with a morphism $A \otimes U \to U$, $a \otimes u \mapsto au$, that satisfies the Leibniz rule

$$\partial(au) = \partial(a)u + (-1)^{|a|}a\partial(u) \qquad \text{for} \quad a \in A \text{ and } u \in U$$

and is unitary and associative in the obvious sense. A *module* is a DG module with zero differential; $U^\natural$ is a module over $A^\natural$, and $\mathrm{H}(U)$ is a module over $\mathrm{H}(A)$.

Let U and V be DG modules over A.

A homomorphism $\beta\colon U \to V$ of the underlying complexes is *A–linear* if $\beta(au) = (-1)^{|\beta||a|}a\beta(u)$ for all $a \in A$ and $u \in U$. The A–linear homomorphisms form a subcomplex $\mathrm{Hom}_A(U, V) \subseteq \mathrm{Hom}_{\Bbbk}(U, V)$. The action

$$(a\beta)(u) = a(\beta(u)) = (-1)^{|a||\beta|}\beta(au)$$

turns it into a DG module over A. Two A–linear chain maps $\beta, \beta'\colon U \to V$ that are homotopic by an A–linear homotopy are said to be *homotopic over* A. Thus, $\mathrm{H}_d(\mathrm{Hom}_A(U, V))$ is the set of *homotopy classes* of A–linear, degree d chain maps. DG modules over A and their A–linear morphisms are, respectively, the objects and morphisms of the *category of DG modules* over A.

The residue complex $U \otimes_A V$ of $U \otimes_{\Bbbk} V$ by the subcomplexspanned by all elements $au \otimes_{\Bbbk} v - (-1)^{|a||u|}u \otimes_{\Bbbk} av$, has an action

$$a(u \otimes_A v) = au \otimes_A v = (-1)^{|a||u|}u \otimes_A av\,.$$

It is naturally a DG module over A, and has the usual universal properties.

The shift ΣU becomes a DG module over A by setting $a\Sigma^U(u) = (-1)^{|a|}\Sigma^U(au)$; the map $\Sigma^U\colon U \to \Sigma U$ is then an A–linear homomorphism.

The mapping cone of a morphism $\beta\colon U \to V$ of DG modules over A is a DG module over A, and the maps in the mapping cone sequence are A–linear.

Semi-free modules. A bounded below DG module F over A is *semi-free* if its underlying $A^\natural$–module $F^\natural$ has a basis[9] $\{e_\lambda\}_{\lambda\in\Lambda}$. Thus, for each $f \in F$ there are unique $a_\lambda \in A$ with $f = \sum_{\lambda\in\Lambda} a_\lambda e_\lambda$; we set $\Lambda_n = \{\lambda \in \Lambda : |e_\lambda| = n\}$.

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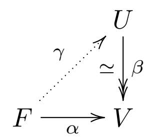
[8]This convention reduces the length of the exposition by 1.713%, as it trims from each argument all sentences starting with 'We may assume that the element x is homogeneous'; note that by Remark 1.2.10 above, a *graded ring* is the usual thing.

[9]Over a ring, a such a DG module is simply a bounded below complex of free modules. For arbitrary DG modules over any graded associative DG algebras, the notion is defined by a different condition: cf. [33], where the next three propositions are established in general.

Note that F is *not* a free object on $\{e_\lambda\}$ in the category of DG modules over A: As $\partial(e_\lambda)$ is a linear combination of basis elements e_μ of lower degree, the choice for the image of e_λ is restricted by the choices already made for the images of the e_μ. What freeness remains is in the important *lifting property*:

Proposition 1.3.1. *If F is a semi-free DG module over a DG algebra A, then each diagram of morphisms of DG modules over A represented by solid arrows*



with a surjective quasi-isomorphism β can be completed to a commutative diagram by a morphism γ, that is defined uniquely up to A–linear homotopy.

Remark. A degree d chain map $F \longrightarrow V$ is nothing but a morphism $F \longrightarrow \Sigma^{-d}V$, so the proposition provides also a 'unique lifting property' for chain maps.

Proof. Note that $F^n = \bigoplus_{|e_\lambda| \leqslant n} Ae_\lambda$ is a DG submodule of F over A, and $F^n = 0$ for $n \ll 0$. By induction on n, we may assume that $\gamma^n \colon F^n \longrightarrow U$ has been constructed, with $\alpha|_{F^n} = \beta\circ\gamma^n$.

For each $\lambda \in \Lambda_{n+1}$, we have $\partial(\alpha\partial(e_\lambda)) = \alpha(\partial^2(e_\lambda)) = 0$, so $\alpha\partial(e_\lambda)$ is a cycle in V. Since β is a surjective quasi-isomorphism, there exists a *cycle* $z'_\lambda \in U$, such that $\beta(z'_\lambda) = \alpha\partial(e_\lambda)$. Thus, $z_\lambda = \gamma^n\partial(e_\lambda) - z'_\lambda \in U$ satisfies

$$\partial(z_\lambda) = \gamma^n\partial^2(e_\lambda) - \partial(z'_\lambda) = 0 \qquad \text{and} \qquad \beta(z_\lambda) = \alpha\partial(e_\lambda) - \beta(z'_\lambda) = 0\,,$$

that is, z_λ is a cycle in $W = \operatorname{Ker}\beta$. The homology exact sequence of the short exact sequence of DG modules $0 \longrightarrow W \longrightarrow U \longrightarrow V \longrightarrow 0$ shows that $\mathrm{H}(W) = 0$, hence $z_\lambda = \partial(y_\lambda)$ for some $y_\lambda \in W$. In view of our choices, the formula

$$\gamma^{n+1}\Bigg(f + \sum_{\lambda\in\Lambda_{n+1}} a_\lambda e_\lambda\Bigg) = \gamma^n(f) + \sum_{\lambda\in\Lambda_{n+1}} a_\lambda y_\lambda \qquad \text{for} \quad f \in F^n$$

defines a morphism of DG modules $\gamma^{n+1} \colon F^{n+1} \longrightarrow U$, with $\gamma^{n+1}|_{F^n} = \gamma^n$, and completes the inductive construction. As $F = \bigcup_{n\in\mathbb{Z}} F^n$, setting $\gamma(f) = \gamma^n(f)$ whenever $f \in F^n$, we get a morphism $\gamma \colon F \longrightarrow U$ with $\alpha = \beta\gamma$.

If $\gamma' \colon F \longrightarrow U$ is a morphism with $\alpha = \beta\gamma'$, then $\beta(\gamma - \gamma') = 0$, hence there exists a morphism $\delta \colon F \longrightarrow W$ such that $\gamma - \gamma' = \iota\delta$, where $\iota \colon W \subseteq U$ is the inclusion. Again, we assume by induction that a homotopy $\sigma^n \colon F^n \longrightarrow W$ between $\delta|_{F^n}$ and 0 is available: $\delta|_{F^n} = \partial\sigma^n + \sigma^n\partial$. As

$$\begin{aligned}\partial(\delta(e_\lambda) - \sigma^n\partial(e_\lambda)) &= \delta\partial(e_\lambda) - (\partial\sigma^n)(\partial(e_\lambda)) \\ &= (\delta - \partial\sigma^n)(\partial(e_\lambda)) = (\sigma^n\partial)(\partial(e_\lambda)) = 0\end{aligned}$$

and $\mathrm{H}(W) = 0$, there is a $w_\lambda \in W$ such that $\partial(w_\lambda) = \delta(e_\lambda) - \sigma^n\partial(e_\lambda)$. Now

$$\sigma^{n+1}\Bigg(f + \sum_{\lambda\in\Lambda_{n+1}} a_\lambda e_\lambda\Bigg) = \sigma^n(f) + \sum_{\lambda\in\Lambda_{n+1}} (-1)^{|a_\lambda|} a_\lambda w_\lambda$$

is a degree 1 homomorphism $\sigma^{n+1} \colon F^{n+1} \longrightarrow W$, with $\delta|_{F^{n+1}} = \partial\sigma^{n+1} + \sigma^{n+1}\partial$, and $\sigma^{n+1}|_{F^n} = \sigma^n$. In the limit, we get a homotopy $\sigma \colon F \longrightarrow W$ from δ to 0, and then $\sigma' = \iota\sigma \colon F \longrightarrow U$ is a homotopy from γ to γ'. □

Proposition 1.3.2. *If F is a semi-free DG module, then each quasi-isomorphism $\beta\colon U \to V$ of DG modules over A induces quasi-isomorphisms*

$$\operatorname{Hom}_A(F,\beta):\ \operatorname{Hom}_A(F,U) \to \operatorname{Hom}_A(F,V)\ ; \qquad F\otimes_A \beta\colon F\otimes_A U \to F\otimes_A V\,.$$

Proof. To prove that $\operatorname{Hom}_A(F,\beta)$ is a quasi-isomorphism, we show the exactness of its mapping cone, which is isomorphic to $\operatorname{Hom}_A(F,\mathrm{C}(\beta))$. Thus, we want to show that each chain map $F \to \mathrm{C}(\beta)$ is homotopic to 0. Such a chain map is a lifting of $F \to 0$ over the quasi-isomorphism $\mathrm{C}(\beta) \to 0$. Since $0\colon F \to \mathrm{C}(\beta)$ is another such lifting, they are homotopic by Proposition 1.3.1.

To prove that $\beta\otimes_A F$ is a quasi-isomorphism, we use the exact sequences $0 \to F^n \to F^{n+1} \to \overline{F}^{n+1} \to 0$ of DG modules over A, involving the submodules F^n from the preceding proof. The sequences split over $A^\natural$, and so induce commutative diagrams with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U\otimes_A F^n & \longrightarrow & U\otimes_A F^{n+1} & \longrightarrow & U\otimes_A \overline{F}^{n+1} & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle \beta\otimes_A F^n} & & \big\downarrow{\scriptstyle \beta\otimes_A F^{n+1}} & & \big\downarrow{\scriptstyle \beta\otimes_A \overline{F}^{n+1}} & & \\
0 & \longrightarrow & V\otimes_A F^n & \longrightarrow & V\otimes_A F^{n+1} & \longrightarrow & V\otimes_A \overline{F}^{n+1} & \longrightarrow & 0\,.
\end{array}$$

By induction on n, we may assume that $\beta\otimes_A F^n$ is a quasi-isomorphism. As

$$\overline{F}^{n+1} \cong \bigoplus_{\lambda\in\Lambda_{n+1}} Ae_\lambda \qquad \text{with} \quad \partial(e_\lambda)=0 \text{ for all } \lambda\in\Lambda_{n+1}\,,$$

the map $\beta\otimes_A \overline{F}^{n+1}$ is a quasi-isomorphism. By the Five-Lemma, $\beta\otimes_A F^{n+1}$ is one as well, hence so is $\beta\otimes_A F = \beta\otimes(\operatorname{inj\,lim}_n F^n) = \operatorname{inj\,lim}_n(\beta\otimes_A F^n)$. □

Proposition 1.3.3. *Let U be a DG module over a DG algebra A. Each quasi-isomorphism $\gamma\colon F \to G$ of semi-free modules induces quasi-isomorphisms*

$$\operatorname{Hom}_A(\gamma,U):\ \operatorname{Hom}_A(G,U) \to \operatorname{Hom}_A(F,U)\ ; \qquad \gamma\otimes_A U\colon F\otimes_A U \to G\otimes_A U.$$

Proof. The mapping cone $C = \mathrm{C}(\gamma)$ is exact. It is semi-free, so by the preceding proposition the quasi-isomorphism $C \to 0$ induces a quasi-isomorphism $\operatorname{Hom}_A(C,C) \to 0$. Thus, there is a homotopy σ from id^C to 0^C. It is easily verified that $\operatorname{Hom}_A(\sigma,U)$ and $\sigma\otimes_A U$ are null-homotopies on $\operatorname{Hom}_A(C,U)$ and $C\otimes_A U$, so these complexes are exact. They are isomorphic, respectively, to $\Sigma^{-1}\mathrm{C}(\operatorname{Hom}_A(\gamma,U))$ and $\mathrm{C}(\gamma\otimes_A U)$, which are therefore exact. We conclude that $\operatorname{Hom}_A(\gamma,U)$ and $\gamma\otimes_A U$ are quasi-isomorphisms. □

The preceding results have interesting applications even for complexes over a ring. A first illustration occurs in the proof of Proposition 1.1.2. Another one is in the following proof of the classical *Künneth Theorem*.

Proposition 1.3.4. *If G is a bounded below complex of free R–modules, such that $F = \mathrm{H}(G)$ is free, then the Künneth map*

$$\kappa^{GU}:\ \mathrm{H}(G)\otimes_R \mathrm{H}(U) \to \mathrm{H}(G\otimes_R U)\,,$$
$$\kappa^{GU}(\mathrm{cls}(g)\otimes\mathrm{cls}(u)) = \mathrm{cls}(g\otimes u)\,,$$

is an isomorphism for each complex of R–modules U.

Proof. Set $F = \mathrm{H}(G)$. The composition of an $R-$ linear splitting of the surjection $\mathrm{Z}(G) \to F$ with the injection $\mathrm{Z}(G) \to G$ is a quasi-isomorphism $\gamma\colon F \to G$ of semi-free DG modules over R. By the last proposition, so is $\gamma \otimes_R U\colon F \otimes_R U \to G \otimes_R U$. The Künneth map being natural, it suffices to show that κ^{FU} is bijective. As $\partial^F = 0$ and each F_n is free, this is clear. □

2. Multiplicative structures on resolutions

Is it possible to 'enrich' resolutions over a commutative ring Q, by endowing them with DG module or DG algebra structures? The rather complete—if at first puzzling—answer comes in three parts:

- For residue rings of Q, algebra structures are carried by essentially all resolutions of length ≤ 3; for finite Q–modules, DG module structures exist on all resolutions of length ≤ 2.
- Beyond these bounds, not all resolutions support such structures.
- There always exist resolutions, that do carry the desired structure.

In this chapter we present in detail the results available on all three counts. Most developments in the rest of the notes are built on resolutions that comply with the Fourth Commandment[10].

2.1. **DG algebra resolutions.** Let Q be a commutative ring and let R be a Q–algebra. A *DG algebra resolution* of R over Q consists of a (commutative) DG algebra A, such that A_i is a projective Q–module for each i, and a quasi-isomorphism $\epsilon^A\colon A \to R$ of DG algebras over Q.

The next example is the grandfather of all DG algebra resolutions.

Example 2.1.0. If $R = Q/(\boldsymbol{f})$ for a Q–regular sequence $\boldsymbol{f}$, then the Koszul complex on $\boldsymbol{f}$ is a DG algebra resolution of R over Q.

'Short' projective resolutions often carry DG algebra structures.

Example 2.1.1. If $R = Q/I$ has a resolution A of length 1 of the form

$$0 \longrightarrow F_1 \longrightarrow Q \longrightarrow 0$$

then the only product that makes it a graded algebra over Q is defined by the condition $F_1 \cdot F_1 = 0$; it clearly makes A into a DG algebra.

If ϕ is a matrix, then for $J, K \subset \mathbb{N}$ we denote ϕ_J^K the submatrix obtained by deleting the rows with indices from J and the columns with indices from K.

Example 2.1.2. If $R = Q/I$ has a free resolution of length 2, then by the Hilbert-Burch Theorem there exist a non-zero-divisor a and an $r \times (r-1)$ matrix ϕ, such that a free resolution of R over Q is given by the complex

$$A\colon \quad 0 \longrightarrow \bigoplus_{k=1}^{r-1} Qf_k \xrightarrow{\partial_2} \bigoplus_{j=1}^{r} Qe_j \xrightarrow{\partial_1} Q \longrightarrow 0$$

$$\partial_2 = \phi \qquad \partial_1 = a\big(\det(\phi_1), \dots, (-1)^{j-1}\det(\phi_j), \dots, (-1)^{r-1}\det(\phi_r)\big)$$

Herzog [86] shows that there exists a unique DG algebra structure on A, namely:

$$e_j \cdot e_k = -e_k \cdot e_j = -a \sum_{\ell=1}^{r-1} (-1)^{j+k+\ell} \det(\phi_{jk}^{\ell}) f_\ell \qquad \text{for} \quad j < k\,; \qquad e_j \cdot e_j = 0\,.$$

Example 2.1.3. An ideal I in a local ring $(Q, \mathfrak{n})$ is *Gorenstein* if $R = Q/I$ has $\operatorname{pd}_Q R = p < \infty$, $\operatorname{Ext}_Q^n(R, Q) = 0$ for $n \ne p$, and $\operatorname{Ext}_Q^p(R, Q) \cong R$; thus, when Q is regular, I is Gorenstein if and only if R is a Gorenstein ring.

[10] Resolve in kind!

If I is Gorenstein, $\operatorname{pd}_Q R = 3$, and I is minimally generated by r elements, then J. Watanabe [157] proves that the number r is odd, and Buchsbaum-Eisenbud [47] show that there exists an alternating $r \times r$ matrix ϕ with elements in $\mathfrak{n}$, such that a minimal free resolution A of R over Q has the form

$$A\colon \quad 0 \longrightarrow Qg \xrightarrow{\partial_3} \bigoplus_{k=1}^{r} Qf_k \xrightarrow{\partial_2} \bigoplus_{j=1}^{r} Qe_j \xrightarrow{\partial_1} Q \longrightarrow 0$$

$$\partial_2 = \phi \qquad \partial_1 = \big(\operatorname{pf}(\phi_1^1), \dots, (-1)^{j-1}\operatorname{pf}(\phi_j^j), \dots, (-1)^{r-1}\operatorname{pf}(\phi_r^r)\big) = \partial_3^*$$

where $\operatorname{pf}(\alpha)$ is the Pfaffian of α. A DG algebra structure on A is given by

$$e_j \cdot e_k = -e_k \cdot e_j = \sum_{\ell=1}^{r} (-1)^{j+k+\ell} \rho_{jk\ell} \operatorname{pf}(\phi_{jk\ell}^{jk\ell}) f_\ell \qquad \text{for} \quad j < k\,;$$

$$e_j \cdot e_j = 0\,; \qquad e_j \cdot f_k = f_k \cdot e_j = \delta_{jk} g\,,$$

where $\rho_{jk\ell}$ is equal to -1 if $j < \ell < k$, and to 1 otherwise, cf. [21].

More generally, Buchsbaum and Eisenbud [47] prove that DG algebra structures always exist in projective dimension ≤ 3:

Proposition 2.1.4. *If A is a projective resolution of a Q–module R, such that $A_0 = Q$ and $A_n = 0$ for $n \geq 4$, then A has a structure of DG algebra.*

Proof. For the construction, consider the complex $\mathrm{S}^2(A)$, that starts as

$$\dots \longrightarrow (A_1 \otimes A_3) \oplus \mathrm{S}^2(A_2) \oplus A_4 \xrightarrow{\delta_4} (A_1 \otimes A_2) \oplus A_3$$
$$\xrightarrow{\delta_3} (\textstyle\bigwedge^2 A_1) \oplus A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\delta_1} Q \longrightarrow 0$$

with differentials defined by the condition $\delta_n|_{A_n} = \partial_n$ and the formulas

$$\delta_2(a \wedge b) = \partial_1(a)b - \partial_1(b)a\,; \qquad \delta_3(a \otimes b) = -a \wedge \partial_2(b) + \partial_1(a)b\,;$$
$$\delta_4(a \otimes b) = \partial_1(a)b - a * \partial_3(b)\,; \qquad \delta_4(a * b) = \partial_2(a) \otimes b + \partial_2(b) \otimes a\,,$$

where $*$ denotes the product in the symmetric algebra. The complex $\mathrm{S}^2(A)$ is projective and naturally augmented to R, so by the Lifting Theorem there is a morphism $\mu\colon \mathrm{S}^2(A) \to A$ that extends the identity map of R.

Define a product on A (temporarily denoted $\cdot$) by composing the canonical projection $A \otimes A \to \mathrm{S}^2(A)$ with μ. With the unit $1 \in Q = A_0$, one has all the properties required from a DG algebra except, possibly, associativity. Because $A_n = 0$ for $n \geq 4$, this may be an issue only for a product of three elements a, b, c, of degree 1. For them we have

$$\begin{aligned}\partial_3((a \cdot b) \cdot c) &= \partial_2(a \cdot b) \cdot c + (a \cdot b)\partial_1(c)\\ &= (\partial_1(a)b) \cdot c - (\partial_1(b)a) \cdot c + (a \cdot b)\partial_1(c)\\ &= \partial_1(a)(b \cdot c) - \partial_1(b)(a \cdot c) + \partial_1(c)(a \cdot b)\,.\end{aligned}$$

A similar computation of $\partial_3(a \cdot (b \cdot c))$ yields the same result.

As ∂_3 is injective, we conclude that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. □

Next we describe two existence results in projective dimension 4.

Example 2.1.5. If Q is local, $\operatorname{pd}_Q Q/I = 4$, and I is Gorenstein, then Kustin and Miller [101] prove that the minimal free resolution of R over Q has a DG algebra structure if $Q \ni \frac{1}{2}$, a restriction removed later by Kustin [97].

Example 2.1.6. If I is a grade 4 perfect ideal generated by 5 elements in a local ring Q containing $\frac{1}{2}$, then Kustin [98], building on work of Palmer [127], constructs a DG algebra structure on the minimal free resolution of Q/I.

The question naturally arises whether it is possible to put a DG algebra structure on each *minimal* resolution of a residue ring of a local ring. As far as the projective dimension is concerned, the list above turns out to be essentially complete: for (perfect) counterexamples in dimension 4, cf. Theorem 2.3.1.

On the positive side, each Q–algebra has *some* DG algebra resolution, obtained by a universal construction: Given a cycle z in a DG algebra A, we embed A into a DG algebra A' by freely adjoining a variable y such that $\partial y = z$. In A' the cycle z has been killed: it has become a boundary.

Construction 2.1.7. Exterior variable. When $|z|$ is even, $\Bbbk[y]$ is the *exterior algebra* over $\Bbbk$ of a free $\Bbbk$–module on a generator y of degree $|z|+1$; the differential of $A[y]^\natural = A^\natural \otimes_\Bbbk \Bbbk[y]$ is given by

$$\partial(a_0 + a_1 y) = \partial(a_0) + \partial(a_1) y + (-1)^{|a_1|} a_1 z\,;$$

thus, when A is concentrated in degree zero, $A[y]$ is the Koszul complex $K(z;A)$.

Construction 2.1.8. Polynomial variable. When $|z|$ is odd, $\Bbbk[y]$ is the *polynomial ring* over $\Bbbk$ on a variable y of degree $|z|+1$, $A[y]^\natural = A^\natural \otimes_\Bbbk \Bbbk[y]$, and

$$\partial\bigg(\sum_i a_i y^i\bigg) = \sum_i \partial(a_i) y^i + \sum_i (-1)^{|a_i|} i a_i z y^{i-1}\,.$$

In either case, ∂ is the unique differential on $A[y]^\natural$ that extends the differential on A, satisfies $\partial(y) = z$, and the Leibniz formula; we call y a *variable* over A, and often use the more complete notation, $A[y \,|\, \partial(y) = z]$.

Let $u = \operatorname{cls}(z) \in \mathrm{H}(A)$ be the class of z. The quotient complex $A[y]/A$ is trivial in degrees $n \le |z|$ and is equal to $A_0 y$ in degree $n = |z|+1$, so the homology exact sequence shows that the inclusion $A \hookrightarrow A[y]$ induces a morphism of graded algebras $\mathrm{H}(A)/u\,\mathrm{H}(A) \to \mathrm{H}(A[y])$ that is bijective in degrees $\le |z|$.

A *semi-free extension* of A is a DG algebra A' obtained by repeated adjunction of free variables. If Y is the set of all variables adjoined in the process, then we write $A[Y]$ for A'; we also set $Y_n = \{y \in Y \mid |y| = n\}$ and $Y_{\leqslant n} = \bigcup_{i=0}^n Y_i$. Semi-free algebra extensions have a *lifting property*:

Proposition 2.1.9. *If $A[Y]$ is a semi-free extension of a DG algebra A, then each diagram of morphisms of DG algebras over A represented by solid arrows*

$$\begin{array}{ccc} & & B \\ & {\scriptstyle\gamma}\nearrow & \simeq\big\downarrow{\scriptstyle\beta} \\ A[Y] & \xrightarrow[\alpha]{} & C \end{array}$$

with a surjective quasi-isomorphism β can be completed to a commutative diagram by a morphism γ, that is defined uniquely up to A–linear homotopy.

Proof. Set $A^i = A[Y_{\leqslant i}]$. Starting with the structure map $A \to B$, we assume that for some $n \ge -1$ we have a morphism $\gamma^n\colon A^n \to B$ of DG algebras over A, with $\beta\gamma^n = \alpha|_{A^n}$. Over A^n, the set $\{1\} \cup Y_{n+1}$ generates a semi-free submodule F^{n+1} of $A[Y]$. By Theorem 1.3.1, γ^n extends to a morphism $\delta^{n+1}\colon F^{n+1} \to B$ of DG modules over A^n. The graded commutative algebra $(A^{n+1})^\natural$ is freely generated over $(A^n)^\natural$ by Y_{n+1}, so δ^{n+1} extends uniquely to a homomorphisms of graded algebras $\gamma^{n+1}\colon A^{n+1} \to B$. Since δ^{n+1} is a morphism of DG modules, γ^{n+1} is necessarily a morphism of DG algebras. In the limit, one gets a morphism of DG algebras $\gamma\colon A[Y] \to B$, with the desired properties. □

A *resolvent* of Q–algebra R is a DG algebra resolution of R over Q, that is a semi-free DG algebra extension of Q.

Proposition 2.1.10. *Each (surjective) homomorphism $\psi\colon Q \to R$ has a resolvent $Q[Y]$ (with $Y_0 = \varnothing$). When Q is noetherian and R is a finitely generated Q–algebra there exists a resolvent with Y_n finite for each n.*

Proof. Factor ψ as an inclusion $Q \hookrightarrow Q[Y_0]$ into a polynomial ring on a set Y_0 of variables and a surjective morphism $\psi'\colon Q[Y_0] \twoheadrightarrow R$ that maps Y_0 to a set of generators of the Q–algebra R. The Koszul complex $Q \hookrightarrow Q[Y_0][[Y_{\leqslant 1}]$ on a set of generators of $\operatorname{Ker}\psi'$ is a semi-free extension of Q, with $\mathrm{H}_0(Q[Y_{\leqslant 1}]) \cong R$.

By induction on i, assume that consecutive adjunctions to $Q[Y_{\leqslant 1}]$ of sets Y_j of variables of degrees $j = 2, \dots, n$ have produced an extension $Q \hookrightarrow Q[Y_{\leqslant n}]$ with $\mathrm{H}_i(Q[Y_{\leqslant n}]) = 0$ for $0 < i < n$. Adjoin to $Q[Y_{\leqslant n}]$ a set Y_{n+1} of variables of degree $n+1$ that kill a set of generators of the $Q[Y_0]$–module $\mathrm{H}_n(Q[Y_{\leqslant n}])$. As observed above, we then get $\mathrm{H}_i(Q[Y_{\leqslant n+1}]) = 0$ for $0 < i < n+1$.

Going over the induction procedure with a noetherian hypothesis in hand, it is easy to see that a finite set Y_n suffices at each step. □

2.2. **DG module resolutions.** There is nothing esoteric about DG module structures on resolutions of modules. In fact, they offer a particularly adequate framework for important commutative algebra information.

Remark 2.2.1. Let U be a free resolution of a Q–module M. If $f \in (0 :_Q M)$, then both $f\,\mathrm{id}^U$ and 0^U induce the zero map on M, hence they are homotopic, say $f\,\mathrm{id}^U = \partial\sigma + \sigma\partial$. A homotopy σ such that $\sigma^2 = 0$ exists if and only if U can be made a DG module over the Koszul complex $A = Q[y \mid \partial(y) = f]$: just set $yu = \sigma(u)$, and note that the homotopy condition for σ translates precisely into the Leibniz rule $fu = \partial(yu) + y\partial(u)$ for the action of y.

In some cases one can prove the existence of a square-zero homotopy.

Proposition 2.2.2. *If $(Q, \mathfrak{n}, k)$ is a local ring, $f \in \mathfrak{n} \smallsetminus \mathfrak{n}^2$, and M is a Q–module such that $fM = 0$, then the minimal free resolution U of M over Q has a structure of semi-free DG module over the Koszul complex $A = Q[y \mid \partial(y) = f]$.*

Proof. Setting $f_j = f\,\mathrm{id}^{U_j}$, we restate the desired assertion as follows: For each j there is a homomorphism $\sigma_j\colon U_j \to U_{j+1}$, such that:

$$\partial_{j+1}\sigma_j + \sigma_{j-1}\partial_j = f_j\,; \qquad \sigma_{j-1}(U_{j-1}) = \operatorname{Ker}\sigma_j\,;$$
$$\sigma_{j-1}(U_{j-1}) \text{ is a direct summand of } U_j\,.$$

Indeed, by the preceding remark the first two conditions define on U a structure of DG module over A; the third one is then equivalent to an isomorphism of $A^\natural$–modules $U^\natural \cong A^\natural \otimes_Q V$, with $V = U^\natural / yU^\natural$.

The map $\sigma_j = 0$ has the desired properties when $j < 0$, so we assume by induction that σ_j has been constructed for $j \le i$, with $i \ge -1$. Since

$$\partial_{i+1}(f_{i+1} - \sigma_i\partial_{i+1}) = f\partial_{i+1} - \partial_{i+1}\sigma_i\partial_{i+1} = f\partial_{i+1} - f\partial_{i+1} + \sigma_{i-1}\partial_i\partial_{i+1} = 0$$

and U is acyclic, there exists a map σ_{i+1} such that $\partial_{i+2}\sigma_{i+1} = f_{i+1} - \sigma_i\partial_{i+1}$.

Furthermore, as $\sigma_i\sigma_{i-1} = 0$ by the induction hypothesis, we have

$$(f_{i+1} - \sigma_i\partial_{i+1})\sigma_i = f\sigma_i - \sigma_i\partial_{i+1}\sigma_i = f\sigma_i - f\sigma_i + \sigma_i\sigma_{i-1}\partial_i = 0\,.$$

Thus, we can arrange for σ_{i+1} to be zero on the direct summand $\operatorname{Im}\sigma_i$ of U_{i+1}.

Let V_{i+1} be a complementary direct summand of $\operatorname{Im}\sigma_i$ in U_{i+1}. For $v \in V_{i+1} \smallsetminus \mathfrak{n}V_{i+1}$, we have $\partial_{i+2}\sigma_{i+1}(v) = fv - \sigma_i\partial_{i+1}(v)$. The two terms on the right lie in distinct direct summands, and $fv \notin \mathfrak{n}^2V_{i+1}$, so $\partial_{i+2}\sigma_{i+1}(v) \notin \mathfrak{n}^2U_{i+1}$, and thus $\sigma_{i+1}(v) \notin \mathfrak{n}U_{i+2}$. This shows that $\sigma_{i+1} \otimes_Q k \colon V_{i+1} \otimes_Q k \to U_{i+2} \otimes_Q k$ is injective, so σ_{i+1} is a split injection, completing the induction step. □

The construction of σ above is taken from Shamash [143]; it is implicit in Nagata's [124] description of the syzygies of M over $Q/(f)$ in terms of those over Q, presented next; neither source uses DG module structures.

Theorem 2.2.3. *Let $(Q, \mathfrak{n}, k)$ be a local ring, let $f \in \mathfrak{n} \smallsetminus \mathfrak{n}^2$ be Q–regular, and let M be a finite module over $R = Q/(f)$. If U is a minimal free resolution of M over Q, then there exists a homotopy σ from id^U to 0^U, such that*

$$U'\colon \qquad \dots \to \frac{U_n}{fU_n + \sigma(U_{n-1})} \to \dots \to \frac{U_1}{fU_1 + \sigma(U_0)} \to \frac{U_0}{fU_0} \to 0 \qquad (*)$$

is a minimal R–free resolution of M and $\operatorname{rank}_Q U_n = \operatorname{rank}_R U'_n + \operatorname{rank}_R U'_{n-1}$.

Proof. Set $A = Q[y \,|\, \partial(y) = f]$. By the preceding proposition, U is a semi-free DG module over $A = Q[y \,|\, \partial(y) = f]$. Let σ be the homotopy given by left multiplication with y. By Proposition 1.3.2 the quasi-isomorphism $A \to R$ induces a quasi-isomorphism $U \to U \otimes_A R = U'$. The complex $U/(f,y)U = U/\big(fU + \sigma(U)\big) = U'$ is obviously minimal, so we are done. □

DG module structures are more affordable than DG algebra structures.

Remark 2.2.4. Let $S = Q/J$, and let B be a DG algebra resolution of S over Q. If A is the Koszul complex on a sequence $\boldsymbol{f} \subset J$, then by Proposition 2.1.9 the canonical map $A \to Q/(\boldsymbol{f}) \to S$ lifts over the surjective quasi-isomorphism $B \to S$ to a morphism $A \to B$ of DG algebras over Q.

Thus, any DG algebra resolution of S over Q is a DG module over each Koszul complex $K(\boldsymbol{f}; Q)$. However, DG module structures may exist even when DG algebra structures do not: for an explicit example, cf. Srinivasan [146].

Short projective resolutions always carry DG module structures: Iyengar [92] notes that a modification of the argument for Proposition 2.1.4 yields

Proposition 2.2.5. *Let $R = Q/I$, and let M be an R–module. If U is a projective resolution of M, and $U_n = 0$ for $n \geq 3$, then U has a structure of DG module over each DG algebra resolution A of R over Q.* □

On the other hand, not all minimal resolutions of length ≥ 3 support DG module structures over DG algebras $A \neq Q$, cf. Theorem 2.3.1.

Let A be a DG algebra resolution of R over Q, and let M be an R–module. A *DG module resolution* of M over A is a quasi-isomorphism $\epsilon^U : U \to M$ of DG modules over A, such that for each n the Q–module U_n is projective. To construct such resolutions in general, we describe a 'linear' adjunction process.

Construction 2.2.6. Adjunction of basis elements. Let V be a DG module over A, and $Z = \{z_\lambda \in V\}_{\lambda\in\Lambda}$ be a set of cycles. For a linearly independent set $Y = \{y_\lambda : |y_\lambda| = |z_\lambda| + 1\}_{\lambda\in\Lambda}$ over the graded algebra $A^\natural$ underlying A, set

$$\partial\Bigg(v + \sum_{\lambda\in\Lambda} a_\lambda y_\lambda\Bigg) = \partial(v) + \sum_{\lambda\in\Lambda} \partial(a_\lambda) y_\lambda + (-1)^{|a_\lambda|} \sum_{\lambda\in\Lambda} a_\lambda z_\lambda \ .$$

This is the unique differential on $V \bigoplus_{\lambda\in\Lambda} Ay_\lambda$ which extends that of V, satisfies the Leibniz rule, and has $\partial(y_\lambda) = z_\lambda$ for $\lambda \in \Lambda$.

Proposition 2.2.7. *If A is a DG algebra resolution of R over Q and M is an R–module, then M has a semi-free resolution U over A.*

Proof. Pick a surjective homomorphism $F \to M$ from a free Q–module, and extend it to a chain map of DG modules $\epsilon^0 \colon U^0 = A \otimes_Q F \to M$. Clearly, $\mathrm{H}_0(\epsilon^0)$ is surjective. If Z^0 is a set of cycles whose classes generate $\operatorname{Ker} \mathrm{H}_0(\epsilon^0)$, then let U^1 be the semi-free extension of U^0, obtained by adjunction of a linearly independent set Y^1 that kills Z^0. Extend ϵ^0 to $\epsilon^1 \colon U^1 \to M$ by $\epsilon^1(Y^1) = 0$, and note that $\mathrm{H}_0(\epsilon^1)$ is an isomorphism. Successively adjoining linearly independent sets Y^n, of elements of degree $n = 2, 3, \dots$ that kill sets Z^{n-1} of cycles generating $\mathrm{H}_{n-1}(U^{n-1})$, we get a semi-free DG module $U = \bigcup_n U^n$ over A, with a quasi-isomorphism $\epsilon^U \colon U \to M$. □

In an important case, the constructions are essentially finite.

Proposition 2.2.8. *Let Q be a noetherian ring.*

If R is a finite Q–algebra and M is a finite R–module, then there exist a DG algebra resolution A of R over Q and a DG module resolution U of M over A, such that the Q–modules $\operatorname{Coker}(\eta^A \colon Q \to A_0)$, *$A_n$, and U_n, are finite projective for all n and are trivial for $n >$* $\max\{\operatorname{pd}_Q R, \operatorname{pd}_Q M\}$.

Proof. If $r_1, \dots, r_s$ generate R as a Q–module, then each r_j is a root of a monic polynomial $f_j \in Q[x_j]$, hence R is a residue of $Q' = Q[x_1, \dots, x_s]/(f_1, \dots, f_s)$, which is a free Q–module. Use Proposition 2.1.10 to pick a resolvent $A' = Q'[Y]$ such that each A'_n is a finite free module over Q'. Then use Construction 2.2.7 to get a semi-free resolution U' of M over A' with each U'_n a finite free Q'–module; in particular, A'_n and U'_n are finite free Q–modules.

If $\max\{\operatorname{pd}_Q R, \operatorname{pd}_Q M\} = m < \infty$, then define a Q–submodule V of U' by setting $V_{<m} = 0$, $V_m = \partial(U'_{m+1})$, and $V_{>m} = U'_{>m}$. It is easy to check that $V = \{V_n\}$ is a DG A'–submodule with $\mathrm{H}(V) = 0$, hence $U = U'/V$ has $\mathrm{H}(U) = M$. The assumption on $\operatorname{pd}_Q M$ implies that the Q–module U_m is projective, so U is a DG module resolution of M over A'. Similarly, one sees that $J \subseteq A'$ defined by $J_{<m} = 0$, $J_m = \partial(A'_{m+1})$, and $J_{>m} = A'_{>m}$, is a DG ideal of A, such that $A = A'/J$ is a DG algebra resolution of R. Finally, the Leibniz formula shows that $JU' \subseteq V$, so U is a DG module over A.

The fact that $\operatorname{Coker} \eta$ is projective can be checked locally; Nakayama's Lemma then shows that $\operatorname{Im} \eta$ is a direct summand of the free Q–module A_0. □

2.3. **Products versus minimality.** Our goal is the following *non-existence*

Theorem 2.3.1. *Let k be a field, and Q be the polynomial ring $k[s_1, s_2, s_3, s_4]$ with the usual grading, or the power series ring $k[[s_1, s_2, s_3, s_4]]$. There exists no DG algebra structure on the minimal Q–free resolution U of the residue ring*

$$S = Q/I \qquad \textit{where} \quad I = (s_1^2\,,\, s_1s_2\,,\, s_2s_3\,,\, s_3s_4\,,\, s_4^2)$$

or on the minimal Q–free resolution U' of the Cohen-Macaulay residue ring

$$S' = Q/I' \qquad \textit{where} \quad I' = I + (s_1s_3^6\,,\, s_2^7\,,\, s_2^6s_4\,,\, s_3^7)\,.$$

If A is a DG algebra over Q, and U or U' is a DG module over A, then $A = Q$.

Remark. To prove the theorem, we check by a direct computation the non-vanishing of certain obstructions introduced by Avramov [22], and described in Theorem 3.2.6 below. Both the examples and the computations simplify those appearing in [22], and were developed in conversations with S. Iyengar.

As in [22], the examples can be used to generate, in any local ring Q with $\operatorname{depth} Q = g \geq 4$ (respectively, ≥ 6) perfect ideals of prescribed grades between 4 and g (respectively, Gorenstein ideals of prescribed grades between 6 and g), whose minimal free resolution admits no DG algebra structure.

Gorenstein ideals of grade 5 with this property had been missing, until the paper of Srinivasan [147]. The last open question, whether the minimal resolution of each non-cyclic module of projective dimension 3 (recall Proposition 2.1.4 and Proposition 2.2.5) carries a structure of DG module over some DG algebra $A \neq Q$, was answered by Iyengar [92] with perfect counter-examples.

Construction 2.3.2. Tor algebras. Let $S \leftarrow R \rightarrow k$ be homomorphisms of rings. If D is a DG algebra resolution D of S over R that is a resolution of S by free R–modules, cf. Proposition 2.1.10, then $\operatorname{Tor}^R(S,k) = \operatorname{H}(D \otimes_R k)$ inherits a structure of graded algebra. It can be computed also from a DG algebra resolution E of the second argument, or from resolvents of both arguments, due to the quasi-isomorphisms of DG algebras

$$D \otimes_R k \xleftarrow{D\otimes_R \epsilon^E} D \otimes_R E \xrightarrow{\epsilon^D \otimes_R E} S \otimes_R E\,.$$

Varying in these isomorphisms one varies one of D or E, while keeping the other fixed, one sees that the algebra structure on Tor does not depend on the choice of a DG algebra resolution. It can even be computed[11] from projective resolutions D' and E' with no multiplicative structure: the unique up to homotopy lifting of $\mu^S \colon S \otimes_R S \rightarrow S$ to a morphism $\mu^{D'} \colon D' \otimes_R D' \rightarrow D'$, that conspires with the Künneth map of Proposition 1.3.4 to produce

$$\begin{aligned}\operatorname{H}(D' \otimes_R k) \otimes_R \operatorname{H}(D' \otimes_R k) &\xrightarrow{\kappa} \operatorname{H}\left((D' \otimes_R k) \otimes_R (D' \otimes_R k)\right)\\ &\cong \operatorname{H}\left((D' \otimes_R D') \otimes_R (k \otimes_R k)\right) \xrightarrow{\operatorname{H}(\mu^{D'} \otimes_R \mu^k)} \operatorname{H}(D' \otimes_R k)\,.\end{aligned}$$

As the multiplication $\mu^D \colon D \otimes_R D \rightarrow D$ also is a comparison map, the unique isomorphism $\operatorname{H}(D' \otimes_R k) \cong \operatorname{H}(D \otimes_R k)$ transforms products into each other.

There is a related structure in the case of R–modules.

[11] In fact, this is how they were originally *introduced* by Cartan and Eilenberg [51].

Construction 2.3.3. Tor modules. Let $\psi\colon Q \to R$ and $R \to k$ be homomorphisms of rings, and let M be an R–module.

Choose a DG algebra resolution $\epsilon^A\colon A \to R$ over Q, by Proposition 2.1.10, and a semi-free resolution $\epsilon^U : U \to M$ over A, by Proposition 2.2.7. As both A and U are free over Q, we see that $\operatorname{Tor}^Q(M,k) = \mathrm{H}(U \otimes_Q k)$ is a module over the graded algebra $\operatorname{Tor}^Q(R,k) = \mathrm{H}(A \otimes_Q k)$ from the preceding construction. Recycling the discussion there, we verify that this structure is unique, and natural with respect to the module arguments.

The constructions also have a less well known naturality with respect to the *ring* argument; it is the one that we need.

Construction 2.3.4. Naturality. If $\psi\colon Q \to R$ is a ring homomorphism, then k becomes a Q–algebra, so pick a DG algebra resolution C of k over Q. By Proposition 2.1.9, there is a morphism of DG algebras $\gamma\colon C \to E$ over the identity map of k, that is unique up to Q–linear homotopy. The induced map $\mathrm{H}(M \otimes_\psi \gamma)\colon \mathrm{H}(M \otimes_Q C) \to \mathrm{H}(M \otimes_R E)$ is linear over $\mathrm{H}(R \otimes_Q C)$, and does not depend on the choice of γ. Thus, there is a natural homomorphism $\operatorname{Tor}^\psi(M,k)\colon \operatorname{Tor}^Q(M,k) \to \operatorname{Tor}^R(M,k)$ of $\operatorname{Tor}^Q(R,k)$–modules.

If $U' \to M \leftarrow V'$ and $C' \to k \leftarrow E'$ are arbitrary free resolutions, respectively over Q and over R, and $\beta'\colon U' \to V'$ and $\gamma'\colon C' \to E'$ are morphisms inducing id^M and id^k, then $\operatorname{Tor}^\psi(M,k) = \mathrm{H}(\beta' \otimes_\psi k) = \mathrm{H}(M \otimes_\psi \gamma')$.

Remark 2.3.5. Let $(Q,\mathfrak{n},k)$ be a local (or graded) ring, let $f \in \mathfrak{n}$ be a (homogeneous) regular element, let $\psi\colon Q \to Q/(f) = R$ be the natural projection, and let A be the Koszul complex $Q[y \mid \partial(y) = f]$.

For a DG module resolution U of M over A, set $\overline{U}\langle x\rangle = \bigoplus_{i\geqslant 0} Rx^{(i)} \otimes_Q \overline{U}$, with $|x^{(i)}| = 2i$ and $\partial(x^{(i)} \otimes u) = x^{(i-1)} \otimes yu + x^{(i)} \otimes \partial(u)$. In Example 3.1.2, we show that $\overline{U}\langle x\rangle$ is a resolution of M over R. By Theorem 3.2.6 and Remark 3.2.7, if U is minimal, then $\operatorname{Ker}(\operatorname{Tor}^\psi_n(M,k)) = \operatorname{cls}(y)\operatorname{Tor}^Q_{n-1}(M,k)$ for $n \geq 1$.

Proof of Theorem 2.3.1. First we look at the residue ring S. By Remark 2.2.4, it suffices to prove that its minimal free resolution U, over $Q = k[s_1,s_2,s_3,s_4]$ or over $Q = k[[s_1,s_2,s_3,s_4]]$, has no DG module structure over the Koszul complex $A = Q[y \mid \partial(y) = f]$, where $f = s_1^2 + s_4^2$. By the preceding remark, this will follow from $\operatorname{cls}(y)\operatorname{Tor}^Q_3(S,k) \nsubseteq \operatorname{Ker}(\operatorname{Tor}^\psi_4(S,k))$. The Tor's involved do not change under completion, so we restrict to the graded polynomial ring.

The Koszul resolvent $C = Q[y_1,y_2,y_3,y_4 \mid \partial(y_i) = s_i]$ of k over Q is a DG algebra over A, via the map $y \mapsto s_1y_1 + s_4y_4$; by Remark 2.3.5, the complex $\overline{C}\langle x\rangle$, is a resolution of k over $R = Q/(f)$. For $K = S \otimes_Q C$, we have

$$\operatorname{Tor}^Q(S,k) = \mathrm{H}(U \otimes_Q k) \cong \mathrm{H}(K) \qquad (*)$$

as modules over $\operatorname{Tor}^Q(R,k) = k[y \mid \partial(y) = 0]$, cf. Construction 2.3.3. The isomorphisms takes $\operatorname{Ker}\operatorname{Tor}^\psi(S,k)$ to $\operatorname{Ker}\mathrm{H}(\iota)$, where ι is the inclusion $K \subset L = S \otimes_R \overline{C}\langle x\rangle$. So it suffices to exhibit an element

$$w \in \operatorname{Ker}\mathrm{H}_4(\iota) \smallsetminus \operatorname{cls}(y)\,\mathrm{H}_3(K)\,. \qquad (\dagger)$$

It is easy to check that the cycles

$$z_1 = (s_1s_4)\, y_1 \wedge y_2 \wedge y_3\,, \qquad z_2 = (s_1s_4)\, y_1 \wedge y_2 \wedge y_4\,,$$
$$z_3 = (s_1s_4)\, y_1 \wedge y_3 \wedge y_4\,, \qquad z_4 = (s_2s_4)\, y_1 \wedge y_3 \wedge y_4\,,$$

are linearly independent modulo $S\partial(y_1 \wedge \cdots \wedge y_4)$. MACAULAY [40] shows that the minimal graded resolution U of the S over Q has the form

$$0 \to Q(-6) \xrightarrow{\partial_4} Q(-5)^4 \xrightarrow{\partial_3} Q(-4)^3 \oplus Q(-3)^4 \xrightarrow{\partial_2} Q(-2)^5 \xrightarrow{\partial_1} Q \to 0 \qquad (\ddagger)$$

with differentials given by the matrices

$$\partial_1 = \begin{pmatrix} s_1^2 & s_1s_2 & s_2s_3 & s_3s_4 & s_4^2 \end{pmatrix}$$

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & -s_2 & -s_4^2 & 0 & -s_3s_4 \\ 0 & 0 & -s_3 & s_1 & 0 & -s_4^2 & 0 \\ 0 & -s_4 & s_1 & 0 & 0 & 0 & 0 \\ -s_4 & s_2 & 0 & 0 & 0 & 0 & s_1^2 \\ s_3 & 0 & 0 & 0 & s_1^2 & s_1s_2 & 0 \end{pmatrix}$$

$$\partial_3 = \begin{pmatrix} 0 & 0 & s_1s_2 & s_1^2 \\ s_1^2 & 0 & s_1s_4 & 0 \\ s_1s_4 & 0 & s_4^2 & 0 \\ s_3s_4 & s_4^2 & 0 & 0 \\ 0 & -s_2 & 0 & -s_3 \\ 0 & s_1 & -s_3 & 0 \\ -s_2 & 0 & 0 & s_4 \end{pmatrix} \qquad \partial_4 = \begin{pmatrix} -s_4 \\ s_3 \\ s_1 \\ -s_2 \end{pmatrix}$$

From $(*)$ and $(\ddagger)$ we get $\operatorname{rank}_k \mathrm{H}_3(K) = 4$, so $\operatorname{cls}(z_1), \dots, \operatorname{cls}(z_4)$ is a basis of $\mathrm{H}_3(K)$. As $yz_i = (s_1y_1 + s_4y_4)z_i = 0$ for $i = 1, \dots, 4$, we get $\operatorname{cls}(y)\,\mathrm{H}_3(K) = 0$. Now

$$z = (s_1s_4)\, y_1 \wedge y_2 \wedge y_3 \wedge y_4 \in K_4$$

is a non-zero cycle, and so not a boundary in K, but becomes one in L:

$$z = \partial\big(s_4\, y_2 \wedge y_3 \wedge y_4\, y + s_2\, y_3\, x^{(2)}\big) \in L_4\,.$$

We have proved that $w = \operatorname{cls}(z)$ satisfies $(\dagger)$.

Turning to $S' = S/(s_1s_3^6, s_2^7, s_2^6s_4, s_3^7)$, consider the commutative square

$$\begin{array}{ccccccc} & & K & \xrightarrow{\iota} & L & & \\ & & {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} & & \\ K' & = & S' \otimes_S K & \xrightarrow{\iota'} & S' \otimes_S L & = & L' \end{array}$$

of morphisms of complexes, with $\iota' = S' \otimes_S \iota$. For $w' = \mathrm{H}(\pi)(w)$ it yields

$$\mathrm{H}(\iota')(w') = \mathrm{H}(\pi')\,\mathrm{H}(\iota)(w) = 0\,.$$

Assume $w' \in \operatorname{cls}(y)\,\mathrm{H}_3(K')$, set $\mathfrak{m} = (t_1, \dots, t_4)$, and consider the subcomplex

$$J\colon \quad 0 \to \mathfrak{m}^3K_4 \to \mathfrak{m}^4K_3 \to \mathfrak{m}^5K_2 \to \mathfrak{m}^6K_1 \to \mathfrak{m}^7K_0 \to 0$$

of K. Since $(*)$ and $(\ddagger)$ show that the non-zero homology of K is concentrated in internal degrees ≤ 6, cf. Remark 1.2.10, we conclude that $\mathrm{H}(J) = 0$, so the projection $\xi\colon K \to K/J$ is a quasi-isomorphism. On the other hand, it is clear that J is a DG ideal of K, such that $\operatorname{Ker}\pi = \mathfrak{m}^7K \subset J$. Thus, $\xi = \rho\pi$, where $\rho\colon K' \to K/J$ is the canonical map. By the surjectivity of $\mathrm{H}(\xi)$, we have

$$\mathrm{H}(\xi)(w) = \mathrm{H}(\rho)(w') \subseteq \operatorname{cls}(y)\,\mathrm{H}(\xi)\big(\mathrm{H}_3(K)\big) = \mathrm{H}(\xi)(\operatorname{cls}(y)\,\mathrm{H}_3(K))\,.$$

The injectivity of $\mathrm{H}(\xi)$ implies $w \in \operatorname{cls}(y)\,\mathrm{H}_3(K)$, violating $(\dagger)$.

Thus, we have found $w' \in \operatorname{Ker}\mathrm{H}_4(\iota') \smallsetminus \operatorname{cls}(y)\,\mathrm{H}_3(K')$. As above, this implies that U' carries no structure of DG module over A. $\square$

3. Change of rings

Fix a homomorphism of rings $\psi\colon Q \to R$, and an R–module M.

We consider various aspects of the problem: How can homological information on R and M over Q be used to study the module M over R?

3.1. **Universal resolutions.** A recent result of Iyengar [92] addresses this problem on the level of resolutions[12].

Theorem 3.1.1. *Let $\epsilon^A\colon A \to R$ be a DG algebra resolution over Q with structure map $\eta^A\colon Q \to A$, and let $\epsilon^U\colon U \to M$ be a DG module resolution of M over A. With $\overline{A} = (R \otimes_Q \operatorname{Coker}\eta^A)$ and $\overline{U} = (R \otimes_Q U)$, set*

$$F_n(A,U) = \bigoplus_{p+i_1+\cdots+i_p+j=n} \overline{A}_{i_1} \otimes_R \cdots \otimes_R \overline{A}_{i_p} \otimes_R \overline{U}_j\,;$$

$$\partial'(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes \overline{u}) = \sum_{r=1}^{p} (-1)^{r+i_1+\cdots+i_{r-1}}\, \overline{a}_1 \otimes \cdots \otimes \partial(\overline{a}_r) \otimes \cdots \otimes \overline{a}_p \otimes u$$
$$+ (-1)^{p+i_1+\cdots+i_p}\, \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes \overline{\partial(u)}\,;$$

$$\partial''(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes \overline{u}) = \sum_{r=1}^{p-1} (-1)^{r+i_1+\cdots+i_r}\, \overline{a}_1 \otimes \cdots \otimes \overline{a_r a}_{r+1} \otimes \cdots \otimes \overline{a}_p \otimes u$$
$$+ (-1)^{p+i_1+\cdots+i_{p-1}}\, \overline{a}_1 \otimes \cdots \otimes \overline{a}_{p-1} \otimes \overline{a_p u}\,.$$

The homomorphisms $\partial = \partial' + \partial''\colon F_n(A,U) \to F_{n-1}(A,U)$ make $\mathrm{F}(A,U)$ into a complex of R–modules. If the Q–modules $\operatorname{Coker}\eta^A_i$ and U_i are free for all i, then it is a free resolution of M over R.

When Q is a field, $A = A_0$, and U is an A–module, this is the well known *standard resolution*. The First Commandment[13] points the way to generalizations: this is the philosophy of the proof presented at the end of this section.

Example 3.1.2. Let $R = Q/(f)$ for a non-zero-divisor f. If U has a homotopy $\partial\sigma + \sigma\partial = f\,\mathrm{id}^U$ with $\sigma^2 = 0$, then by Remark 2.2.1 it is a DG module over $A = Q[y \,|\, \partial(y) = f]$. As $\overline{A}_1 = R\overline{y}$ and $\overline{A}_i = 0$ for $i \neq 1$, all but the last summands in the expressions for ∂' and ∂'' vanish, so $\mathrm{F}(A,U)$ takes the form

$$\dots \to \bigoplus_i Rx^{(i)} \otimes_Q U_{n-2i} \xrightarrow{\partial} \bigoplus_i Rx^{(i)} \otimes_Q U_{n-1-2i} \to \dots$$

where $x^{(i)} = \overline{y} \otimes \cdots \otimes \overline{y}$ (i copies), and $\partial\big(x^{(i)} \otimes u\big) = x^{(i-1)} \otimes \sigma(u) + x^{(i)} \otimes \partial(u)$.

In the setup of the example, a resolution of M over R can be constructed even if no square-zero homotopy is available: this is the contents of the next theorem, due to Shamash [143]; the proof we present is from [25].

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[12] In view of Proposition 2.2.8, if Q is noetherian and finite projective Q–modules are free, then a resolution of M over R is *finitistically* determined by matrix data over Q, namely, the multiplication tables of the algebra A and the module U, and the differentials in these finite complexes. With the help of computer algebra systems, such as MACAULAY [40], these data can be *effectively* gathered, at least when Q is a polynomial ring over a (small) field.

[13] Resolve!

Theorem 3.1.3. *If $R = Q/(f)$ for a non-zero-divisor f, M is an R–module, and U is a resolution of M by free Q–modules, then there exists a family of Q–linear homomorphisms $\boldsymbol{\sigma} = \big(\sigma^{[j]} \in \operatorname{Hom}_Q(U,U)_{2j-1}\big)_{j \geqslant 0}$, such that*

$$\sigma^{[0]} = \partial\,; \qquad \sigma^{[0]}\sigma^{[1]} + \sigma^{[1]}\sigma^{[0]} = f\,\mathrm{id}^U\,; \qquad \sum_{j=0}^{n} \sigma^{[j]}\sigma^{[n-j]} = 0 \quad \textit{for } n \geq 2\,.$$

If $\{x^{(i)} : |x^{(i)}| = 2i\}_{i \geqslant 0}$ is a linearly independent set over R, then

$$\dots \longrightarrow \bigoplus_{i=0}^{n} Rx^{(i)} \otimes_Q U_{n-2i} \xrightarrow{\partial} \bigoplus_{i=0}^{n-1} Rx^{(i)} \otimes_Q U_{n-1-2i} \longrightarrow \dots$$

$$\partial\big(x^{(i)} \otimes u\big) = \sum_{j=0}^{i} x^{(i-j)} \otimes \sigma^{[j]}(u)$$

is a free resolution $\mathrm{G}(\boldsymbol{\sigma}, U)$ of M over R.

Remark. Clearly, $\sigma = \sigma^{[1]}$ is a homotopy between $f\,\mathrm{id}^U$ and 0^U. If $\sigma^2 = 0$, then one can take $\sigma^{[n]} = 0$ for $n \geq 2$, and both proposition and example yield the same resolution. In general, rewriting the condition for $n = 2$ in the form $\partial\sigma^{[2]} + \sigma^{[2]}\partial = -\sigma^2$, we see that $\sigma^{[2]}$ is a *homotopy* which corrects the failure of σ^2 to be actually 0. A similar interpretation applies to all $\sigma^{[n]}$ with $n \geq 3$, so $\boldsymbol{\sigma}$ is a *family of higher homotopies* between $f\,\mathrm{id}^U$ and 0^U.

Proof. Note that $\sigma^{[0]}$ is determined, let $\sigma^{[1]}$ be any homotopy such that $f\,\mathrm{id}^U = \partial\sigma^{[1]} + \sigma^{[1]}\partial$, and assume by induction that maps $\sigma^{[j]}$ with the desired properties have been defined when $1 \leq j < n$ for some $n \geq 2$. Setting $\tau^{[1]} = f\,\mathrm{id}^U$ and $\tau^{[j]} = -\sum_{h=1}^{j-1} \sigma^{[h]}\sigma^{[j-h]}$ for $j \geq 2$, we have $\partial\sigma^{[j]} = \tau^{[j]} - \sigma^{[j]}\partial$, whence

$$\partial\sigma^{[j]}\sigma^{[n-j]} = \tau^{[j]}\sigma^{[n-j]} - \sigma^{[j]}\tau^{[n-j]} + \sigma^{[j]}\sigma^{[n-j]}\partial \quad \text{for } j = 1, \dots, n-1\,.$$

Summing up these equalities, we are left with $\partial\tau^{[n]} = \tau^{[n]}\partial$, so $\tau^{[n]}$ is a cycle of degree $2n-2$ in the complex $\operatorname{Hom}_Q(U,U)$. By Proposition 1.3.2, $U \xrightarrow{\simeq} M$ induces a quasi-isomorphism $\operatorname{Hom}_Q(U,U) \to \operatorname{Hom}_Q(U,M)$; the latter complex is zero in positive degrees, hence $\tau^{[n]}$ is a boundary. Thus, there is a homomorphism $\sigma^{[n]} \colon U \to U$ of degree $2n-1$, such that $\tau^{[n]} = \partial\sigma^{[n]} + \sigma^{[n]}\partial$. This finishes the inductive construction of the family $\boldsymbol{\sigma}$.

A direct computation shows that $\partial^2 = 0$. Set $G = \mathrm{G}(\boldsymbol{\sigma}, U)$, and note that there is an exact sequence $0 \to R \otimes_Q U \to G \to \Sigma^2 G \to 0$ of complexes of free R–modules. As $\mathrm{H}_i(R \otimes_Q U) \cong \operatorname{Tor}_i^Q(R,M) = 0$ for $i \neq 0,1$, it yields

$$M \cong \mathrm{H}_0(R \otimes_Q U) \cong \mathrm{H}_0(G)\,, \quad \mathrm{H}_{n+2}(G) = \mathrm{H}_n(\Sigma^2 G) \cong \mathrm{H}_n(G) \quad \text{for } n \geq 1\,,$$

and an exact sequence

$$0 \to \mathrm{H}_2(G) \to \mathrm{H}_2(\Sigma^2 G) \xrightarrow{\eth} \mathrm{H}_1(R \otimes_Q U) \to \mathrm{H}_1(G) \to 0\,.$$

Acyclicity of G will follow by induction on n, once we prove that $\eth$ is bijective.

If $z \in (\Sigma^2 G)_2 = Rx \otimes_Q U_0$ is a cycle, then $\eth(\mathrm{cls}(z))$ is the class of $\partial(x \otimes z) = 1 \otimes \sigma^{[1]}(z) \in R \otimes_Q U_1$. To show that $\mathrm{H}_0(R \otimes_Q \sigma^{[1]}) \colon \mathrm{H}_0(R \otimes_Q U) \to \mathrm{H}_1(R \otimes_Q U)$ is bijective, note that $\sigma^{[1]}$ is a homotopy between $f\,\mathrm{id}^U$ and 0^U, so we may replace U by any Q–free resolution V of M, and show that for some homotopy σ between $f\,\mathrm{id}^V$ and 0^V the map $\mathrm{H}_0(R \otimes \sigma)$ is bijective. Take V to be a semi-free resolution of M over $A = Q[y \,|\, \partial(y) = f]$, and σ to be left multiplication by y, so that $\mathrm{H}_0(R \otimes \sigma)$

is the action of $1 \otimes y \in \mathrm{H}_1(R \otimes_Q A) = \mathrm{Tor}_1^Q(R,R)$ on $\mathrm{H}_0(R \otimes_Q V) = \mathrm{Tor}_0^Q(R,M)$. By Construction 2.3.3, it can be computed from the resolution A of R over Q, as the multiplication of $\mathrm{H}(A \otimes_Q M) = R[y] \otimes_R M$ with $-(y \otimes 1) \in \mathrm{H}(A \otimes_Q R) = R[y]$; this is obviously bijective. □

We now turn to the proof of Theorem 3.1.1. It uses a nice tool popular with algebraic topologists, cf. [50], [115], but neglected by commutative algebraists.

Construction 3.1.4. Bar construction. Consider a DG module U over a DG algebra A (as always, defined over $\Bbbk$), and set $\widetilde{A} = \mathrm{Coker}(\eta^A \colon \Bbbk \to A)$. Let

$$S_p^{\Bbbk}(A,U) = A \otimes_{\Bbbk} \underbrace{\widetilde{A} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \widetilde{A}}_{p \text{ times}} \otimes_{\Bbbk} U\,, \qquad \text{for} \quad p \geq 0\,,$$

be the DG module, with action of A on the leftmost factor and tensor product differential ∂^p, and set $S_p^{\Bbbk}(A,U) = 0$ for $p < 0$. The expression on the right in

$$\begin{aligned}\delta^p(a \otimes \widetilde{a}_1 \otimes \cdots \otimes \widetilde{a}_p \otimes u) =& (aa_1) \otimes \widetilde{a}_2 \otimes \cdots \otimes \widetilde{a}_p \otimes u \\ &+ \sum_{i=1}^{p-1} (-1)^i a \otimes \widetilde{a}_1 \otimes \cdots \otimes (\widetilde{a_i a_{i+1}}) \otimes \cdots \otimes \widetilde{a}_p \otimes u \\ &(-1)^p a \otimes \widetilde{a}_1 \otimes \cdots \otimes \widetilde{a}_{p-1} \otimes (a_p u)\end{aligned}$$

is easily seen to be well-defined. Another easy verification yields

$$\delta^{p+1}\delta^p = 0 \qquad \text{and} \qquad \partial^{p-1}\delta^p = \delta^p\partial^p \qquad \text{for all} \quad p\,.$$

Thus, $(\mathrm{S}^{\Bbbk}(A,U),\delta)$ is a complex of DG modules[14], called the *standard complex*. It comes equipped with $\Bbbk$-linear maps

$$\begin{gathered}\pi' \colon S_0^{\Bbbk}(A,U) \to U\,, \quad a \otimes u \mapsto au\,; \\ \iota' \colon U \to S_0^{\Bbbk}(A,U)\,, \quad u \mapsto 1 \otimes u\,; \\ \sigma^p \colon S_p^{\Bbbk}(A,U) \to S_{p+1}^{\Bbbk}(A,U)\,, \\ a \otimes \widetilde{a}_1 \otimes \cdots \otimes \widetilde{a}_p \otimes u \mapsto 1 \otimes \widetilde{a} \otimes \widetilde{a}_1 \otimes \cdots \otimes \widetilde{a}_p \otimes u\,,\end{gathered}$$

that (are seen by another direct computation to) satisfy the relations

$$\begin{gathered}\delta^1\sigma^0 = \mathrm{id}^{S_0^{\Bbbk}(A,U)} - \iota'\pi'\,; \\ \delta^{p+1}\sigma^p + \sigma^{p-1}\delta^p = \mathrm{id}^{S_p^{\Bbbk}(A,U)} \qquad \text{for} \quad p \geq 1\,.\end{gathered}$$

In particular, when A and U have trivial differentials, $\mathrm{H}(\mathrm{S}^{\Bbbk}(A,U)) \cong U$. If, furthermore, the $\Bbbk$–modules $\widetilde{A}_i$ and U_i are free for all i, then this *free* resolution is known as the *standard resolution* of U over A.

Returning to the DG context, we reorganize $\mathrm{S}^k(A,U)$ into a DG module over A, by the process familiar 'totaling' procedure. The resulting DG module, with the action of A defined in Section 1.3, is the (*normalized*) *bar construction*:

$$\mathrm{B}^{\Bbbk}(A,U) = \bigoplus_{p=0}^{\infty} \Sigma^p\big(S_p^{\Bbbk}(A,U)\big) \qquad \text{with} \quad \partial = \partial' + \partial''$$

[14]Of course, a complex in the category of DG modules is a sequence of morphisms of DG modules $\delta^p \colon C^p \longrightarrow C^{p-1}$, such that $\delta^{p-1}\delta^p = 0$.

where ∂' denotes the differential of the DG module $\Sigma^p S_p^{\Bbbk}(A,U)$, and ∂' is the degree -1 map induced by the boundary δ of the complex of DG modules $\mathrm{S}^k(A,U)$; the equality $\partial^2 =$ results from the relations

$$\partial'\partial' = 0\,, \qquad \partial''\partial'' = 0\,, \qquad \partial'\partial'' + \partial''\partial' = 0\,,$$

of which the first two are clear, and the last one is due to the difference by a factor $(-1)^p$ of the differentials of $S_p^{\Bbbk}(A,U)$ and $\Sigma^p S_p^{\Bbbk}(A,U)$. Furthermore, the maps π', ι', σ^p, total to maps

$$\pi\colon\ \mathrm{B}^{\Bbbk}(A,U) \to U\,, \qquad \iota\colon U \to \mathrm{B}^{\Bbbk}(A,U)\,, \qquad \sigma\colon\ \mathrm{B}^{\Bbbk}(A,U) \to \mathrm{B}^{\Bbbk}(A,U)\,.$$

Clearly, π is a morphism of DG modules over A, ι is a morphism of complexes over $\Bbbk$, σ is a degree 1 morphism of complexes over $\Bbbk$, and they are related by

$$\pi\iota = 1_U \qquad \text{and} \qquad \partial\sigma + \sigma\partial = \mathrm{id}^{\mathrm{B}^{\Bbbk}(A,U)} - \iota\pi\,.$$

Thus, $\mathrm{H}(\pi)$ and $\mathrm{H}(\iota)$ are inverse isomorphisms, so π is a quasi-isomorphism.

The canonical isomorphism of DG modules over the DG algebra A,

$$\Sigma^p\big(S_p^{\Bbbk}(A,U)\big) \cong A \otimes_k \underbrace{\Sigma(\widetilde{A}) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \Sigma(\widetilde{A})}_{p \text{ times}} \otimes_{\Bbbk} U\,,$$

expresses the degree n component of the bar construction as

$$B_n(A,U) = \bigoplus_{h+p+i_1+\cdots+i_p+j=n} A_h \otimes_{\Bbbk} \widetilde{A}_{i_1} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \widetilde{A}_{i_p} \otimes_{\Bbbk} U_j\,.$$

The signs arising from the application of the shift, cf. Section 1.3, then yield the following expressions for the two parts of the differential:

$$\begin{aligned}
\partial'(a \otimes \widetilde{a}_1 \otimes \ldots \otimes \widetilde{a}_p \otimes u) &= \partial(a) \otimes \widetilde{a}_1 \otimes \cdots \otimes \widetilde{a}_p \otimes u \\
&\quad + \sum_{r=1}^{p} (-1)^{r+h+i_1+\cdots+i_{r-1}} a \otimes \widetilde{a}_1 \otimes \cdots \otimes \partial(\widetilde{a}_r) \otimes \cdots \otimes \widetilde{a}_p \otimes u \\
&\quad + (-1)^{p+h+i_1+\cdots+i_p} a \otimes \widetilde{a}_1 \otimes \cdots \otimes \widetilde{a}_p \otimes \partial(u) \\
\partial''(a \otimes \widetilde{a}_1 \otimes \ldots \otimes \widetilde{a}_p \otimes u) &= (-1)^h (aa_1) \otimes \widetilde{a}_2 \otimes \cdots \otimes \widetilde{a}_p \otimes u \\
&\quad + \sum_{r=1}^{p-1} (-1)^{r+h+i_1+\cdots+i_r} a \otimes \widetilde{a}_1 \otimes \cdots \otimes \widetilde{a_r a_{r+1}} \otimes \cdots \otimes \widetilde{a}_p \otimes u \\
&\quad + (-1)^{p+h+i_1+\cdots+i_{p-1}} a \otimes \widetilde{a}_1 \otimes \cdots \otimes \widetilde{a}_{p-1} \otimes (a_p u)\,.
\end{aligned}$$

This finishes our description of the bar construction.

Proof of Theorem 3.1.1. We apply Construction 3.1.4 to the DG algebra A and its DG module U, considered as complexes over the base ring $\Bbbk = Q$. Thus, we get quasi-isomorphisms

$$\mathrm{B}^Q(A,U) \xrightarrow{\pi} U \xrightarrow{\epsilon^U} M\,.$$

On the other hand, as the DG module $\mathrm{B}^Q(A,U)$ is semi-free over A, so

$$\mathrm{B}^Q(A,U) = \mathrm{B}^Q(A,U) \otimes_A A \xrightarrow{\mathrm{B}^Q(A,U)\otimes_A \epsilon^A} \mathrm{B}^Q(A,U) \otimes_A R$$

is a quasi-isomorphism by Proposition 1.3.2. Viewed as a complex of R–modules, $\mathrm{B}^Q(A,U) \otimes_A R$ is precisely the complex $\mathrm{F}(A,U)$ described in the statement of the theorem, so $\mathrm{F}(A,U)$ is a free resolution of M over R. □

3.2. **Spectral sequences.** Various spectral sequences relate (co)homological invariants of M over R and over Q. Those presented below are *first quadrant*, that is, have ${}^r\mathrm{E}_{p,q} = 0$ when $p < 0$ or $q < 0$, and *of homological type of* , meaning that their differentials follow the pattern ${}^r d_{p,q} \colon {}^r\mathrm{E}_{p,q} \to {}^r\mathrm{E}_{p-r,q+r-1}$.

For starters, here is a classical *Cartan-Eilenberg spectral sequence* [51].

Proposition 3.2.1. *For each Q–module N there exists a spectral sequence*

$$ {}^2\mathrm{E}_{p,q} = \mathrm{Tor}_p^R\left(M, \mathrm{Tor}_q^Q(R,N)\right) \implies \mathrm{Tor}_{p+q}^Q(M,N)\,. $$

Proof. Let $V \to M$ be a free resolution over R, and $W \to N$ be a free resolution over Q. By Proposition 1.3.2, the induced map $V \otimes_Q W \to M \otimes_Q W$ is a quasi-isomorphism, hence $\mathrm{H}(V \otimes_Q W) \cong \mathrm{Tor}^Q(M,N)$. As $V \otimes_Q W \cong V \otimes_R (R \otimes_Q W)$, the filtration $(V_{\leqslant p}) \otimes_R (R \otimes_Q W)$ yields a spectral sequence

$$ {}^2\mathrm{E}_{p,q} = \mathrm{H}_p(V \otimes_R \mathrm{H}_q(R \otimes_Q W)) \implies \mathrm{Tor}_{p+q}^Q(M,N) $$

where $\mathrm{H}_q(R \otimes_Q W) = \mathrm{Tor}_q^Q(R,N)$ and $\mathrm{H}_p(V \otimes_R L) = \mathrm{Tor}_p^R(M,L)$. □

In simple cases, this spectral sequence degenerates to an exact sequence. This may be used to prove the next result, but we take a direct approach.

Proposition 3.2.2. *If f is a non-zero-divisor on Q, and N is a module over $R = Q/(f)$, then there is a long exact sequence*

$$ \begin{aligned} \dots \to \mathrm{Tor}_{n-1}^R(M,N) \to \mathrm{Tor}_n^Q(M,N) &\xrightarrow{\mathrm{Tor}_n^{\psi}(M,N)} \mathrm{Tor}_n^R(M,N) \\ \xrightarrow{\vartheta_n} \mathrm{Tor}_{n-2}^R(M,N) \to \mathrm{Tor}_{n-1}^Q(M,N) &\to \dots\,. \end{aligned} $$

Proof. In the description of $\mathrm{F}(A,U)$ given in example 3.1.2, define a morphism $\iota\colon U \to \mathrm{F}(A,U)$ with $\iota(u) = x^{(0)} \otimes u$, and note that $\mathrm{Coker}\,\iota \cong \Sigma^2\,\mathrm{F}(A,U)$. Thus, we have a short exact sequence of complexes of free R–modules

$$ 0 \to R \otimes_Q U \xrightarrow{R\otimes\iota} \mathrm{F}(A,U) \xrightarrow{\vartheta} \Sigma^2\,\mathrm{F}(A,U) \to 0\,. $$

Tensoring it with N over R, and writing down the homology exact sequence of the resulting short exact sequence of complexes, we get what we want. □

The next spectral sequence is introduced by Lescot [107].

Proposition 3.2.3. *If ψ is surjective and k is a residue field of R, then there is a spectral sequence*

$$ {}^2\mathrm{E}_{p,q} = \mathrm{Tor}_p^Q(k,M) \otimes_k \mathrm{Tor}_q^R(k,k) \implies \left(\mathrm{Tor}^Q(k,k) \otimes_k \mathrm{Tor}^R(M,k)\right)_{p+q}\,. $$

Proof. Choose free resolutions: $U \to k$ over Q; $V \to M$ and $W \to k$ over R. As $U \otimes_Q V$ is a bounded below complex of free R–modules, Proposition 1.3.2 yields the first isomorphisms below; the third ones comes from Proposition 1.3.4:

$$ \begin{aligned} \mathrm{H}(U \otimes_Q V \otimes_R W) &\cong \mathrm{H}((U \otimes_Q V) \otimes_R k) \cong \mathrm{H}((U \otimes_Q k) \otimes_k (V \otimes_R k)) \\ &\cong \mathrm{H}(U \otimes_Q k) \otimes_k \mathrm{H}(V \otimes_R k) = \mathrm{Tor}^Q(k,k) \otimes_k \mathrm{Tor}^R(M,k)\,. \end{aligned} $$

Thus, the spectral sequence of the filtration $(U \otimes_Q V) \otimes_R (W_{\leqslant p})$ has

$$ {}^2\mathrm{E}_{p,q} = \mathrm{H}_p(\mathrm{H}_q(U \otimes_Q V) \otimes_R W) \implies \left(\mathrm{Tor}^Q(k,k) \otimes_k \mathrm{Tor}^R(M,k)\right)_{p+q}\,. $$

Since U is a bounded below complex of free Q–modules, $U \otimes_Q V \to U \otimes_Q M$ is a quasi-isomorphism, so ${}^2\mathrm{E}_{p,q} \cong \mathrm{H}_p(\mathrm{Tor}_q^Q(k,M) \otimes_R W)$, and that module is equal to $\mathrm{Tor}_p^R\left(\mathrm{Tor}_q^Q(k,M), k\right) \cong \mathrm{Tor}_p^Q(k,M) \otimes_k \mathrm{Tor}_q^R(k,k)$. □

For us, the preceding sequences have the drawback of going in the 'wrong' direction: they require input of data over the ring of interest, R. Next we describe a sequence where the roles of Q and R are reversed. It belongs to the family of *Eilenberg-Moore spectral sequences*, cf. [122].

By Construction 2.3.3, $\operatorname{Tor}^Q(M,k)$ is a module over the graded algebra $\operatorname{Tor}^Q(R,k)$. A homogeneous free resolution of the former over the latter provides an 'approximation' of a resolution of M over R: this is the contents of the following special case of a result of Avramov [22].

Proposition 3.2.4. *When k is a residue field of R, there is a spectral sequence*

$$ {}^2\mathrm{E}_{p,q} = \operatorname{Tor}_p^{\operatorname{Tor}^Q(R,k)}\left(\operatorname{Tor}^Q(M,k),k\right)_q \implies \operatorname{Tor}^R_{p+q}(M,k)\,. $$

Proof. In the notation of Construction 3.1.4, set $B = A\otimes_Q k$ and $V = U\otimes_Q k$. The filtration $\bigoplus_{i\leqslant p}\left(\widetilde{B}^{\otimes i}\otimes V\right)$ of $\mathrm{B}^k(B,V)\otimes_B k$ yields a spectral sequence

$$ {}^0\mathrm{E}_{p,q} = \left(S^k_p(B,V)\otimes_B k\right)_q \implies \mathrm{H}_{p+q}(\mathrm{B}^k(B,V)\otimes_B k) $$

with ${}^0d_{p,q}$ equal to the tensor product differential. By Künneth,

$$ {}^1\mathrm{E}_{p,q} = \mathrm{H}_q(\mathrm{S}^k_p(B,V)\otimes_B k)_q \cong \left(\mathrm{S}^k_p(\mathrm{H}(B),\mathrm{H}(V))\otimes_{\mathrm{H}(B)} k\right)_q\,, \quad {}^1d_{p,q} = \delta^{[p]}_q\otimes_R k\,. $$

As k is a field, $\mathrm{S}^k(\mathrm{H}(B),\mathrm{H}(V))$ is a resolution of $\mathrm{H}(V)$ over $\mathrm{H}(B)$, so

$$ {}^2\mathrm{E}_{p,q} = \operatorname{Tor}_p^{\mathrm{H}(B)}(\mathrm{H}(V),k)_q\,. $$

Since $\mathrm{H}(B) = \operatorname{Tor}^Q(R,k)$ and $\mathrm{H}(V) = \operatorname{Tor}^Q(M,k)$, the second page of the spectral sequence has the desired form. The isomorphisms

$$ \mathrm{B}^k(B,V)\otimes_B k \cong \left(\mathrm{B}^Q(A,U)\otimes_A R\right)\otimes_R k = \mathrm{F}(A,U)\otimes_R k $$

and Theorem 3.1.1 identify its abutment as $\operatorname{Tor}^R(M,k)$. □

Corollary 3.2.5. *If ψ is surjective, then for each $n\geq 0$ there is an inclusion*

$$ \sum_{i=1}^n \operatorname{Tor}^Q_i(R,k)\cdot\operatorname{Tor}^Q_{n-i}(M,k) \subseteq \operatorname{Ker}\operatorname{Tor}^\psi_n(M,k)\,. $$

Proof. The natural map $U\otimes_Q k\to \mathrm{F}(A,U)$, that is, $V\to \mathrm{B}^k(B,V)\otimes_B k$, identifies V with $F^1\subseteq \mathrm{B}^k(B,V)\otimes_B k$. Thus, the map $\operatorname{Tor}^\psi_n(M,k)$ that it induces in homology factors through

$$ \nu_n\colon \operatorname{Tor}^Q_n(M,k) = \mathrm{H}_n(V)\to {}^2\mathrm{E}_{0,n} = \frac{\operatorname{Tor}^Q_n(M,k)}{\sum_{i=1}^n \operatorname{Tor}^Q_i(R,k)\cdot\operatorname{Tor}^Q_{n-i}(M,k)}\,. $$

We get $\operatorname{Ker}\nu_n\subseteq\operatorname{Ker}\operatorname{Tor}^\psi_n(M,k)$, which is the desired inclusion. □

The next theorem shows that the vector spaces

$$ o^\psi_n(M) = \frac{\operatorname{Ker}\operatorname{Tor}^\psi_n(M,k)}{\sum_{i=1}^n \operatorname{Tor}^Q_i(R,k)\cdot\operatorname{Tor}^Q_{n-i}(M,k)} $$

are *obstructions* to DG module structures; they were found in [22].

Theorem 3.2.6. *Let $(Q, \mathfrak{n}, k)$ be a local ring, let $\psi\colon Q \to R$ be a surjective homomorphism of rings, and let M be a finite R–module.*

If the minimal free resolution A of R over Q has a structure of DG algebra, and the minimal free resolution U of M over Q admits a structure of DG module over A, then $o_n^{\psi}(M) = 0$ for all n.

Proof. Under the hypotheses of the theorem, $\partial' \otimes_Q k = 0$ in the bar construction $\mathrm{B}^k(B, V) \otimes_B k$ of Proposition 3.2.4, so the only non-zero differential in the spectral sequence constructed there acts on the first page. Thus, the sequence stops on the second page, yielding $\operatorname{Ker} \operatorname{Tor}_n^{\psi}(M,k) = \operatorname{Ker} \nu_n$. □

Remark 3.2.7. Let $\boldsymbol{f}$ be a Q–regular sequence. Computing the Tor algebra for $R = Q/(\boldsymbol{f})$ with the help of the Koszul complex $A = K(\boldsymbol{f}; Q)$ we get

$$\operatorname{Tor}^Q(R,k) = \mathrm{H}(A \otimes_R k) = A \otimes_R k = \bigwedge (A_1 \otimes_R k) = \bigwedge \operatorname{Tor}_1^Q(R,k)$$

and hence $\sum_{i=1}^n \operatorname{Tor}_i^Q(R,k) \cdot \operatorname{Tor}_{n-i}^Q(M,k) = \operatorname{Tor}_1^Q(R,k) \cdot \operatorname{Tor}_{n-1}^Q(M,k)$.

3.3. **Upper bounds.** In this section $\psi\colon Q \to R$ is a finite homomorphism of local rings that induces the identity on their common residue field k, and M is a finite R–module. We relate the Betti numbers of M over R and Q.

Often, such relations are expressed in terms of the formal power series

$$\mathrm{P}_M^R(t) = \sum_{n=0}^{\infty} \beta_n^R(M) t^n \in \mathbb{Z}[[t]]\,,$$

known as the *Poincaré series* of M over R, and the corresponding series over Q. Results then take the form of coefficientwise inequalities (denoted $\preccurlyeq$ and $\succcurlyeq$) of formal power series; equalities are significant.

Spectral sequence generate inequalities, by an elementary observation:

Remark 3.3.1. In a spectral sequence of vector spaces ${}^r\mathrm{E}_{p,q} \implies E$, $r \geq a$, the space ${}^{r+1}\mathrm{E}_{p,q}$ is a subquotient of ${}^r\mathrm{E}_{p,q}$ for $r \geq a$, and the spaces ${}^{\infty}\mathrm{E}_{p,q}$ are the subfactors of a filtration of E_{p+q}. Thus, there are (in)equalities

$$\dim_k E_n = \sum_{p+q=n} \operatorname{rank}_k {}^{\infty}\mathrm{E}_{p,q} \leq \sum_{p+q=n} \operatorname{rank}_k {}^{r}\mathrm{E}_{p,q} \leq \sum_{p+q=n} \operatorname{rank}_k {}^{a}\mathrm{E}_{p,q}\ .$$

Multiplying the n'th one by t^n, and summing in $\mathbb{Z}[[t]]$, we get inequalities

$$\sum_{n \geqslant 0} \dim_k E_n t^n \preccurlyeq \sum_n \Big(\sum_{p+q=n} \operatorname{rank}_k {}^{r}\mathrm{E}_{p,q} \Big) t^n \qquad \text{for} \quad r \geq a\,.$$

The next result was initially deduced by Serre from the sequence in Proposition 3.2.1. It is more expedient to get it from that in Proposition 3.2.4.

Proposition 3.3.2. *There is an inequality* $\mathrm{P}_M^R(t) \preccurlyeq \dfrac{\mathrm{P}_M^Q(t)}{1 - t\big(\mathrm{P}_R^Q(t) - 1\big)}$.

Proof. The spectral sequence of Proposition 3.2.4 has

$$\begin{aligned}\sum_n\Big(\sum_{p+q=n}\operatorname{rank}_k {}^1\mathrm{E}_{p,q}\Big)t^n &= \sum_p\Big(\sum_q \operatorname{rank}_k {}^1\mathrm{E}_{p,q}\, t^q\Big)t^p\\ &= \sum_p\Big(\big(\mathrm{P}^Q_R(t)-1\big)^p\,\mathrm{P}^Q_M(t)\Big)t^p\\ &= \mathrm{P}^Q_M(t)\sum_p\big(\mathrm{P}^Q_R(t)-1\big)^p t^p = \frac{\mathrm{P}^Q_M(t)}{1-t\big(\mathrm{P}^Q_R(t)-1\big)}\end{aligned}$$

so the desired inequality follows from the preceding remark. □

Remark. If equality holds with $M = k$, then ψ is called a *Golod homomorphism*. These maps, introduced by Levin [109], are studied in detail in [110], [24]; they are used in many computations of Poincaré series. The 'absolute case', when Q is a regular local ring, is the subject of Chapter 5.

Directly from the spectral sequence in Proposition 3.2.3, we read off

Proposition 3.3.3. *There is an inequality* $\mathrm{P}^R_M(t)\,\mathrm{P}^Q_k(t) \preccurlyeq \mathrm{P}^Q_M(t)\,\mathrm{P}^R_k(t)$. □

Remark. If equality holds, then the module M is said to be *inert* by ψ: these modules are introduced and studied by Lescot [107].

In special cases, universal bounds can be sharpened.

Recall [46] that a finite Q–module N has *rank* if for each prime ideal $\mathfrak{q} \in \operatorname{Ass} Q$ the $Q_\mathfrak{q}$–module $N_\mathfrak{q}$ is free, and its rank does not depend on $\mathfrak{q}$. The common rank of these free modules is called the Q–rank of N, and denoted $\operatorname{rank}_Q N$; we write $\operatorname{rank}_Q N \geq 0$ to indicate that the rank of N is defined.

Proposition 3.3.4. *If f is Q–regular and $R = Q/(f)$, then*

$$0 \preccurlyeq \sum_{n=1}^{\infty}\operatorname{rank}_Q \operatorname{Syz}^Q_n(M)\,t^{n-1} = \frac{\mathrm{P}^Q_M(t)}{(1+t)} \preccurlyeq \mathrm{P}^R_M(t) \preccurlyeq \frac{\mathrm{P}^Q_M(t)}{(1-t^2)}\,.$$

Proof. Let U be a minimal resolution of M over Q. For each $\mathfrak{q} \in \operatorname{Ass} Q$ we have $f \notin \mathfrak{q}$. Thus, $M_\mathfrak{q} = 0$, and so for each n there is an exact sequence

$$0 \to \operatorname{Syz}^Q_{n+1}(M)_\mathfrak{q} \to (U_n)_\mathfrak{q} \to \dots \to (U_0)_\mathfrak{q} \to 0\,.$$

It follows that $\operatorname{Syz}^Q_{n+1}(M)_\mathfrak{q}$ is free of rank $\sum_{i\geqslant 0}(-1)^i\beta^Q_{n-i}(M)$: this establishes the equality, and the first inequality.

For the second inequality, apply the exact sequence of Proposition 3.2.2, to get $\beta^Q_n(M) \leq \beta^R_{n-1}(M) + \beta^R_n(M)$ for all n.

The third inequality results from counting the ranks of the free modules in the resolution of M over R given by Theorem 3.1.3. □

There are useful sufficient conditions for equalities. One is essentially contained in Nagata [124]; the other, from Shamash [143], is given a new proof.

Proposition 3.3.5. *Let f be a Q–regular element, and $R = Q/(f)$.*

1. *If $f \notin \mathfrak{n}^2$, then* $\mathrm{P}^R_M(t) = \mathrm{P}^Q_M(t)/(1+t)$.
2. *If $f \in \mathfrak{n}(0 :_Q M)$, then* $\mathrm{P}^R_M(t) = \mathrm{P}^Q_M(t)/(1-t^2)$.

Proof. (1) This is just the last assertion of Theorem 2.2.3.

(2) Let $s_1,\dots,s_e$ be a minimal set of generators of $\mathfrak{n}$, and write $f=\sum_{j=1}^{e} a_js_j$ with $a_j\in(0\colon_Q M)$. In a DG algebra resolution $V\to k$ over Q, pick $y_1,\dots,y_e\in V_1$ such that $\partial(y_j)=s_j$. As $\partial\big(\sum_{j=1}^{e} a_jy_j\big)=f$, Example 3.1.2 yields a resolution G of k over R, with $G_n=\bigoplus_{i=0}^{n} Rx^{(i)}\otimes_Q V_{n-2i}$ and

$$\partial\big(x^{(i)}\otimes_Q v\big)=\sum_{j=1}^{e} a_jx^{(i-1)}\otimes_Q y_jv+x^{(i)}\otimes_Q \partial(v)\,.$$

For $b\in M$, the induced differential of $M\otimes_R G$ then satisfies

$$\partial\big(bx^{(i)}\otimes_Q v\big)=\sum_{j=1}^{e} a_jbx^{(i-1)}\otimes_Q y_jv+bx^{(i)}\otimes_Q \partial(v)=bx^{(i)}\otimes_Q \partial(v)\,,$$

so $M\otimes_R G\cong\bigoplus_{i=0}^{\infty}\Sigma^{2i}(M\otimes_Q V)$ as complexes of R–modules. This yields

$$\operatorname{Tor}^R(M,k)=\operatorname{H}(M\otimes_R G)=\bigoplus_{i=0}^{\infty}\Sigma^{2i}\operatorname{Tor}^Q(M,k)\,.$$

The desired equality of Poincaré series is now obvious. □

Remark. The resolution $\operatorname{G}(\boldsymbol{\sigma},U)$ of Theorem 3.1.3 has

$$\sum\operatorname{rank}_R G_nt^n=\big(\sum\operatorname{rank}_Q U_it^i\big)/(1-t^2)\,.$$

Thus, if U is a minimal resolution of M over Q and $f\in\mathfrak{n}(0:_Q M)$, then by (2) $\operatorname{G}(\boldsymbol{\sigma},U)$ is a *minimal* R–free resolution of M. Another case of minimality is given by Construction 5.1.2, which shows that if $\operatorname{pd}_Q M=1$, then $\operatorname{Syz}_n^R(M)$ has a minimal resolution of that form for each $n\geq 1$. Quite the opposite happens 'in general': it is proved in [32] that if R is a complete intersection and the Betti numbers of M are not bounded, then $\operatorname{Syz}_n^R(M)$ has such a minimal resolution for at most one value of n.

4. Growth of resolutions

The gap between regularity and singularity widens to a chasm in homological local algebra: Minimal resolutions are always finite over a regular local ring, and (very) rarely over a singular one. A bridge[15] is provided by the Cohen Structure Theorem: the completion of each local ring is a residue of a regular ring, so by change of rings techniques homological invariants over the singular ring may be approached from those—essentially finite—over the regular one.

To describe and compare resolutions of modules over a singular local ring, we go beyond the primitive dichotomy of finite versus infinite projective dimension, and analyze infinite sequences of integers, such as ranks of matrices, or Betti numbers. For that purpose there in no better choice than to follow the time-tested approach of calculus, and compare sizes of resolutions to the functions we[16] understand best: polynomials and exponentials.

4.1. **Regular presentations.** Let I be an ideal in a noetherian ring R. Recall that the *minimal number of generators* $\nu_R(I)$, the height, and the depth of I are always related by inequalities, due to Rees and to Krull:

$$\operatorname{depth}_R(I,R) \leq \operatorname{height} I \leq \nu_R(I)\,.$$

For the rest of this section, $(R,\mathfrak{m},k)$ is a local ring; $\nu_R(\mathfrak{m})$ is then known as its *embedding dimension*, denoted $\operatorname{edim} R$, and the inequalities read

$$\operatorname{depth} R \leq \dim R \leq \operatorname{edim} R\,.$$

Discrepancies between these numbers provide measures of irregularity:

- $\operatorname{cmd} R = \dim R - \operatorname{depth} R$ is the *Cohen-Macaulay defect* of R;
- $\operatorname{codim} R = \operatorname{edim} R - \dim R$ is the *codimension*[17] of R;
- $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$ is the *codepth*[17] of R.

We use the vanishing of $\operatorname{codepth} R$ to define[18] the *regularity* of R. Thus, if R is regular, then each minimal generating set of $\mathfrak{m}$ is a regular sequence.

Two cornerstone results of commutative ring theory determine the role of regular rings in the study of free resolutions.

The *Auslander-Buchsbaum-Serre Theorem* describes them homologically.

Theorem 4.1.1. *The following conditions are equivalent.*

(i) *R is regular.*
(ii) $\operatorname{pd}_R M < \infty$ *for each finite R–module M.*
(iii) $\operatorname{pd}_R k < \infty$.

[15] Warning: crossing may take an infinite time.

[16] Algebraists.

[17] Because 'codimension' has been used to denote depth, and 'codepth' to denote Cohen-Macaulay defect, the notions described here are sometimes qualified by 'embedding'; it would be too cumbersome to stick to that terminology, and to devise new notation.

[18] This clearly implies the usual definition, in terms of the vanishing of $\operatorname{codim} R$. Conversely, if R is regular, then the associated graded ring $S = \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is the quotient of a polynomial ring in $e = \operatorname{edim} R$ variables. Since S and R have equal Hilbert-Samuel functions, $\dim S = \dim R = e$, so S is the polynomial ring on the classes in S_1 of a minimal set of generators $\boldsymbol{t}$ of $\mathfrak{m}$. In particular, these classes form an S–regular sequence, and then a standard argument shows that $\boldsymbol{t}$ is an R–regular sequence.

Proof. (i) $\implies$ (iii). By Example 1.1.1, the Koszul complex on a minimal set of generators for $\mathfrak{m}$ is a free resolution of k.

(iii) $\implies$ (ii). As $\operatorname{Tor}_n^R(M,k) = 0$ for $n \gg 0$, apply Proposition 1.2.2.

(ii) $\implies$ (i). We prove that $\operatorname{codepth} R = 0$ by induction on $d = \operatorname{depth} R$. If $d = 0$, then k is free by Proposition 1.2.7.1, hence $\mathfrak{m} = 0$. If $d > 0$, then a standard prime avoidance argument yields a regular element $g \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. Set $R' = R/(g)$, and note that $\operatorname{codepth} R = \operatorname{codepth} R'$. As $\operatorname{pd}_{R'} k < \infty$ by Theorem 2.2.3, we have $\operatorname{codepth} R' = 0$ by the induction hypothesis. □

Corollary 4.1.2. *If R is regular, then so is $R_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. By the theorem, $R/\mathfrak{p}$ has a finite R–free resolution F; then $F_\mathfrak{p}$ is a finite $R_\mathfrak{p}$–free resolution of $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, so $R_\mathfrak{p}$ is regular by the theorem. □

The *Cohen Structure Theorem* establishes the dominating position of regular rings. A *regular presentation* of R is an isomorphism $R \cong Q/I$, where Q is a regular local ring. Many local rings (for example, all those arising in classical algebraic or analytic geometry), come equipped with such a presentation. By Cohen's theorem, *every complete local ring has a regular presentation*.

Here is how the two theorems above apply to the study of resolutions.

Since the $\mathfrak{m}$-adic completion $\widehat{R}$ is a faithfully flat R–module, a complex of R–modules F is a (minimal) free resolution of M over R if and only if $\widehat{F} = \widehat{R} \otimes_R F$ is a (minimal) free resolution of $\widehat{M} = \widehat{R} \otimes_R M$ over $\widehat{R}$; in particular, $\mathrm{P}_M^R(t) = \mathrm{P}_{\widehat{M}}^{\widehat{R}}(t)$. The point is: as $\widehat{M}$ iand $\widehat{R}$ have *finite* free resolutions over the ring Q, all change of rings results apply with *finite* entry data.

If $(Q, \mathfrak{n}, k)$ is a regular local ring, $R = Q/I$, and $f \in I \smallsetminus \mathfrak{n}^2$, then $Q/(f)$ is also regular, and maps onto R. Iterating, one sees that if R has some regular presentation, then it has a *minimal* one, with $\operatorname{edim} R = \operatorname{edim} Q$.

Minimal presentations often fade into the background, because some of their invariants can be computed directly over the ring R, from Koszul complexes on minimal sets of generators $\boldsymbol{t}$ of $\mathfrak{m}$. Different choices of $\boldsymbol{t}$ lead to isomorphic complexes, so when we do not need to make an explicit choice of generators, we write K^R instead of $K(\boldsymbol{t}; R)$, and set $K^M = K^R \otimes_R M$.

Lemma 4.1.3. *If K^R is a Koszul complex on a minimal set of generators of $\mathfrak{m}$, and $\widehat{R} = Q'/I'$ (respectively, $R = Q/I$) is a minimal regular presentation, then*

(1) $\mathrm{H}(K^M) \cong \mathrm{H}(K^{\widehat{M}}) \cong \operatorname{Tor}^{Q'}(\widehat{M}, k) \ \big(\cong \operatorname{Tor}^Q(M,k)\big)$.

(2) $\sup\{i \mid \mathrm{H}_i(K^M) \neq 0\} = \sup\{i \mid \mathrm{H}_i(K^{\widehat{M}}) \neq 0\} = \operatorname{pd}_{Q'} \widehat{M} \ \big(= \operatorname{pd}_Q M\big)$.

(3) $\mathrm{H}_1(K^R) \cong \mathrm{H}_1(K^{\widehat{R}}) \cong I' \otimes_{Q'} k \ \big(\cong I \otimes_Q k\big)$.

Proof. The fact that $\widehat{R}$ is faithfully flat over R, and $\mathfrak{m}\widehat{R}$ is its maximal ideal yields the relations on both ends, so we argue for those in the middle. By Example 1.1.1, $K^{Q'}$ is a minimal free resolution of k over Q': this yields (1), and then (2) follows from Proposition 1.2.2. For (3), use the long exact sequence of $\operatorname{Tor}^{Q'}(-,k)$ applied to the exact sequence $0 \to I' \to Q' \to \widehat{R} \to 0$. □

In view of the lemma, Proposition 3.3.2 translates into:

Proposition 4.1.4. *For each finite R–module M there is an inequality*

$$\mathrm{P}_M^R(t) \preccurlyeq \frac{\displaystyle\sum_{i=0}^{\operatorname{edim} R - \operatorname{depth} M} \operatorname{rank}_k \mathrm{H}_i(K^M)t^i}{1 - \displaystyle\sum_{j=1}^{\operatorname{codepth} R} \operatorname{rank}_k \mathrm{H}_j(K^R)t^{j+1}}\,. \qquad \square$$

Corollary 4.1.5. *There is an $\alpha \in \mathbb{R}$, such that $\beta_n^R(M) \le \alpha^n$ for $n \ge 1$.* $\square$

A local ring homomorphism $\varphi\colon R \to R'$, such that $\mathfrak{m}R'$ is the maximal ideal of R', may be lifted—in more than one way—to a morphism of DG algebras $K^\varphi\colon K^R \to K^{R'}$ (we use a 'functorial' notation, because in the cases treated below the choice of a specific lifting will be of no consequence, while it helps to distinguish K^φ from $K^R \otimes_R \varphi\colon K^R \to K^R \otimes_R R'$).

The following easily proved statements have unexpectedly strong consequences. The second is the key to Serre's original proof that $\operatorname{pd}_R k$ characterizes regularity in [141]. The third, also due to Serre, is important in the study of multiplicities, cf. [142], [17]; its proof below is from Eagon and Fraser [54].

Lemma 4.1.6. *The complexes K^R and K^M have the following properties.*

(1) *If $g \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ is R–regular, then the homomorphism $\varphi\colon R \to R/(g)$ induces a surjective quasi-isomorphism $K^\varphi\colon K^R \to K^{R/(g)}$.*
(2) *If $a \in K^R$ satisfies $\partial(a) \in \mathfrak{m}^2K^R$, then $a \in \mathfrak{m}K^R$.*
(3) *For each finite R–module M there is an integer s such that the complex*
$$C^i\colon \quad 0 \to \mathfrak{m}^{i-e}K_e^M \to \ldots \to \mathfrak{m}^{i-1}K_1^M \to \mathfrak{m}^iK_0^M \to 0 \qquad (*)$$
is exact for $i \ge s$; for each i, C^i is a DG submodule of K^M over K^R.

Proof. (1) Choose $y_1, \ldots, y_e \in K_1^R$, such that $\{\partial(y_i) = t_i \mid j = 1, \ldots, e\}$ is a minimal generating set for $\mathfrak{m}$. We may assume that $g = t_e$, and set $D = R[y \mid \partial(y) = g]$. As K^R is a semi-free DG module over D, by Proposition 1.3.2 the quasi-isomorphism $D \to R/(g)$ induces a quasi-isomorphism $K^R \to R/(g) \otimes_D K^R = K^{R/(g)}$.

(2) The statement may be rephrased as follows: the map $K_n^R/\mathfrak{m}K_n^R \to \mathfrak{m}K_{n-1}^R/\mathfrak{m}^2K_{n-1}^R$ induced by the differential ∂_n of K^R is injective for all $n \ge 1$. This is a direct consequence of the formula for the Koszul differential, and the minimality of the generating set $t_1, \ldots, t_e$.

(3) As $C^1 = \operatorname{Ker}(K^R \to k) \subset K^R$ is a DG ideal, $C^i = (C^1)^iK^M$ is a DG submodule of K^M. For each $1 \le n \le e$ and $i \gg 0$, we have equalities
$$\mathrm{Z}_n(C^i) = \mathrm{Z}_n(K^M) \cap \mathfrak{m}^{i-n}K_n^M = \mathfrak{m}\big(\mathrm{Z}_n(K^M) \cap \mathfrak{m}^{i-n-1}K_n^M\big)$$
(the first by definition, the second by Artin-Rees). Increasing i, we may assume they hold simultaneously for all n. Thus, each $z \in \mathrm{Z}_n(C^i)$ can be written as $z = \sum_{j=1}^e t_jv_j$ with $v_j \in \mathfrak{m}^{i-n-1}K_n^M$, so $z = \partial y$ for $y = \sum_{j=1}^e y_jv_j \in C_{n+1}^i$. $\square$

Next we present a result of [21], which shows that the minimal resolution of each R–module M is part of that of 'most' of its residue modules. It is best stated in terms of a property of minimal complexes.

Remark 4.1.7. Let F be a minimal complex of free R–modules, with $F_n = 0$ for $n < 0$. If $\epsilon\colon F \to N$ is a morphism to a finite R–module N, let $\alpha\colon F \to G$ be a lifting of ϵ to a minimal resolution G of N. Any two liftings are homotopic, so the homomorphism $k\otimes_R\alpha = \mathrm{H}(k\otimes_R\alpha)$ depends only on ϵ.

We say that ϵ is *essential* if $k\otimes_R\alpha_n$ is injective for each n. In that case, each α_n is a split injection, hence α maps F isomorphically onto a subcomplex of G, that splits off as a graded R–module; the entire resolution of N is obtained then from $\alpha(F)$ by adjunction of basis elements, as in Construction 2.2.6.

When $\mathrm{pd}_R M$ is finite the next result is a straightforward application of the Artin-Rees Lemma. Replacing resolutions over R by resolutions over K^R, we make Artin-Rees work simultaneously in infinitely many dimensions.

Theorem 4.1.8. *Let F be a minimal free resolution of a finite R–module M. There exists an integer $s\geq 1$, such that for each submodule $M'\subseteq \mathfrak{m}^s M$ the augmentation $\epsilon\colon F\longrightarrow M''=M/M'$ is essential. In particular,*

$$\mathrm{P}^R_{M''}(t)=\mathrm{P}^R_M(t)+t\,\mathrm{P}^R_{M'}(t)\,.$$

Proof. Choose s as in Lemma 4.1.6.3, so that $C^s\subseteq K^M$ is an exact DG submodule. The projection $\rho\colon K^M\longrightarrow K^M/C^s$ is then a quasi-isomorphism of DG modules over K^R. Let U be a semi-free resolution of k over K^R, let M' be a submodule in $\mathfrak{m}^s M$, let $\pi\colon M\longrightarrow M''$ be the canonical map. The composition

$$U\otimes_{K^R}\left(K^R\otimes_R M\right)\xrightarrow{\pi'} U\otimes_{K^R}\left(K^R\otimes_R M''\right)\longrightarrow U\otimes_{K^R}\left(K^M/C^s\right)$$

of morphisms of DG modules, where $\pi'=U\otimes_{K^R}(K^R\otimes_R\pi)$, is equal to $U\otimes_{K^R}\rho$, and so is a quasi-isomorphism by Proposition 1.3.2. It follows that $\mathrm{H}(\pi')$ is injective. As U is a free resolution of k over R and $U\otimes_{K^R}(K^R\otimes_R\pi)=U\otimes_R\pi\colon U\otimes_R M\longrightarrow U\otimes_R M''$, we see that $\mathrm{H}(\pi')=\mathrm{Tor}^R(k,\pi)$. If F'' is a minimal free resolution of M'', then $k\otimes_R\pi'\colon k\otimes_R F\longrightarrow k\otimes_R F''$ is another avatar of the same map, so it is injective, as desired. For the equality of Poincaré series, apply $\mathrm{Tor}^R(k,-)$ to $0\longrightarrow M'\longrightarrow M\longrightarrow M''\longrightarrow 0$. ☐

In a precise sense, the residue field has the 'largest' resolution.

Corollary 4.1.9. *For each finite R–module M there exists an integer $\ell\geq 1$, such that $\mathrm{P}^R_M(t)\preccurlyeq \ell\,\mathrm{P}^R_k(t)$.*

Proof. By the theorem, $\mathrm{P}^R_M(t)\preccurlyeq \mathrm{P}^R_{M/\mathfrak{m}^s M}(t)$ for some s, so we may assume that $\mathrm{length}_R M<\infty$. The obvious induction on length, using the exact sequence of $\mathrm{Tor}^R(-,k)$ then establishes the inequality with $\ell=\mathrm{length}_R M$. ☐

For another manifestation of the ubiquity of $\mathrm{P}^R_k(t)$, cf. Theorem 6.3.6.

4.2. **Complexity and curvature.** In this section we introduce and begin to study measures for the asymptotic size of resolutions.

On the polynomial scale, the *complexity* of M over R is defined by

$$\mathrm{cx}_R M=\inf\left\{d\in\mathbb{N}\;\middle|\;\begin{matrix}\text{there exists a real number }\beta\text{ such that}\\ \beta^R_n(M)\leq\beta n^{d-1}\text{ for }n\gg 0\end{matrix}\right\};$$

this is an adaptation from [25], [26], of a concept originally introduced by Alperin and Evens [2] to study modular representations of finite groups; clearly, one gets the same concept by requiring inequalities for $n\gg 0$.

Example 4.2.1. (1) If $R' = k[s_1]/(s_1^2)$, and $t_1 \in R'$ is the image of s_1, then

$$F' : \qquad \dots \to R' \xrightarrow{t_1} R' \xrightarrow{t_1} \dots R' \xrightarrow{t_1} R' \xrightarrow{t_1} R' \to 0\,. \tag{$*$}$$

is a minimal free resolution of $k = R'/(t_1)$. Thus, $\beta_n^{R'}(k) = 1$ for $n \geq 0$, and $\operatorname{cx}_{R'} k = 1$. Each finite module M over the principal ideal ring $k[t_1]$ is a finite direct sum of copies of k and of R', hence $\beta_n^{R'}(M) = \beta_1^{R'}(M)$ for $n \geq 1$.

(2) Set $R = k[s_1, s_2]/(s_1^2, s_2^2) = k[t_1, t_2]$. As R is a free module over its subring R', the complex $R \otimes_{R'} F'$ is a minimal resolution of the cyclic module $M = R/(t_1)$ over R. Thus, $\beta_n^R(M) = 1$ for $n \geq 0$, and $\operatorname{cx}_R M = 1$.

On the other hand, note that $\operatorname{Syz}_1^R(R/(t_1t_2)) \cong k$, and let F'' be the complex corresponding to F' over $R'' = k[t_2]$. By the Künneth Theorem 1.3.4, $F' \otimes_k F''$ is a resolution of $k \otimes_k k = k$ over $R = R' \otimes_k R''$, and is obviously minimal. Thus, $\beta_n^R(k) = n+1$ for $n \geq 0$, and $\operatorname{cx}_R \big(R/(t_1t_2)\big) = \operatorname{cx}_R \big(R/(t_1, t_2)\big) = 2$.

In particular, over the ring R there exist modules of complexity 0, 1, and 2; in fact, these are all the possible values: cf. Proposition 4.2.5.4.

Complexity may itself be infinite.

Example 4.2.2. Let $R = k[s_1, s_2]/(s_1^2, s_1s_2, s_2^2)$. The isomorphism $\mathfrak{m} \cong k^2$ and the exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ show that $\beta_{n+1}^R(k) = 2\beta_n^R(k)$ for $n \geq 1$, hence $\beta_n^R(k) = 2^n$ for $n \geq 0$, and $\operatorname{cx}_R k = \infty$. If M is any finite R–module, then $\operatorname{Syz}_1^R(M) \subseteq \mathfrak{m}F_0$ is isomorphic to a finite direct sum of copies of k, so $\operatorname{cx}_R M = \infty$ unless M is free.

For modules of infinite complexity, the exponential scale is used in [29] to introduce a notion of *curvature*[19] by the formula

$$\operatorname{curv}_R M = \limsup_{n\to\infty} \sqrt[n]{\beta_n^R(M)}\,.$$

Thus, in the last example, either M is free or $\operatorname{curv}_R M = 2$.

Some relations between these *asymptotic invariants* follow directly from the definitions, except for (5), which is a consequence of Corollary 4.1.5:

Remark 4.2.3. For a finite R–module M the following hold:

(1) $\operatorname{pd}_R M < \infty \iff \operatorname{cx}_R M = 0 \iff \operatorname{curv}_R M = 0\,.$
(2) $\operatorname{pd}_R M = \infty \iff \operatorname{cx}_R M \geq 1 \iff \operatorname{curv}_R M \geq 1\,.$
(3) $\operatorname{cx}_R M \leq 1 \iff M$ has bounded Betti numbers.
(4) $\operatorname{cx}_R M < \infty \implies \operatorname{curv}_R M \leq 1\,.$
(5) $\operatorname{curv}_R M < \infty\,.$

With respect to change of modules, properties of complexity and curvature mirror well known properties of projective dimension.

Proposition 4.2.4. *When M is a finite R–module the following hold.*

(1) $\operatorname{cx}_R M \leq \operatorname{cx}_R k \qquad$ *and* $\qquad \operatorname{curv}_R M \leq \operatorname{curv}_R k\,.$
(2) *For each n there are equalities*

$$\operatorname{cx}_R M = \operatorname{cx}_R \operatorname{Syz}_n^R(M) \qquad \text{and} \qquad \operatorname{curv}_R M = \operatorname{curv}_R \operatorname{Syz}_n^R(M)\,.$$

[19]So called because it is the *inverse of the radius* of convergence of $\mathrm{P}_M^R(t)$.

(3) *If M' and M'' are R–modules, then*

$$\operatorname{cx}_R(M' \oplus M'') = \max\{\operatorname{cx}_R M', \operatorname{cx}_R M''\};$$
$$\operatorname{curv}_R(M' \oplus M'') = \max\{\operatorname{curv}_R M', \operatorname{curv}_R M''\}.$$

(4) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R–modules, then*

$$\operatorname{cx}_R M \le \operatorname{cx}_R M' + \operatorname{cx}_R M'' \quad \textit{and} \quad \operatorname{curv}_R M \le \operatorname{curv}_R M' + \operatorname{curv}_R M''.$$

(5) *If a sequence $\boldsymbol{g} \subset R$ is regular on R and on M, then*

$$\operatorname{cx}_R(M/(\boldsymbol{g})M) = \operatorname{cx}_R M \qquad \textit{and} \qquad \operatorname{curv}_R(M/(\boldsymbol{g})M) = \operatorname{curv}_R M.$$

(6) *If N is a finite R–module such that $\operatorname{Tor}_n^R(M,N) = 0$ for $n > 0$, then*

$$\max\{\operatorname{cx}_R M, \operatorname{cx}_R N\} \le \operatorname{cx}_R(M \otimes_R N) \le \operatorname{cx}_R M + \operatorname{cx}_R N;$$
$$\operatorname{curv}_R(M \otimes_R N) = \max\{\operatorname{curv}_R M, \operatorname{curv}_R N\}.$$

Proof. (1) comes from Proposition 4.1.9; (2), (3), and (4) are clear.

(5) By Example 1.1.1, $\operatorname{Tor}_n^R(M, R/(\boldsymbol{g})) = \operatorname{H}_n(K(\boldsymbol{g}; M))$, and the latter module vanishes for $n > 0$. Thus, the desired equalities follow from (6).

(6) Let F and G be minimal free resolutions of M and N, respectively; as $\operatorname{H}_n(F \otimes_R G) = \operatorname{Tor}_n^R(M,N) = 0$ for $n \ge 1$, the complex $F \otimes_R G$ is a free resolution of $M \otimes_R N$. It is obviously minimal, so

$$\max\{\beta_n^R(M), \beta_n^R(N)\} \le \sum_{p+q=n} \beta_p^R(M)\beta_q^R(N) = \beta_n^R(M \otimes_R N).$$

The inequalities for complexities follow. To get the equality for curvatures, rewrite the relations above in terms of Poincaré series:

$$\max\{\operatorname{P}_M^R(t), \operatorname{P}_N^R(t)\} \preccurlyeq \operatorname{P}_M^R(t)\operatorname{P}_N^R(t) = \operatorname{P}_{M \otimes_R N}^R(t).$$

A product converges in the smaller of the circles of convergence of its factors, so the equality of power series yields $\operatorname{curv}_R(M \otimes_R N) \le \max\{\operatorname{curv}_R M, \operatorname{curv}_R N\}$; the converse inequality comes from the inequality of power series. ☐

The finiteness of projective dimension of a module is notoriously unstable under change of rings. Complexity and curvature fare better:

Proposition 4.2.5. *Let M be a finite R–module, let $\varphi \colon R \to R'$ be a homomorphism of local rings, and set $M' = R' \otimes_R M$.*

(1) *For each prime ideal $\mathfrak{p}$ in R there are inequalities*

$$\operatorname{cx}_{R_\mathfrak{p}} M_\mathfrak{p} \le \operatorname{cx}_R M \qquad \textit{and} \qquad \operatorname{curv}_{R_\mathfrak{p}} M_\mathfrak{p} \le \operatorname{curv}_R M.$$

(2) *If φ is a flat local homomorphism, then*

$$\operatorname{cx}_{R'} M' = \operatorname{cx}_R M \qquad \textit{and} \qquad \operatorname{curv}_{R'} M' = \operatorname{curv}_R M.$$

(3) *If $\operatorname{Tor}_n^R(R', M) = 0$ for $n \gg 0$, then*

$$\operatorname{cx}_{R'} M' \le \operatorname{cx}_R M \qquad \textit{and} \qquad \operatorname{curv}_{R'} M' \le \operatorname{curv}_R M.$$

(4) *If $\boldsymbol{g}$ is an R–regular sequence of length r and $R' = R/(\boldsymbol{g})$, then*

$$\operatorname{cx}_R M \le \operatorname{cx}_{R'} M' \le \operatorname{cx}_R M + r;$$
$$\operatorname{curv}_{R'} M' = \operatorname{curv}_R M \quad \textit{when} \quad \operatorname{pd}_R M = \infty.$$

Proof. Both (1) and (2) are immediate. (3) results from Proposition 1.2.3. (4) follows from the inequalities of Poincaré series in Proposition 3.3.4. ☐

Finally, we note a result specific to Cohen-Macaulay rings; it fails in general, as shown by the ring $R = k[[s_1, s_2]]/(s_1^2, s_1 s_2)$ of multiplicity 1.

Proposition 4.2.6. *If R is Cohen-Macaulay, then* $\operatorname{curv}_R M \leq \operatorname{mult} R - 1$*; equality holds when R has minimal multiplicity* (*defined in Example 5.2.8*).

This follows from a lemma, that sharpens the result of Ramras [135].

Lemma 4.2.7. *If R is Cohen-Macaulay,* $\operatorname{mult} R = l$*, and* $\operatorname{type} R = s$*, then*

$$(l-1)\beta_n^R(M) \;\geq\; \beta_{n+1}^R(M) \;\geq\; \frac{s}{l-s}\,\beta_n^R(M) \quad \textit{for } n > \operatorname{depth} R - \operatorname{depth} M\,.$$

Proof. By Remark 1.2.9, we may assume that R is artinian, with $\operatorname{length} R = l$, and $\operatorname{length}(0 :_R \mathfrak{m}) = s$. In a minimal resolution F of M, $\operatorname{Syz}_{n+1}^R(M) \subseteq \mathfrak{m}F_n$, so

$$(l-1)\beta_n^R(M) = \operatorname{length} \mathfrak{m}F_n \geq \operatorname{length} \operatorname{Syz}_{n+1}^R(M) \geq \beta_{n+1}^R(M) \quad \text{for } n \geq 1\,.$$

As $\partial\big((0 :_R \mathfrak{m})F_i\big) \subseteq (0 :_R \mathfrak{m})\mathfrak{m}F_{i-1} = 0$, we have $(0 :_R \mathfrak{m})F_i \subseteq \operatorname{Syz}_{i+1}^R(M)$, and hence $\operatorname{length} \operatorname{Syz}_{i+1}^R(M) \geq s\beta_i^R(M)$. The exact sequence

$$0 \longrightarrow \operatorname{Syz}_{n+2}^R(M) \longrightarrow F_{n+1} \longrightarrow \operatorname{Syz}_{n+1}^R(M) \longrightarrow 0$$

now yields $l\beta_{n+1}^R(M) = \operatorname{length} F_{n+1} \geq s\beta_{n+1}^R(M) + s\beta_n^R(M)$. □

4.3. **Growth problems.** As of today, a distinctive feature of the state of knowledge of infinite free resolutions is that tantalizing questions on the behavior of basic invariants can be stated in very simple terms. We present four groups of interrelated problems, that set benchmarks for many results in the text. Some are (variants) of problems discussed in more detail in [27], others are new. They could be viewed in the broader context of growth of algebraic structures, to which the survey of Ufnarovskij [153] is a good introduction.

For the rest of this section, $(R, \mathfrak{m}, k)$ is a local ring, and M is a finite R–module; to avoid distracting special cases, we assume that $\operatorname{pd}_R M = \infty$.

All problems discussed below have positive answers for modules over Golod rings and over complete intersections. These two cases, for which exhaustive information is available, are treated in detail in later chapters.

Growth. The easily proved inequality over Cohen-Macaulay rings suggests

Problem 4.3.1. Is $\displaystyle\limsup_{n\longrightarrow\infty} \frac{\beta_{n+1}^R(M)}{\beta_n^R(M)}$ always finite?

There is no problem when the Betti numbers of M are bounded, but little is known on when such modules occur. The only general construction that I am aware of, presented in Example 5.1.3, yields modules with periodic of period 2 minimal resolutions. Such a periodicity implies constant Betti numbers, but not conversely, cf. Remark 5.1.4. On the other hand, there exist rings over which all infinite Betti sequences are unbounded, cf. Theorem 5.3.3.

The simplest pattern of bounded Betti numbers is highlighted in

Problem 4.3.2. Is a bounded sequence $\{\beta_n^R(M)\}$ eventually constant?

Problem 4.3.2, proposed in [134], is subsumed in the next one, from [23]:

Problem 4.3.3. Is the sequence $\{\beta_n^R(M)\}$ eventually non-decreasing?

An early class of artinian examples is given by Gover and Ramras [75]. Many more can be found in the papers discussed after Problem 4.3.9.

Complexity. Complexity is a measure for infinite projective dimensions, in the polynomial range of the growth spectrum. It is very interesting to know whether it satisfies an analogue of the Auslander-Buchsbaum result: $\operatorname{depth} R$ is an upper bound for all *finite* projective dimensions. More precisely, we propose

Problem 4.3.4. Does $\operatorname{cx}_R M < \infty$ imply $\operatorname{cx}_R M \le \operatorname{codepth} R$?

Over a complete intersection the inequality holds for all modules, and all values between 0 and $\operatorname{codepth} R$ occur. More generally, for modules of finite CI-dimension[20] the problem has a positive solution, that comes with a bonus: if R is Cohen-Macaulay (respectively, Gorenstein), but *not* a complete intersection, then $\operatorname{cx}_R M \le \operatorname{codepth} R - 2$ (respectively, $\le \operatorname{codepth} R - 3$).

Finite complexity only imposes an upper bound on the Betti numbers, prescribing no asymptote. As in calculus, when $\{b_n\}$ and $\{c_n\}$ are sequences of positive real numbers, we write $b_n \sim c_n$ to denote $\lim_n b_n/c_n = 1$.

Problem 4.3.5. If $\operatorname{cx}_R M = d < \infty$, is then $\beta_n^R(M) \sim \alpha n^{d-1}$ for some $\alpha \in \mathbb{R}$?

Would Betti sequences turn up that have subpolynomial but not asymptotically polynomial growth, then complexity should be refined by the number $\limsup_n \ln\big(\beta_n^R(M)\big)/\ln(n)$, modeled on[21] *Gelfand-Kirillov dimension*, cf. [71].

Curvature. By d'Alembert's convergence criterion, $\operatorname{curv}_R M$ is contained between $\liminf_n \beta_{n+1}^R(M)/\beta_n^R(M)$ and $\limsup_n \beta_{n+1}^R(M)/\beta_n^R(M)$. To quantify the observation that 'at infinity' Betti numbers display uniform behavior, we risk

Problem 4.3.6. Is $\operatorname{curv}_R M = \lim\limits_{n\longrightarrow\infty} \dfrac{\beta_{n+1}^R(M)}{\beta_n^R(M)}$?

A positive answer would solve Problem 4.3.2, and also Problem 4.3.3 when $\operatorname{curv}_R M > 1$. If $\operatorname{curv}_R M = 1$, then the Betti sequence of M grows subexponentially; none is known to grow superpolynomially, so we propose

Problem 4.3.7. Does $\operatorname{curv}_R M = 1$ imply $\operatorname{cx}_R M < \infty$?

For k the answer is positive, cf. Theorem 8.2.1. The dichotomy implied by a general positive answer would be all the more remarkable for the fact, that intermediate growth does occur in most (even finitely presented!) algebraic systems: associative algebras, Lie algebras, and groups, cf. [153].

Remark 4.2.3 and Proposition 4.2.4 show that the curvatures of all R–modules (of infinite projective dimension) lie on $[1, \operatorname{curv}_R k]$, but their actual distribution is a mystery. Obviously, the first question to ask is:

Problem 4.3.8. Is the set $\{\operatorname{curv}_R M \mid M \text{ a finite } R\text{–module}\}$ finite?

We also propose an exponential version of Problem 4.3.5:

Problem 4.3.9. If $\operatorname{curv}_R M = \beta > 1$, is then $\beta_n^R(M) \sim \alpha\beta^n$ sor some $\alpha \in \mathbb{R}$?

---

[20] That class, introduced by Avramov, Gasharov, and Peeva [32], includes the modules of finite virtual projective dimension of [25], and hence all modules over a complete intersection.

[21] This *is* a GK-dimension: that of $\operatorname{Ext}_R(M,k)$ over $\operatorname{Ext}_R(k,k)$, cf. Chapter 10.

This is proved by Sun [150] over generalized Golod rings, cf. Theorem 10.3.3.2. Intermediate results are known in many other cases. Gasharov and Peeva [70], [130], prove that if R is Cohen-Macaulay of multiplicity ≤ 8, then there is a real number $\gamma > 1$, such that $\beta_{n+1}^R(M) > \gamma\beta_n^R(M)$ for $n \gg 0$. This property is called 'termwise exponential growth' by Fan [63], who extends some methods of [70] to handle artinian rings of 'large' embedding dimension. For a Cohen presentation $\widehat{R} = Q/I$ Choi [52], [53] shows that $c = \operatorname{rank}_K \big(\mathfrak{n}(I : \mathfrak{n})/\mathfrak{n}I\big)$ is an invariant of R, and that $\beta_{n+2}^R(M) > c\beta_n^R(M)$ for $n \geq 1$.

A weaker property, called 'strongly[22] exponential growth' in [26], is established when $\mathfrak{m}^3 = 0$ by Lescot [106]: for each M there is a real number $\gamma > 1$, such that $\beta_n^R(M) \geq \gamma^n$ for $n \geq 0$. This special case is significant, as Anick and Gulliksen [13] prove that each $\mathrm{P}_k^R(t)$ is rationally related to $\mathrm{P}_k^S(t)$ for some artinian k–algebra $(S, \mathfrak{n}, k)$ with $\mathfrak{n}^3 = 0$.

Rationality. The study of infinite resolutions over local rings was triggered by a question, variously and appropriately linked to the names of Kaplansky, Kostrikin, Serre, and Shafarevich: *Does* $\mathrm{P}_k^R(t)$ *represent a rational function?*

Gulliksen [79] raised the stakes, proving that a positive answer for all $(R, \mathfrak{m}, k)$ would imply that $\mathrm{P}_M^R(t)$ is rational for all R and all M. Anick [10], [11] answered the original question, with a graded artinian local k–algebra[23], such that $\mathrm{P}_k^R(t)$ is transcendental, but the following is widely open:

Problem 4.3.10. Over which R do all modules have rational Poincaré series?

Anick's construction, as reworked by Löfwall and Roos [114], is described in detail by Roos [138], Babenko [39], and Ufnarovskij [153]; these surveys, and that of Anick [12], also describe a finite CW complex, whose loop space homology has a transcendental Poincaré series. Bøgvad [44] uses the artinian examples to produces a Gorenstein ring with irrational $\mathrm{P}_k^R(t)$. Fröberg, Gulliksen and Löfwall [69] show that the rationality of the Poincaré series of the residue fields is not preserved in flat families of local rings.

Jacobsson [93] provides the shortest path to irrational Poincaré series. He also shows that the rationality of $\mathrm{P}_k^R(t)$ does not imply that of all $\mathrm{P}_M^R(t)$, hence:

Problem 4.3.11. Do all rational $\mathrm{P}_M^R(t)$ over R have a common denominator?

A result of Levin [111], cf. Corollary 6.3.7, shows that for the last two problems it suffices to consider modules of finite length. Positive solutions are obtained over generalized Golod rings in [28], cf. Theorem 10.3.3.1.

Besides the aesthetic of the formula in 'closed form' that it embodies, a rational expression for a Poincaré series has practical applications. First, it provides a recurrent relation for Betti numbers that can be useful in constructing a minimal resolution. Second, it allows for efficient estimates of the asymptotic behavior of Betti sequences; for instance, rationality implies a positive solution to Problem 4.3.7, and yields some information on Problem 4.3.2: a bounded Betti sequence is eventually periodic, cf. e.g. [26].

[22] The terminological discrepancy reflects the growth of expectations, over a decade.

[23] Of rank 13.

5. Modules over Golod rings

In this chapter $(R, \mathfrak{m}, k)$ is a local ring.

Golod [74] characterized those rings R over which the resolution of k has the fastest growth allowed by Proposition 4.1.4, that is, which have

$$\mathrm{P}_k^R(t) = \frac{(1+t)^{\operatorname{edim} R}}{1 - \sum_{j=1}^{\operatorname{codepth} R} \operatorname{rank}_k \mathrm{H}_j(K^R) t^{j+1}} \ . \tag{5.0.1}$$

They are now known as *Golod rings*. We present some highlights of the abundant information available on resolutions of modules over them. It neatly splits into two pieces, corresponding to $\operatorname{codepth} R \le 1$ and $\operatorname{codepth} R \ge 2$.

5.1. **Hypersurfaces.** A local ring with $\operatorname{codepth} R \le 1$ is called a *hypersurface*.

To account for the name, consider a minimal presentation $\widehat{R} \cong Q/I$. Lemma 4.1.3.2 yields $\operatorname{pd}_Q Q/I \le 1$, so I is principal, say $I = (f)$; in particular, regular rings are hypersurfaces. If R is a hypersurface, then $\mathrm{P}_k^R(t) = (1+t)^{\operatorname{edim} R}$ by Example 1.1.1 when it is regular, and $\mathrm{P}_k^R(t) = (1+t)^{\operatorname{edim} R}/(1-t^2)$ by Proposition 3.3.5.2 when it is singular. Thus, a hypersurface is a Golod ring.

Resolutions over hypersurface rings have a nice periodicity, discovered by Eisenbud [57]. In particular, $\operatorname{cx}_R M \le 1$ for each R–module M. Remark 8.1.1.3 contains a strong converse: if $\operatorname{cx}_R k \le 1$, then R is a hypersurface.

Theorem 5.1.1. *If M is a finite module over a hypersurface ring R, then $\beta_{n+1}^R(M) = \beta_n^R(M)$ for $n > m = \operatorname{depth} R - \operatorname{depth} M$. The minimal free resolution of M becomes periodic of period 2 after at most $m+1$ steps. It is periodic if and only if M is maximal Cohen-Macaulay without free direct summand.*

Proof. By the uniqueness of the minimal free resolution F, to say that it is periodic of period 2 after s steps amounts to saying that the syzygy modules $\operatorname{Syz}_s^R(M)$ and $\operatorname{Syz}_{s+2}^R(M)$ are isomorphic. We may test the isomorphism of these finite modules after tensoring by the faithfully flat R–module $\widehat{R}$. Since $\widehat{R} \otimes_R \operatorname{Syz}_n^R(M) \cong \operatorname{Syz}_n^{\widehat{R}}(\widehat{M})$, we may assume that the ring R is complete, and hence of the form $Q/(f)$ for some regular local ring Q.

For $m = \operatorname{depth} R - \operatorname{depth} M + 1$, Proposition 1.2.8 and Corollary 1.2.5 show that $\operatorname{Syz}_m^R(M)$ is maximal Cohen-Macaulay without free direct summand; this establishes the 'only if' part of the last assertion. As Q is regular, and $\operatorname{depth} \operatorname{Syz}_m^R(M) = \operatorname{depth} R = \operatorname{depth} Q - 1$, we have $\operatorname{pd}_Q \operatorname{Syz}_m^R(M) = 1$, so the other assertions follow from the next construction. □

Construction 5.1.2. Periodic resolutions. Let $(Q, \mathfrak{n}, k)$ be a local ring, let $f \in \mathfrak{n}$ be a regular element, and let M be a finite module over $R = Q/(f)$, with $\operatorname{pd}_Q M = 1$. By Proposition 1.2.7, its minimal resolution U over Q is of the form $0 \to U_1 \to U_0 \to 0$ with $U_1 \cong U_0 \cong Q^b$. Since $fM = 0$, the homothety $f \operatorname{id}_{U_0}$ lifts to a homomorphism $\sigma \colon U_0 \to U_1$.

Considered as a degree 1 map of complexes $\sigma \colon U \to U$, it is a homotopy between $f \operatorname{id}^U$ and 0. If $\delta \colon U_1 \to U_0$ is the differential of U, this means that $\delta\sigma = f \operatorname{id}^{U_0}$ and $\sigma\delta = f \operatorname{id}^{U_1}$. Thinking of σ and δ as $b \times b$ matrices, we get matrix equalities $\delta\sigma = fI_b = \sigma\delta$, called by Eisenbud [57] a *matrix factorization* of f. Clearly, this defines an infinite complex of free R–modules

$$F(\delta,\sigma)\colon\quad \dots \xrightarrow{R\otimes_Q\delta} R\otimes_Q U_0 \xrightarrow{R\otimes_Q\sigma} R\otimes_Q U_1 \xrightarrow{R\otimes_Q\delta} R\otimes_Q U_0 \to \cdots \qquad (*)$$

If $1\otimes u_1 \in \operatorname{Ker}(R\otimes\delta)$ for $u_1\in U_1$, then $\delta(u_1)=fu_0=\delta\sigma(u_0)$ with $u_0\in U_0$. As δ is injective, $u_1=\sigma(u_0)$, and thus $\operatorname{Ker}(R\otimes\delta)\subseteq\operatorname{Im}(R\otimes\sigma)$. By a symmetric argument $\operatorname{Ker}(R\otimes\sigma)\subseteq\operatorname{Im}(R\otimes\delta)$, so $F(\delta,\sigma)$ resolves $\operatorname{Coker}(R\otimes_Q\delta)=M$.

When M has a free direct summand, its minimal resolution is not periodic, so $F(\delta,\sigma)$ is not minimal. Conversely, if $F(\delta,\sigma)$ is not minimal, then $N=\operatorname{Im}(R\otimes\sigma)\nsubseteq\mathfrak{m}(R\otimes_Q U_1)$, so U_1 has a basis element in N. Thus, N has a free direct summand; since $N\cong\operatorname{Coker}(R\otimes\delta)\cong M$, so does M.

Iyengar [92] notes an alternative approach to the preceding construction:

Remark. The shortness of U forces $\sigma^2=0$; reversing Remark 2.2.1, we define a DG module structure of U over the Koszul complex $A=Q[y\,|\,\partial(y)=f]$, by setting $yu=\sigma(u)$ for $u\in U$. Comparison of formulas then shows that $F(\delta,\sigma)$ coincides with the resolution $F(A,U)$ of Example 3.1.2.

Modules with periodic resolutions exist over all non-linear *hypersurface sections*: the relevant example, due to Buchweitz, Greuel, and Schreyer [48], is adapted for the present purpose by Herzog, Ulrich, and Backelin [87].

Example 5.1.3. In the notation of Construction 5.1.2, assume that $f\in\mathfrak{n}^2$, so that $f=\sum_{i=1}^{e}a_is_i$, with $a_i,s_i\in\mathfrak{m}$. Let K be the Koszul complex on $s_1,\dots,s_e$, let $x_1,\dots,x_e$ be a basis of K_1, such that $\partial(x_i)=s_i$ for $i=1,\dots,e$. Left multiplication with $\sum_{i=1}^{e}a_ix_i\in K_1$ yields a map τ such that $\partial\tau+\tau\partial=f\,\mathrm{id}_K$, cf. 2.2.1. Setting $U_0=\bigoplus_i K_{2i}$ and $U_1=\bigoplus_i K_{2i+1}$, one then sees that

$$\delta=\sum_i(\partial_{2i+1}+\tau_{2i+1})\colon U_1\to U_0\qquad\text{and}\qquad\sigma=\sum_i(\partial_{2i}+\tau_{2i})\colon U_0\to U_1$$

is a matrix factorization of f over Q, so the preceding remark provides a periodic of period 2 minimal resolution of the R–module $\operatorname{Coker}(R\otimes_Q\delta)=M$.

Remark 5.1.4. A ring R *satisfies the Eisenbud conjecture* if each module of complexity 1 has a resolution that is eventually periodic of period 2. This is known for various R: complete intersections (Eisenbud, [57]); with $\operatorname{codepth} R\le 3$ (Avramov, [26]); with $\operatorname{codepth} R\le 4$ that are Gorenstein ([26]) or Cohen-Macaulay almost complete intersections (Kustin and Palmer [102]); Cohen-Macaulay of multiplicity ≤ 7 or Gorenstein of multiplicity ≤ 11 (Gasharov and Peeva, [70]); some 'determinantal' cases ([26]; Kustin [99], [100]).

On the other hand, Gasharov and Peeva [70] introduce a series of graded rings of embedding dimension 4 and multiplicity 8, setting

$$R=k[s_1,s_2,s_3,s_4]\big/\big(as_1s_3+s_2s_3\,,\ s_1s_4+s_2s_4\,,\ s_3s_4\,,\ s_1^2\,,\ s_2^2\,,\ s_3^2\,,\ s_4^2\big)$$

for some non-zero element a in a field k. It is easy to check that

$$\dots\to R^2(-n)\xrightarrow{\partial_n}R^2(-n+1)\to\dots\qquad\text{with}\quad\partial_n=\begin{pmatrix}t_1 & a^nt_3+t_4\\ 0 & t_2\end{pmatrix}$$

is a minimal exact complex of graded R–modules, hence $M=\operatorname{Coker}\partial_1$ has $\beta_n^R(M)=2$ for all $n\ge 0$. It is proved in [70] that this complex is periodic of period q if and only if q is the order of a in the multiplicative group of k, so

Eisenbud's conjecture fails when $q > 2$; similar examples over Gorenstein rings of embedding dimension 5 and multiplicity 12 are also given.

5.2. **Golod rings.** To investigate the rings satisfying (5.0.1), Golod [74] introduced in commutative algebra certain higher order homology operations, that have become an important tool for the construction and study of resolutions.

In this section, we use the shorthand notation $\overline{a} = (-1)^{|a|+1}a$.

Remark 5.2.1. Let A be a DG algebra, with $\mathrm{H}_0(A) \cong k$. We say that A admits a *trivial Massey operation*, if for some k–basis $\boldsymbol{b} = \{h_\lambda\}_{\lambda\in\Lambda}$ of $\mathrm{H}_{\geqslant 1}(A)$ there exists a function $\mu\colon \bigsqcup_{i=1}^{\infty} \boldsymbol{b}^i \to A$, such that

$$\mu(h_\lambda) = z_\lambda \in \mathrm{Z}(A) \qquad \text{with} \quad \mathrm{cls}(z_\lambda) = h_\lambda\,;$$
$$\partial\mu(h_{\lambda_1},\dots,h_{\lambda_p}) = \sum_{j=1}^{p-1}\overline{\mu(h_{\lambda_1},\dots,h_{\lambda_j})}\mu(h_{\lambda_{j+1}},\dots,h_{\lambda_p})\,.$$

The fact that A admits a trivial Massey operation means that the algebra structure on $\mathrm{H}(A)$ is highly trivial. For example, $z_{\lambda_1}z_{\lambda_2} = \pm\partial\mu(h_{\lambda_1},h_{\lambda_2})$ implies that $\mathrm{H}_{\geqslant 1}(A)\cdot\mathrm{H}_{\geqslant 1}(A) = 0$. Furthermore, as

$$|\mu(h_{\lambda_1},\dots,h_{\lambda_p})| = |h_{\lambda_1}| + \cdots + |h_{\lambda_p}| + p - 1\,,$$

if $\mathrm{H}_i(A) = 0$ for $i \gg 0$, then there are only finitely many obstructions—known as *Massey products*—to the construction of a trivial Massey operation.

Theorem 5.2.2. *If the Koszul complex K^R of a local ring $(R,\mathfrak{m},k)$ admits a trivial Massey operation μ, defined on a basis $\boldsymbol{b} = \{h_\lambda\}_{\lambda\in\Lambda}$ of $\mathrm{H}_{\geqslant 1}(K^R)$, then*

$$G_n = \bigoplus_{p+h+i_1+\cdots+i_p=n} K^R_h \otimes_R V_{i_1} \otimes_R \cdots \otimes_R V_{i_p}\,,$$

where each V_n is an R–module with basis $\{v_\lambda : n = |v_\lambda| = |h_\lambda| + 1\}_{\lambda\in\Lambda}$, and

$$\partial(a\otimes v_{\lambda_1}\otimes\cdots\otimes v_{\lambda_p}) = \partial(a)\otimes v_{\lambda_1}\otimes\cdots\otimes v_{\lambda_p} + (-1)^{|a|}\sum_{j=1}^{p} a\mu(h_{\lambda_1},\dots,h_{\lambda_j})\otimes v_{\lambda_{j+1}}\otimes\cdots\otimes v_{\lambda_p}$$

is a minimal free resolution (G,∂) of k. In particular, the ring R is Golod.

Remark 5.2.3. The theorem contains one direction of Golod's result: in [74] he also shows that, conversely, the equality of Poincaré series implies that each basis of $\mathrm{H}(K^R)$ has a trivial Massey operation, cf. also [83]. The proof given below is taken from Levin [110], and uses an idea of Ghione and Gulliksen [72].

The condition for R to be Golod means that the differentials ${}^r d_{p,q}$ of the spectral sequence 3.2.4 vanish for $r \geq 1$. Thus, the theorem is 'explained' by Guggenheim and May's description [76] of the differentials of Eilenberg-Moore spectral sequences in terms of certain *matric Massey products*, introduced by May [118]; for proofs using that approach, cf. [18], [24].

Proof. The verification that $\partial^2 = 0$ is direct, using the formulas in Remark 5.2.1. To see that G is exact, note that $K = K^R$ is naturally a subcomplex of G, and $G/K \cong G\otimes_R V$ with $\partial(g\otimes v) = \partial(g)\otimes v$. The exact sequence

$$0 \to K \to G \to G\otimes_R V \to 0$$

of complexes of R–modules has a homology exact sequence

$$\begin{aligned} \dots \to \bigoplus_j \big(\mathrm{H}_{n-j}(G) \otimes_R V_{j+2}\big) &\xrightarrow{\eth_{n+1}} \mathrm{H}_n(K) \to \mathrm{H}_n(G) \\ \to \bigoplus_j \big(\mathrm{H}_{n-j}(G) \otimes_R V_{j+1}\big) &\xrightarrow{\eth_{n}} \mathrm{H}_{n-1}(K) \to \dots \end{aligned}$$

It is clear that $\mathrm{H}_0(G) = k$, $\mathrm{H}_1(G \otimes_R V) = 0$, and that $\eth_{n+1}(1 \otimes v_\lambda) = h_\lambda$. Thus, $\eth_{n+1}$ is surjective, and the homology sequence splits. In the exact sequence

$$0 \to \mathrm{H}_2(G) \to \mathrm{H}_0(K) \otimes_R V_2 \xrightarrow{\eth_1} \mathrm{H}_1(K) \to \mathrm{H}_1(G) \to 0$$

we have $\mathrm{H}_0(K) \otimes_R V_2 \cong \mathrm{H}_1(K)$, hence $\mathrm{H}_1(G) = 0 = \mathrm{H}_2(G)$. Working backwards from here, we see that $\mathrm{H}_n(G) = 0$ for $n \geq 1$.

We induce on p to prove that $\mu(h_{\lambda_1}, \dots, h_{\lambda_p}) \in \mathfrak{m}K^R$ for all sequences $h_{\lambda_1}, \dots, h_{\lambda_p}$. If $p = 1$, then $\mu(h_\lambda) \subseteq \mathrm{Z}(K^R)$, and $\mathrm{Z}(K^R) \subseteq \mathfrak{m}K^R$ by Lemma 4.1.6.2. If $p > 1$, then the definition and the induction hypothesis imply $\partial\mu(h_{\lambda_1}, \dots, h_{\lambda_p}) \in \mathfrak{m}^2 K^R$, so $\mu(h_{\lambda_1}, \dots, h_{\lambda_p}) \in \mathfrak{m}K^R$, again by *loc. cit.*

Thus, G is a minimal resolution of k. A computation identical with the one for the upper bound in the proof of Proposition 3.3.2 shows that $\sum_n \operatorname{rank}_R G_n t^n$ is given by the right hand side of (5.0.1), so R is Golod. □

The following conditions often suffice to recognize a Golod ring,

Proposition 5.2.4. *A local ring R is Golod if some of the following hold:*

(1) $\mathrm{H}_{\geqslant 1}(K^R)$ *is generated by a set of cycles Z, such that $Z^2 = 0$.*
(2) $R/(g)$ *is Golod for a regular element $g \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$.*
(3) $R \cong Q/(f)$ *for a Golod ring $(Q, \mathfrak{n}, k)$ and a regular element $f \in \mathfrak{n} \smallsetminus \mathfrak{n}^2$.*
(4) $\widehat{R} \cong Q/I$, *where $(Q, \mathfrak{n}, k)$ is regular,* $\operatorname{edim} Q = \operatorname{edim} R$, *and the minimal resolution A of $\widehat{R}$ over Q is a DG algebra with $A_{\geqslant 1} \cdot A_{\geqslant 1} \subseteq \mathfrak{n}A$.*

Proof. (1) Select a subset $\{z_\lambda \in Z\}_{\lambda \in \Lambda}$ such that $\boldsymbol{b} = \{\mathrm{cls}(z_\lambda)\}_{\lambda \in \Lambda}$ is a basis of $\mathrm{H}_{\geqslant 1}(K^R)$. A trivial Massey operation can then be defined by setting $\mu(h_{\lambda_1}, \dots, h_{\lambda_j}) = 0$ for $j \geq 2$, so R is Golod by Theorem 5.2.2.

(2) and (3). We note that: $\operatorname{edim} R = \operatorname{edim}(R/(g)) - 1$ (obvious); $\mathrm{P}_k^R(t) = (1+t)\,\mathrm{P}_k^{R/(g)}(t)$ (Proposition 3.3.5.1); $\operatorname{rank}_k \mathrm{H}_n(K^R) = \operatorname{rank}_k \mathrm{H}_n(K^{R/(g)})$ for each n (Lemma 4.1.6.1). Putting these equalities together, we see that R satisfies the defining equality (5.0.1) if and only if $R/(g)$ does.

(4) The Golod conditions for R and $\widehat{R}$ are equivalent, so we may assume that Q/I is a regular presentation of R. By Theorem 5.2.2, it suffices to show that K^R admits a trivial Massey operation.

Choose a set of cycles $\{z_\lambda \in A \otimes_Q K^Q\}_{\lambda \in \Lambda}$, such that $\boldsymbol{b} = \{h_\lambda = \mathrm{cls}(z_\lambda)\}_{\lambda \in \Lambda}$ is a basis of $\mathrm{H}_{\geqslant 1}(A \otimes_Q K^Q)$. Set $\mu(h_\lambda) = z_\lambda$, and assume by induction that a function $\mu\colon \bigsqcup_{i=1}^{p-1} \boldsymbol{b}^i \to A \otimes_Q K^Q$ has been constructed for some $p \geq 2$, and satisfies the conditions of Remark 5.2.1. The element

$$z_{\lambda_1, \dots, \lambda_p} = \sum_{j=1}^{p-1} \overline{\mu(h_{\lambda_1}, \dots, h_{\lambda_j})}\mu(h_{\lambda_{j+1}}, \dots, h_{\lambda_p})$$

is then a cycle in $A \otimes_Q K^Q$. If $\epsilon\colon K^Q \xrightarrow{\simeq} k$ is the augmentation, then

$$(A \otimes_Q \epsilon)(z_{\lambda_1, \dots, \lambda_p}) \in (A \otimes_Q k)_{\geqslant 1} \cdot (A \otimes_Q k)_{\geqslant 1} = 0\,.$$

Since $(A \otimes_Q \epsilon)$ is a quasi-isomorphism, $z_{\lambda_1,\dots,\lambda_p}$ is a boundary, so extend μ to $\bigsqcup_{i=1}^{p} \boldsymbol{b}^i$ by choosing $\mu(h_{\lambda_1},\dots,h_{\lambda_p})$ such that $\partial(\mu(h_{\lambda_1},\dots,h_{\lambda_p})) = z_{\lambda_1,\dots,\lambda_p}$ This completes the inductive construction of a trivial Massey operation on $A \otimes_Q K^Q$. The augmentation $A \to R$ induces a surjective quasi-isomorphism $\phi\colon A \otimes_Q K^Q \to K^R$, so to get a trivial Massey operation on the basis $\mathrm{H}(\phi)(\boldsymbol{b})$ of $\mathrm{H}_{\geqslant 1}(K^R)$, set $\mu'(\phi(h_{\lambda_1}),\dots,\phi(h_{\lambda_p})) = \phi(\mu(h_{\lambda_1},\dots,h_{\lambda_p}))$. □

As a first application, we present a result of Shamash [143].

Proposition 5.2.5. *If* $\operatorname{codim} R \leq 1$*, then* R *is Golod.*

Proof. We may assume that R is a complete, and so has a minimal regular presentation $R \cong Q/I$. As Q is a catenary domain, height $I \leq 1$; since Q is factorial, there is an $f \in \mathfrak{n}$ such that $I = fJ$. In $K^R = R \otimes_Q K^Q$ each cycle of degree ≥ 1 has the form $1 \otimes z$, where $z \in K^Q$ satisfies $\partial(z) = fv$ for some $v \in K^Q$. Also, $f\partial(v) = \partial(fv) = \partial^2(z) = 0$, and so $\partial(v) = 0$. Choosing $u \in K^Q$ such that $\partial(u) = f$, we get $\partial(z - uv) = 0$. As K^Q_1 is acyclic, $z - uv = \partial(w)$ for some $w \in K^Q$. Thus, $\operatorname{cls}(1 \otimes z) = \operatorname{cls}(1 \otimes u)\operatorname{cls}(1 \otimes v)$, so $\mathrm{H}_{\geqslant 1}(K^R)$ is generated by $(1 \otimes u)\mathrm{Z}(K^R)$. As $u^2 = 0$, case (1) of Proposition 5.2.4 applies. □

Remark. It is not clear how the Golod property of a local ring relates to other characteristics of its singularity. For one thing, it does not fit in the hierarchy

$$\text{regular} \implies \text{complete intersection} \implies \text{Gorenstein} \implies \text{Cohen-Macaulay}.$$

The only Golod rings that are Gorenstein are the hypersurfaces: compare Remark 5.2.3 with the fact that if R is Gorenstein, then $\mathrm{H}(K^R)$ has Poincaré duality. On the other hand, a Golod ring may or may not be Cohen-Macaulay: compare the preceding proposition with Example 5.2.8. Furthermore, the Golod condition is not stable under localization, as demonstrated by the next example.

Example 5.2.6. Let Q be the regular ring $k[s_1,s_2,s_3]_{(s_1,s_2,s_3)}$. The ring $R = Q/(s_1^2 s_3, s_2^2 s_3)$ is Golod by Proposition 5.2.5. On the other hand, its localization at $\mathfrak{p} = (s_1,s_2)$ is the ring $\ell[s_1,s_2]/(s_1^2,s_2^2)$, where $\ell = k(s_3)$. By example 4.2.1.2, $\mathrm{P}_\ell^{R_\mathfrak{p}}(t) = 1/(1-t)^2 = (1+t)^2/(1-2t^2+t^4)$, so $R_\mathfrak{p}$ is not Golod.

Golod rings are defined by an extremal property; this might be why they appear frequently as solutions to extremal problems.

Example 5.2.7. Let Q be a graded polynomial ring generated by Q_1 over k, and let S be a residue ring of Q, such that $\operatorname{rank}_k S_1 = \operatorname{rank}_k Q_1 = e$. A (now) famous theorem of Macaulay proves that there exists a *lex-segment* monomial ideal I, such that $R = Q/I$ and S $\operatorname{rank}_k S_n = \operatorname{rank}_k R_n$ for all n, and $\beta_{1j}^Q(S) \leq \beta_{1j}^Q(R)$ for the first graded Betti number, cf˙ Remark 1.2.10.

Bigatti [42] and Hulett [90] in characteristic zero, and Pardue [128] in general, extend Macaulay's theorem to a coefficientwise inequality of Poincaré series in *two variables*

$$\mathrm{P}_S^Q(t,u) = \sum_{n,j} \beta_{nj}^Q(S) t^n u^j \preccurlyeq \sum_{n,j} \beta_{nj}^Q(R) t^n u^j = \mathrm{P}_R^Q(t,u)\,.$$

As I is a *stable* monomial ideal, $\mathrm{P}_R^Q(t,u)$ is known explicitly from the minimal free resolution A of R over Q, given by Eliahou and Kervaire [60].

Peeva [129] constructs a DG algebra structure on A, such that $A_iA_j \subseteq \mathfrak{n}A_{i+j}$ when $i \geq 1$ and $j \geq 1$. By Proposition 5.2.4.4, this implies that R is Golod. The same conclusion is obtained by Aramova and Herzog [14], who verify that condition 5.2.4.1 holds. Thus, there are (in)equalities

$$\mathrm{P}_k^S(t,u) \preccurlyeq \frac{(1+tu)^e}{1+t-t\,\mathrm{P}_S^Q(t,u)} \preccurlyeq \frac{(1+tu)^e}{1+t-t\,\mathrm{P}_R^Q(t,u)} = \mathrm{P}_k^R(t,u)\,.$$

Example 5.2.8. Let R be a Cohen-Macaulay ring of dimension d.

As the residue field of $R' = R[t]_{\mathfrak{m}[t]}$ is infinite, we can choose a regular sequence $\boldsymbol{g} = \{g_1, \dots, g_d\}$ such that the length of $R'' = R'/(\boldsymbol{g})$ is equal to $\operatorname{mult} R$, the multiplicity of R, cf. Remark 1.2.9. Such a sequence is linearly independent modulo $\mathfrak{m}^2$, hence $\operatorname{edim} R'' = \operatorname{edim} R' - d = \operatorname{codim} R' = \operatorname{codim} R$. Thus, $1 + \operatorname{edim} R'' = 1 + \operatorname{length}(\mathfrak{m}''/\mathfrak{m}''^2) = \operatorname{length}(R''/\mathfrak{m}''^2) \leq \operatorname{length} R''$. This translates to an inequality $\operatorname{codim} R \leq \operatorname{mult} R - 1$, noted by Abhyankar [1].

If equality holds, then R is said to have *minimal multiplicity*. Such a ring is Golod. Indeed, it suffices to prove that for R'. Proposition 5.2.4.2 reduces the problem to R''. As $\mathrm{Z}(K^{R''}) \subseteq \mathfrak{m}''K^{R''}$ by Lemma 4.1.6.2 and $\mathfrak{m}''^2 = 0$, Proposition 5.2.4.1 shows that R'' is Golod. Another look at R'' shows that $\operatorname{mult} R = \operatorname{type} R + 1$, so for each M with $\operatorname{pd}_R M = \infty$ Lemma 4.2.7 yields

$$\mathrm{P}_M^R(t) = f(t) + \frac{at^{m+1}}{1 - (\operatorname{mult} R - 1)t}$$

with a positive $a \in \mathbb{Z}$, and $f(t) \in \mathbb{Z}[t]$ of degree $m = \dim R - \operatorname{depth} M$.

5.3. **Golod modules.** A *Golod module* is a finite R–module M, whose Poincaré series reaches the upper bound in the inequality of Proposition 4.1.4:

$$\mathrm{P}_M^R(t) = \frac{\sum_{i \geqslant 0} \operatorname{rank}_k \mathrm{H}_i(K^M)t^i}{1 - \sum_{j \geqslant 1} \operatorname{rank}_k \mathrm{H}_j(K^R)t^{j+1}}\ . \tag{5.3.1}$$

Thus, R is a Golod ring if and only if k is a Golod module. A result of Lescot [107] establishes a tight connection between Golod conditions on a ring and on its modules; part (1) is independently due to Levin [111].

Theorem 5.3.2. (1) *If R has a Golod module $M \neq 0$, then R is a Golod ring.*

(2) *If R is a Golod ring, and M is a finite R–module, then the module* $\operatorname{Syz}_n^R(M)$ *is Golod for $n \geq p = \operatorname{edim} R - \operatorname{depth} M$.*

Proof. (1) Referring successively to the definition, Proposition 3.3.3, and the inequality at the beginning of this section, we get a sequence of (in)equalities

$$\begin{aligned}\frac{\sum_{i \geqslant 0} \operatorname{rank}_k \mathrm{H}_i(K^M)t^i}{1 - \sum_{j \geqslant 1} \operatorname{rank}_k \mathrm{H}_j(K^R)t^{j+1}} \cdot \mathrm{P}_k^Q(t) &= \mathrm{P}_M^R(t) \cdot \mathrm{P}_k^Q(t) \\ &\preccurlyeq \mathrm{P}_M^Q(t) \cdot \mathrm{P}_k^R(t) \preccurlyeq \mathrm{P}_M^Q(t) \cdot \frac{\sum_{i \geqslant 0} \operatorname{rank}_k \mathrm{H}_i(K^R)t^i}{1 - \sum_{j \geqslant 1} \operatorname{rank}_k \mathrm{H}_j(K^R)t^{j+1}}\,.\end{aligned}$$

Lemma 4.1.3 shows that the expressions at the ends are equal, so equalities hold throughout. Cancelling $\mathrm{P}_M^Q(t)$ from both sides of the last equality, we see that $\mathrm{P}_k^R(t)$ satisfies the defining formula for Golod rings.

(2) Set $M_i = \operatorname{Syz}_i^R(M)$. Tensoring the short exact sequence of complexes from the proof of Theorem 5.2.2 with M_i, we get exact sequences

$$0 \to K^{M_i} \to M_i \otimes_R G \to M_i \otimes_R (G \otimes_R V) \to 0\,.$$

Setting $q = \operatorname{codepth} R$, note that

$$\mathrm{H}_n(M_i \otimes_R G) = \operatorname{Tor}_n^R(M_i, k)\ ;$$
$$\mathrm{H}_n((M_i \otimes_R G) \otimes_R V) = \bigoplus_{j=1}^{q} \operatorname{Tor}_{n-j-1}^R(M_i, k) \otimes_k \mathrm{H}_j(K^R)\,.$$

The short exact sequences produce commutative diagrams with exact rows

$$\begin{array}{ccccc}
\operatorname{Tor}_{n+1}^R(M_i,k) & \xrightarrow{\iota_{n+1}^i} & \bigoplus_{j=1}^{q} \operatorname{Tor}_{n-j}^R(M_i,k) \otimes \mathrm{H}_j(K^R) & \longrightarrow & \mathrm{H}_n(K^{M_i}) \\
\Big\downarrow{\scriptstyle \eth_n^i} & & \Big\downarrow{\scriptstyle \bigoplus_j (\eth_j^i \otimes \mathrm{id})} & & \\
\operatorname{Tor}_{n}^R(M_{i+1},k) & \xrightarrow{\iota_{n}^{i+1}} & \bigoplus_{j=1}^{q} \operatorname{Tor}_{n-j-1}^R(M_{i+1},k) \otimes \mathrm{H}_j(K^R) & \longrightarrow & \mathrm{H}_{n-1}(K^{M_{i+1}})
\end{array}$$

where the connecting homomorphisms $\eth_n^i$ come from the exact sequences

$$0 \to M_{i+1} \to R^{\beta_i} \to M_i \to 0\,.$$

Note that $\mathrm{H}_s(K^M) = 0$ for $s > p$, so ι_s^0 is surjective for $s > p+1$. Assume by induction that ι_s^i is surjective for $s > p-i+1$. Since $\eth_j^i$ is surjective for all j, both vertical arrows are onto, so the diagram shows that ι_s^{i+1} is surjective for $s > p-i$. We conclude that ι_s^p is surjective for $s > 1$. Since the complex $G \otimes_R V$ starts in degree 2, the map ι_1^p is surjective as well. Thus,

$$\operatorname{Tor}_n^R(M_p, k) \cong \mathrm{H}_n(K^{M_p}) \oplus \bigoplus_{j=1}^{q} \operatorname{Tor}_{n-j-1}^R(M_p, k) \otimes_k \mathrm{H}_j(K^R) \qquad \text{for} \quad n \geq 0$$

and hence

$$\beta_n^R(M_p) = \operatorname{rank}_k \mathrm{H}_n(K^M) + \sum_{j=1}^{q} \beta_{n-j-1}^R(M_p) \operatorname{rank}_k \mathrm{H}_j(K^R) \qquad \text{for} \quad n \geq 0\,.$$

These numerical equalities add up to the defining equation (5.3.1). □

Betti numbers of modules over Golod rings are well documented in the next theorem. It collects results from work of several authors: Ghione and Gulliksen [72] establish the rational expression in (1); the bound on its numerator, and a proof that $\beta_{n+1}^R(M) > \beta_n^R(M)$ for $n \geq \operatorname{edim} R$, are due to Lescot [107]; the exponential growth result in (5) is established by Peeva [130], while the asymptotic formula in (3) is a consequence of the result of Sun [150].

Hypersurfaces are excluded from the statement of the theorem, as they have been dealt with in Section 5.1.

Theorem 5.3.3. *Let R be a Golod ring with* $\operatorname{codepth} R = q \geq 2$, *and let M be a finite R–module with* $\operatorname{edim} R - \operatorname{depth} M = p$. *If* $\operatorname{pd}_R M = \infty$, *then:*

(1) *There exist polynomials with positive integer coefficients, $p(t)$ of degree $p-1$ and $q(t)$ of degree $\leq q$, such that*

$$\mathrm{P}_M^R(t) = p(t) + \frac{q(t)}{1 - \sum_{j=1}^{q} \operatorname{rank}_k \mathrm{H}_j(K^R) t^{j+1}} \cdot t^p\,.$$

(2) $\operatorname{cx}_R M = \infty$ *and* $\operatorname{curv}_R M = \beta > 1$.

(3) $\beta_n^R(M) \sim \alpha\beta^n$ *for some real number* $\alpha > 0$.

(4) $\displaystyle\lim_{n\to\infty} \frac{\beta_{n+1}^R(M)}{\beta_n^R(M)} = \beta$.

(5) $\displaystyle\frac{\beta_{n+1}^R(M)}{\beta_n^R(M)} \geq \min\left\{ \frac{\beta_{i+1}^R(M)}{\beta_i^R(M)} \;\middle|\; p \leq i < p+q \right\} = \gamma > 1$ *for* $n \geq p$.

Proof. The module $M_p = \operatorname{Syz}_p^R(M)$ is Golod by Theorem 5.3.2.2, and has

$$\operatorname{edim} R - \operatorname{depth} M_p = \operatorname{depth} R - \operatorname{depth} M_p + \operatorname{codepth} R \geq q+2 .$$

Proposition 1.2.8 shows that its depth is equal to $\operatorname{depth} R$. Replacing M by M_p, we see that it suffices to prove the theorem when M is a Golod module, and $p = 0$. For simplicity, we write β_n instead of $\beta_n^R(M)$.

(1) then holds by definition.

(5) Let $\widehat{R} = Q/I$ be a minimal Cohen presentation. With $r_i = \operatorname{rank}_Q \operatorname{Syz}_i^Q(\widehat{M})$ and $s_j = \operatorname{rank}_Q \operatorname{Syz}_j^Q(\widehat{R})$, we have

$$\mathrm{P}_M^R(t) = \frac{\sum_i \operatorname{rank}_k \mathrm{H}_i(K^M)t^i}{1 - \sum_j \operatorname{rank}_k \mathrm{H}_j(K^R)t^{j+1}} = \frac{\mathrm{P}_{\widehat{M}}^Q(t)}{1 + t - t\,\mathrm{P}_{\widehat{R}}^Q(t)} = \frac{\sum_{i=1}^p r_i t^{i-1}}{1 - t\sum_{j=1}^q s_j t^{j-1}}$$

where the first equality holds because M is Golod, the second by Lemma 4.1.3.1, and the third by Proposition 3.3.5.2. This yields numerical relations

$$\beta_{n+1} = \beta_n + s_1\beta_{n-1} + \cdots + s_{q-1}\beta_{n-q+1} + r_{n+1} \quad \text{for} \quad n \geq 0 .$$

Since $s_1 \geq 1$ because R is not a hypersurface, we get $\beta_{n+1} > \beta_n$ for $n \geq 0$. Thus, $\min\{\beta_{i+1}^R(M)/\beta_i^R(M) \mid 0 \leq i < q\} = \gamma > 1$, so assume by induction that $n \geq q-1$ and $\beta_{i+1}/\beta_i \geq \gamma$ for $i \leq n$. As $r_{n+1} = 0$ for $n \geq q-1$, we have

$$\begin{aligned}\beta_{n+1} &= \beta_n + s_1\beta_{n-1} + \cdots + s_{q-1}\beta_{n-q+1} \\ &\geq \gamma\beta_{n-1} + s_1\gamma\beta_{n-2} + \cdots + s_{q-1}\gamma\beta_{n-q} = \gamma\beta_n .\end{aligned}$$

(2) is a direct consequence of (5).

(3) Let ρ be the radius of convergence of $\mathrm{P}_M^R(t)$. Since this is a series with positive coefficients, ρ is a root of the polynomial $g(t) = 1 - \sum_{j=1}^q s_j t^j$. Because $g(0) = 1 > 0$ and $g(1) = 1 - \sum \operatorname{rank}_Q \operatorname{Syz}_{j+1}^Q(R) < 0$, we have $\rho < 1$. As $g'(\rho) = -\sum_{j=1}^q j s_j \rho^{j-1} < 0$, the root ρ is simple. Let $\xi_1, \ldots, \xi_m$ be the remaining roots of $g(t)$; assuming that one of them is equal to $\zeta\rho$ for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $\zeta \neq 1$, we get

$$1 = \left| \sum_{j=1}^q s_j(\zeta\rho)^j \right| < \sum_{j=1}^q \left| s_j(\zeta\rho)^j \right| = \sum_{j=1}^q s_j\rho^j = 1$$

which is absurd. Thus, $\rho < |\xi_i|$ for $i = 1, \ldots, m$. Set $\xi_0 = \rho$, and write $\mathrm{P}_M^R(t)$ as a sum of prime fractions $\alpha_{ih}/(1 - \xi_i^{-1}t)^h$ with $\alpha_{ih} \in \mathbb{C}$ for $i = 0, \ldots, m$. Expanding each one of them by the binomial formula, we see that

$$\beta_n = \alpha\rho^{-n} + \sum_{i=1}^m \sum_{h=1}^{n_i} \alpha_{ih} \binom{h+n-1}{h-1} \xi_i^{-n} \qquad \text{with} \quad \alpha \neq 0 .$$

Thus, $\alpha = \lim_{n\longrightarrow\infty} \beta_n\rho^n > 0$; as $\rho^{-1} = \operatorname{curv}_R M = \beta$, this is what we want.

(4) is a direct consequence of (3). □

The next result is due to Scheja [139]; the proof is from [21].

Proposition 5.3.4. *Let* $e = \operatorname{edim} R$ *and* $r = \operatorname{rank}_k \mathrm{H}_1(K^R)$. *If* $\operatorname{codepth} R = 2$, *then either* R *is Golod, and then* $\mathrm{P}_k^R(t) = (1+t)^{e-1}/(1-t-(r-1)t^2)$, *or* R *is a complete intersection, and then* $\mathrm{P}_k^R(t) = (1+t)^{e-2}/(1-2t+t^2)$.

Proof. Completing if necessary, we may assume that $R \cong Q/I$, with $(Q, \mathfrak{n}, k)$ a regular local ring, and $I \subseteq \mathfrak{n}^2$. The Auslander-Buchsbaum Equality yields $\operatorname{pd}_Q R = \operatorname{depth} Q - \operatorname{depth} R = \operatorname{codepth} R = 2$. By Example 2.1.2, the minimal free resolution A of R over Q is a DG algebra, such that $A_1 A_1 \subseteq \mathfrak{n} A_2$, unless I is generated by a regular sequence. In the latter case R is a complete intersection by definition; in the former, it is Golod by Proposition 5.2.4.4. The Poincaré series come from Proposition 3.3.5.2 and formula (5.0.1), respectively. □

We complement the discussion in Section 1 with Iyengar's [92] construction of minimal resolutions over *Golod* rings of codepth 2; the very different case of complete intersection is treated by Avramov and Buchweitz [31].

Example 5.3.5. Let $R = Q/I$ be a Golod ring with $\operatorname{codepth} R = 2$. If $\operatorname{edim} R - \operatorname{depth} M = p$, then $N = \operatorname{Syz}_p^R(M)$ has $\operatorname{pd}_Q N = 2$ by Proposition 1.2.8 and Lemma 1.2.6. By Proposition 2.2.5, the minimal free resolution U of N over Q is a DG module over the DG algebra A of Example 2.1.2. If $\mathrm{F}(A, U) = F$ is the resolution of N over R given by Theorem 3.1.1, then

$$\sum_{n=0}^{\infty} \operatorname{rank}_R F_n t^n = \frac{\mathrm{P}_N^Q(t)}{1 + t - t\,\mathrm{P}_R^Q(t)} = \mathrm{P}_N^R(t)$$

because the R–module N is Golod by Theorem 5.3.2.2. Thus, F is minimal.

6. Tate resolutions

A process of killing cycles of odd degree by adjunction of divided powers—rather than polynomial—variables, was systematically used by H. Cartan in his spectacular computation [50] of the homology of Eilenberg-MacLane spaces [56]. Their potential for commutative algebra was realized by Tate[24] [151].

Section 1 presents most of Tate's paper. Section 3 contains a major theorem of Gulliksen [77] and Schoeller [140]; it is proved essentially by Gulliksen's arguments, but in a new framework motivated by Quillen's construction of the cotangent complex in characteristic 0 and developed in Section 2.

6.1. **Construction.** Let A be a DG algebra, and let $z \in A$ be a cycle.

Construction 6.1.1. Divided powers variable. The $\Bbbk$–algebra $\Bbbk\langle x\rangle$ on a *divided powers variable* x of positive even degree is the free $\Bbbk$–module with basis $\{x^{(i)} : |x^{(i)}| = i|x|\}_{i\geqslant 0}$ and multiplication table

$$x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)} \qquad \text{for} \quad i, j \geq 0\,;$$

it is customary to set $x^{(1)} = x$, $x^{(0)} = 1$, and $x^{(i)} = 0$ for $i < 0$.

Set $A\langle x\rangle^{\natural} = A^{\natural} \otimes_{\Bbbk} \Bbbk\langle x\rangle$. If $z \in A$ is cycle of *positive odd* degree, then

$$\partial\Big(\sum_i a_i x^{(i)}\Big) = \sum_i \partial(a_i)x^{(i)} + \sum_i (-1)^{|a_i|} a_i z x^{(i-1)}$$

is a differential on $A\langle x\rangle$, that extends that of A, and satisfies the Leibniz rule.

For uniformity of notation, when $|x|$ is odd $A\langle x \,|\, \partial(x) = z\rangle$ stands for the algebra $A[x \,|\, \partial(x) = z]$, described in Construction 2.1.7; when $|x| = 0$ we set $A\langle x \,|\, \partial(x) = 0\rangle = A[x \,|\, \partial(x) = 0]$, cf. Construction 2.1.8.

Example 6.1.2. Let B be a graded commutative algebra. An element $u \in B_d$ is *regular* on B if it is not invertible, and has the smallest possible annihilator: When d is even this means that $(0:_B u) = 0$, the usual concept for commutative rings; when d is odd this means that $(0:_B u) = Bu$, because $u^2 = 0$ implies $uB \subseteq (0:_B u)$.

It is easy to see that u is regular if and only if $\pi\colon B\langle x \,|\, \partial(x) = u\rangle \to B/(u)$, $\pi\big(\sum b_i x^{(i)}\big) = b_0 + (u)$, is a quasi-isomorphism. Indeed, this is classical if d is even. When d is odd, $z = \sum b_i x^{(i)}$ is a cycle precisely when $ub_i = 0$ for $i > 0$; if u is regular, then $b_i = a_i u$ for each i, so $z = b_0 + \partial\big(\sum_i (-1)^{|a_i|} a_i x^{(i+1)}\big)$; else, there is a $v \notin (u)$ such that $uv = 0$, hence $\operatorname{Ker} \mathrm{H}(\pi) \ni \operatorname{cls}(vx) \neq 0$.

A *semi-free Γ-extension* $A\langle X\rangle$ of A is a DG algebra obtained by iterated (possibly, transfinite) sequence of adjunctions of the three types of variables described above; we say that the elements of X are *Γ-variables* over A.

Remark 6.1.3. Let $A \hookrightarrow A\langle X\rangle$ be a semi-free Γ-extension, with $\partial(x_\lambda) = z_\lambda \in A$. Consider the semi-free extension $A[Y \,|\, \partial(y_\lambda) = z_\lambda]$, and let $\alpha\colon A[Y] \to A\langle X\rangle$ be

[24] From the introduction of [151]: 'Our "adjunction of variables" is a naïve approach to the exterior algebras and twisted polynomial rings familiar to topologists, and the ideas involved were clarified in my mind by conversations with John Moore.'

the morphism of DG algebras defined by $\alpha(y_\lambda) = x_\lambda$ for each λ. A simple induction yields $\alpha(y_\lambda^i) = (i!)x_\lambda^{(i)}$. Thus, if $n!$ is invertible in A, then

$$\beta\left(x_{\lambda_1}^{(i_1)}\cdots x_{\lambda_q}^{(i_q)}\right) = \frac{y_{\lambda_1}^{i_1}\cdots y_{\lambda_q}^{i_q}}{(i_1)!\cdots(i_q)!} \qquad \text{for} \quad i_1+\cdots+i_q \le n$$

defines a morphism of complexes $(A\langle X\rangle)_{\leqslant n} \to (A[Y])_{\leqslant n}$, inverse to $\alpha_{\leqslant n}$. When A_0 is a $\mathbb{Q}$–algebra, α and β are inverse isomorphisms of DG algebras.

Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. A factorization of φ through a semi-free Γ-extension $R \hookrightarrow R\langle X\rangle$ followed by a surjective quasi-isomorphism $R\langle X\rangle \to S$ is called a *Tate resolution* of S over R (or, of the R–algebra S). Revisiting the proof of Proposition 2.1.10, this time killing classes of odd degree by adjunction of divided powers variables, one gets

Proposition 6.1.4. *Each R–algebra S has a Tate resolution. If R is noetherian and S is a finitely generated R–algebra* (*and a residue ring of R*), *then such a resolution exists with all X_i finite* (*and $X_0 = \varnothing$*). □

We take a close look at the effect that an adjunctions of Γ-variables has on homology. If $\iota\colon A \hookrightarrow A\langle X\rangle$ is a semi-free Γ-extension, and $\partial(X) \subseteq A$, then $W = \{\mathrm{cls}(\partial(x)) \in \mathrm{H}(A) \mid x \in X\}$ is contained in the kernel of the algebra homomorphism $\mathrm{H}(\iota)\colon \mathrm{H}(A) \to \mathrm{H}(A\langle X\rangle)$, hence there is an induced homomorphism of graded algebras $\bar\iota\colon \mathrm{H}(A)/(W)\,\mathrm{H}(A) \to \mathrm{H}(A\langle X\rangle)$.

Let $A \hookrightarrow A\langle x \mid \partial(x) = z\rangle$ be an extension with $|z| = d$ and $w = \mathrm{cls}(z)$.

Remark 6.1.5. When $d \ge 0$ is even, there is an exact sequence of chain maps

$$0 \to A \xrightarrow{\iota} A\langle x\rangle \xrightarrow{\vartheta} A \to 0 \qquad \text{where} \qquad \vartheta(a + xb) = b\,,$$

has degree $-d-1$, and the homology exact sequence is

$$\dots \to \mathrm{H}_{n-d}(A) \xrightarrow{w} \mathrm{H}_n(A) \xrightarrow{\mathrm{H}_n(\iota)} \mathrm{H}_n(A\langle x\rangle) \xrightarrow{\mathrm{H}_n(\vartheta)} \mathrm{H}_{n-d-1}(A) \to \dots\,.$$

In a special case, it appears in most textbooks on commutative algebra: $A = K(\boldsymbol{f};Q)$ is the Koszul complex on $\boldsymbol{f}$ and $A\langle x\rangle = K(\boldsymbol{f},g;Q)$ is that on $\boldsymbol{f}\cup\{g\}$.

Remark 6.1.6. When $d > 0$ is odd, there is an exact sequence of chain maps

$$0 \to A \xrightarrow{\iota} A\langle x\rangle \xrightarrow{\vartheta} A\langle x\rangle \to 0 \qquad \text{where} \qquad \vartheta\left(\sum_i a_i x^{(i)}\right) = \sum_i a_i x^{(i-1)}$$

of degree $-d-1$, and the homology exact sequence is

$$\dots \to \mathrm{H}_{n-d}(A\langle x\rangle) \xrightarrow{\eth_{n+1}} \mathrm{H}_n(A) \xrightarrow{\mathrm{H}_n(\iota)} \mathrm{H}_n(A\langle x\rangle) \xrightarrow{\mathrm{H}_n(\vartheta)} \mathrm{H}_{n-d-1}(A\langle x\rangle) \to \dots$$

Unlike the preceding case, multiplication by w does not appear as a map in this sequence. To analyze its impact, consider the spectral sequence of the filtration $\sum_{i \leqslant p} Ax^{(i)}$. Its module ${}^2\mathrm{E}_{p,q}$ is the homology of the complex

$$\mathrm{H}_{q-d(p+1)}(A) \xrightarrow{w} \mathrm{H}_{q-dp}(A) \xrightarrow{w} \mathrm{H}_{q-d(p-1)}(A)\,,$$

so for all q there are equalities

$${}^2\mathrm{E}_{0,q} = \frac{\mathrm{H}_q(A)}{w\,\mathrm{H}_{q-d}(A)} \qquad \text{and} \qquad {}^2\mathrm{E}_{p,q} = \frac{(0:_{\mathrm{H}(A)} w)_{q-pd}}{w\,\mathrm{H}_{q-(p+1)d}(A)} \quad \text{when } p \ge 1\,.$$

Setting $s = \inf\{n \mid (0:_{\mathrm{H}(A)} w)_n \neq w\,\mathrm{H}_{n-d}(A)\}$, we get ${}^2\mathrm{E}_{p,q} = 0$ for $p \geq 1$ and $q < dp + s$. Since ${}^r d_{p,q}$ maps ${}^r\mathrm{E}_{p,q}$ to ${}^r\mathrm{E}_{p-r,\,q+r-1}$, we have ${}^2\mathrm{E}_{0,q} = {}^\infty\mathrm{E}_{0,q}$ for $q \leq 2d + s$ and ${}^2\mathrm{E}_{1,q} = {}^\infty\mathrm{E}_{1,q}$ for $q \leq 3d + s$. The spectral sequence converges to $\mathrm{H}(A\langle x\rangle)$, so for $n \leq s$ the inclusion $\iota\colon A \hookrightarrow A\langle X\rangle$ induces isomorphisms

$$\bar\iota_n\colon \frac{\mathrm{H}_n(A)}{w\,\mathrm{H}_{n-d}(A)} \cong \mathrm{H}_n(A\langle X\rangle) \quad \text{for } 0 \leq n \leq d+s\,,$$

and short exact sequences

$$0 \to \frac{\mathrm{H}_n(A)}{w\,\mathrm{H}_{n-d}(A)} \xrightarrow{\bar\iota_n} \mathrm{H}_n(A\langle x\rangle) \to \frac{(0:_{\mathrm{H}(A)} w)_{n-d-1}}{w\,\mathrm{H}_{n-2d-1}(A)} \to 0 \quad \text{for } d{+}s{+}1 \leq n \leq 2d{+}s\,.$$

We extend to the graded commutative setup a basic tool of a commutative algebraist's trade: A sequence $\boldsymbol{u} = u_1, \dots, u_i, \dots \subseteq B$ is *regular* on B if $(\boldsymbol{u}) \neq B$ and the image of u_i is regular on $B/(u_1, \dots, u_{i-1})B$ for each $i \geq 1$. The following result is from the source [151]. When $A = A_0$, the extension $A\langle X\rangle$ is an usual Koszul complex and condition (ii) means that $\mathrm{H}_n(A\langle X\rangle) = 0$ for $n > 0$, so we have an extension of the classical characterization of regular sequences. On the other hand, when $\partial^A = 0$, condition (iii) extends the homological description of regular elements in Example 6.1.2.

Proposition 6.1.7. *Let A be a DG algebra, let $\boldsymbol{w} = w_1, \dots, w_j, \dots$ be a sequence of classes in $\mathrm{H}(A)$, let $\boldsymbol{z}$ be a sequence of cycles such that $\mathrm{cls}(z_j) = w_j$ for each i, and let $\iota\colon A \hookrightarrow A\langle X \mid \partial(X) = \boldsymbol{z}\rangle$ be a semi-free Γ-extension.*

Implications (i) $\implies$ (ii) $\implies$ (iii) *then hold among the conditions:*

(i) *The sequence $\boldsymbol{w}$ is regular in $\mathrm{H}(A)$.*

(ii) *The canonical map* $\bar\iota\colon \dfrac{\mathrm{H}(A)}{(\boldsymbol{w})\,\mathrm{H}(A)} \to \mathrm{H}(A\langle X\rangle)$ *is an isomorphism.*

(iii) *The canonical map* $\mathrm{H}(\iota)\colon \mathrm{H}(A) \to \mathrm{H}(A\langle X\rangle)$ *is surjective.*

The conditions are equivalent if $\boldsymbol{w} = w_1$, or $|w_j| > 0$ for all j, or each $\mathrm{H}_i(A)$ is noetherian over $S = \mathrm{H}_0(A)$ and all w_j of degree zero are in the Jacobson radical of S. In these cases, any permutation of a regular sequence is itself regular.

Proof. Homology commutes with direct limits, so we may assume that the sequence $\boldsymbol{z}$ is finite, say $\boldsymbol{z} = z_1, \dots, z_i$, and induce on i. For $i = 1$, set $z = z_1$, $w = w_1$, and $d = |w|$. When d is even Remark 6.1.5 readily yields (iii) $\implies$ (i) $\implies$ (ii), so assume that d is odd.

(i) $\implies$ (ii). If w is regular on $\mathrm{H}(A)$, then ${}^2\mathrm{E}_{p,q} = 0$ for $p > 0$ in the spectral sequence in Remark 6.1.6, hence

$$\mathrm{H}_q(A)/w\,\mathrm{H}_{q-d}(A) = {}^2\mathrm{E}_{0,q} = {}^\infty\mathrm{E}_{0,q} = \mathrm{H}_q(A\langle X\rangle)\,.$$

(iii) $\implies$ (i). If w is not regular, then by Remark 6.1.6 the sequence

$$\mathrm{H}_{s+d+1}(A) \xrightarrow{\mathrm{H}(\iota)} \mathrm{H}_{s+d+1}(A\langle x\rangle) \to \frac{(0:_{\mathrm{H}(A)} w)_s}{w\,\mathrm{H}_{s-d}(A)} \to 0$$

is exact, with non-trivial quotient. This contradicts the surjectivity of $\bar\iota$.

Let $i > 1$, assume that the proposition has been proved for sequences of length $i-1$, set $X' = x_1, \dots, x_{i-1}$, $X = X' \cup \{x_i\}$, and consider the semi-free Γ-extensions $\iota'\colon A \to A\langle X'\rangle$ and $A\langle X'\rangle \to A\langle X'\rangle\langle x_i\rangle = A\langle X\rangle$.

(i) $\implies$ (ii). If $\boldsymbol{w} = w_1, \dots, w_i$ is regular on $\mathrm{H}(A)$, then so is the sequence $\boldsymbol{w}' = w_1, \dots, w_{i-1}$, hence $\mathrm{H}(A)/(\boldsymbol{w}')\,\mathrm{H}(A) \cong \mathrm{H}(A\langle X'\rangle)$ by the induction hypothesis. As

w_i is regular on $\mathrm{H}(A)/(\boldsymbol{w}')\,\mathrm{H}(A)$, the basis of the induction yields an isomorphism $\mathrm{H}(A)/(\boldsymbol{w})\,\mathrm{H}(A) \cong \mathrm{H}(A\langle X\rangle)$.

(iii) $\implies$ (i). Since the homomorphism $\mathrm{H}(\iota)$ factors as

$$\mathrm{H}(A) \xrightarrow{\alpha} \mathrm{H}(A\langle X'\rangle) \xrightarrow{\beta} \frac{\mathrm{H}(A\langle X'\rangle)}{w_i\,\mathrm{H}(A\langle X'\rangle)} \xrightarrow{\gamma} \mathrm{H}(A\langle X\rangle)\,,$$

where $\alpha = \mathrm{H}(\iota')$, we see that γ is onto. The basis of our induction shows that w_i is regular on $\mathrm{H}(A\langle X'\rangle)$, and that γ is bijective. Thus, $\beta\alpha$ is onto, so

$$\mathrm{H}_n(A\langle X'\rangle) = \alpha(\mathrm{H}_n(A)) + w_i\,\mathrm{H}_{n-d}(A\langle X'\rangle) \qquad \text{for} \quad n \in \mathbb{Z} \text{ and } d = |w_i|\,.$$

When $d > 0$, induction on n shows that α is surjective; when $d = 0$, the same conclusion comes from Nakayama's Lemma, which applies since $\mathrm{H}_n(A\langle X'\rangle)$ is a finite $\mathrm{H}_0(A\langle X'\rangle)$–module. The surjectivity of α and the induction hypothesis imply that $\boldsymbol{w}'$ is regular on $\mathrm{H}(A)$; thus, $\boldsymbol{w}$ is regular on $\mathrm{H}(A)$. □

Theorem 6.1.8. *Let Q be a ring, let B be a DG algebra resolution of $S = Q/(s_1,\dots,s_e)$, and choose $x_1,\dots,x_e \in B_1$ such that $\partial(x_i) = s_i$ for $i = 1,\dots,e$. For $\boldsymbol{f} = f_1,\dots,f_r \in Q$ with $f_j = \sum_{i=1}^{e} a_{ij}s_i$ for $j = 1,\dots,r$, set $z_j = \sum_{i=1}^{e} \overline{a}_{ij}x_i$, where overlines denote images in $R = Q/(\boldsymbol{f})$.*

If $\boldsymbol{f}$ is Q–regular, then $C = \overline{B}\langle x_{e+1},\dots,x_{e+r} \mid \partial(x_{e+j}) = z_j\rangle$ resolves S over R.

Proof. With $A = Q\langle Y \mid \partial(y_j) = f_j\rangle$, we have quasi-isomorphisms of DG algebras $\alpha\colon A \to R$ and $\beta\colon B \to S$, and hence induced quasi-isomorphisms

$$\overline{B} = B\otimes_Q R \xleftarrow{B\otimes_Q\alpha} B\otimes_Q A \xrightarrow{\beta\otimes_Q A} S\otimes_Q A = S\langle y_1,\dots,y_r \mid \partial(y_j) = 0\rangle\ .$$

As $z_j = (B\otimes_Q\alpha)\big(\sum_{i=1}^{e} a_{ij}x_i - y_j\big)$ and $(\beta\otimes_Q A)\big(\sum_{i=1}^{e} a_{ij}x_i - y_j\big) = -y_j$, we see that $\mathrm{H}_1(\overline{B})$ is a free module on $w_1,\dots,w_r$, where $w_j = \mathrm{cls}(z_j)$, and $\mathrm{H}(\overline{B})$ is the exterior algebra $\bigwedge \mathrm{H}_1(\overline{B})$. Thus, the sequence $w_1,\dots,w_r$ is regular on $\mathrm{H}(\overline{B})$, so $\mathrm{H}(C) = \mathrm{H}(\overline{B})/(w_1,\dots,w_r) = S$ by the preceding proposition. □

For $B = Q\langle x_1,\dots,x_e \mid \partial(x_i) = s_i\rangle$ and $t_i = \overline{s}_i \in R$, the theorem yields

Corollary 6.1.9. *If both $\boldsymbol{f}$ and $\boldsymbol{s}$ are Q–regular sequences, then the DG algebra*

$$C = R\langle x_1,\dots,x_{e+r} \mid \partial(x_i) = t_i \text{ for } i = 1,\dots,e\,;\ \partial(x_{e+j}) = z_j \text{ for } j = 1,\dots,r\rangle$$

resolves S over R. □

Remark. A result of Blanco, Majadas, and Rodicio [43] rounds off this circle of ideas: $\mathrm{H}_n(R\langle X_1,X_2\rangle) = 0$ for $n \geq 1$ if and only if $\{\mathrm{cls}(\partial(x)) \mid x \in X_2\}$ is a basis of $H = \mathrm{H}_1(R\langle X_1\rangle)$ over $S = \mathrm{H}_0(R\langle X_1\rangle)$, and the canonical map of graded algebras $\bigwedge_S H \to \mathrm{H}(R\langle X_1\rangle)$ is bijective.

In characteristic 0, the corollary holds with $R\langle X\rangle$ replaced by $R[X]$, cf. Remark 6.1.3; in general, the use of divided powers is essential:

Example 6.1.10. Let $Q = k[[s]]$, where k is a field of characteristic $p > 0$, set $R = Q/(s^{m+1})$ for some $m \geq 1$, and let t be the image of s.

For each $i \geq 0$, the DG algebra $G = R[y_1,y_2 \mid \partial(y_1) = t\,;\, \partial(y_2) = t^m y_1]$ has $G_{2i} = Ry_2^i$ and $G_{2i+1} = Ry_1y_2^i$, with $\partial(y_2^i) = it^m y_1 y^{i-1}$ and $\partial(y_1y_2^i) = ty_2^i$. Thus, $\mathrm{H}_{2ip}(G) \cong \mathrm{H}_{2ip-1}(G) \cong k$ for $i \geq 0$. If $R[Y]$ is a resolvent of k, then it contains a subalgebra isomorphic to G, and so cannot be minimal.

6.2. **Derivations.** Throughout this section, $A \hookrightarrow A\langle X\rangle$ is a semi-free Γ-extension. First, we describe a convenient basis.

Remark 6.2.1. The following conventions are in force: $x^{(i)} = 0$ and $x^{(0)} = 1$ for all $x \in X$ and all $i < 0$; when $|x|$ is odd, $x^{(i)}$ is defined only for $i \leq 1$, and $x^{(1)} = x$; when $|x| = 0$, $x^{(i)}$ stands for x^i.

Order $X = \{x_\lambda\}_{\lambda\in\Lambda}$, first by $x_\lambda < x_\mu$ if $|x_\lambda| < |x_\mu|$, then by well-ordering each X_n. For every sequence of Γ-variables $x_\mu < \cdots < x_\nu$, and every sequence of integers $i_\mu \geq 1, \ldots, i_\nu \geq 1$, the product $x_\mu^{(i_\mu)} \cdots x_\nu^{(i_\nu)}$ is called a *normal Γ-monomial* on X; its degree is $i_\mu|x_\mu| + \cdots + i_\nu|x_\nu|$; by convention, 1 is a normal monomial. The normal monomials form the *standard basis* of $A\langle X\rangle^\natural$ over $A^\natural$.

To contain a proliferation of signs, we use the *canonical bimodule structure* carried by each DG module V over a graded commutative DG algebra B. Namely, B operates on V on the right by $vb = (-1)^{|v||b|}bv$ for all $v \in V$ and $b \in B$. This operation is right associative (that is, $v(bb') = (vb)b'$), distributive, unitary, and commutes with the original action: $(bv)b' = b(vb')$.

Remark 6.2.2. Let U be a module over $A\langle X\rangle^\natural$.

A map of $\Bbbk$–modules $\vartheta\colon A\langle X\rangle \to U$, such that

$$
\begin{aligned}
\vartheta(a) &= 0 && \text{for all} \quad a \in A\,;\\
\vartheta(bb') &= \vartheta(b)b' + (-1)^{|b||\vartheta|}b\vartheta(b') && \text{for all} \quad b, b' \in A\langle X\rangle\,;\\
\vartheta(x^{(i)}) &= \vartheta(x)x^{(i-1)} && \text{for all} \quad x \in X_{\text{even}} \text{ and all } i \in \mathbb{N}\,,
\end{aligned}
$$

is called an *A–linear Γ-derivation*; it is a homomorphism of $A^\natural$–modules.

It is easy to see by induction that

$$
\vartheta(x^{(i)}) = \begin{cases} i\vartheta(x)x^{(i-1)} & \text{if } |x| = 0\,;\\ \vartheta(x)x^{(i-1)} & \text{if } |x| \geq 1\,, \end{cases}
$$

and

$$
\vartheta\left(x_{\lambda_1}^{(i_{\lambda_1})} \cdots x_{\lambda_q}^{(i_{\lambda_q})}\right) = \sum_{j=1}^{q} (-1)^{s_{j-1}} x_{\lambda_1}^{(i_{\lambda_1})} \cdots \vartheta\left(x_{\lambda_j}^{(i_{\lambda_j})}\right) \cdots x_{\lambda_q}^{(i_{\lambda_q})}
$$

where $s_j = |\vartheta|\big(i_{\lambda_1}|x_{\lambda_1}| + \cdots + i_{\lambda_j}|x_{\lambda_j}|\big)$.

In particular, ϑ is determined by its value on X. Conversely, each homogeneous map $X \to U$ extends to a (necessarily unique) A–linear Γ-derivation $\vartheta\colon A\langle X\rangle \to U$. Indeed, define the action of ϑ on the standard basis by the formulas above, and extend it by A–linearity; it suffices to check the Leibniz rule on products of normal monomials, and this is straightforward.

Let $\operatorname{Der}_A^\gamma(A\langle X\rangle, U)$ be the set of all A–linear Γ-derivations from $A\langle X\rangle$ to U. It is easy to see that $\operatorname{Der}_A^\gamma(A\langle X\rangle, U)$ is a submodule of $\operatorname{Hom}_A(A\langle X\rangle, U)$, for the operation of $A\langle X\rangle^\natural$ on the target: $(b\alpha)(b') = b\alpha(b')$ for all $b, b' \in A\langle X\rangle^\natural$.

If U is a DG module over $A\langle X\rangle$, and ϑ is an A–linear Γ-derivation, then so is $\partial\vartheta - (-1)^{|\vartheta|}\vartheta\partial$, hence $\operatorname{Hom}_A(A\langle X\rangle, U)$ contains $\operatorname{Der}_A^\gamma(A\langle X\rangle, U)$ as a DG submodule over $A\langle X\rangle$; we call it the *DG module of A–linear Γ-derivations*. If ϑ is a cycle in $\operatorname{Der}_A^\gamma(A\langle X\rangle, U)$, that is, if $\partial\vartheta = (-1)^{|\vartheta|}\vartheta\partial$, then we say that ϑ is a *chain Γ-derivation*. Clearly, if $\beta\colon U \to V$ is a homomorphism of DG modules over $A\langle X\rangle$, then $\vartheta \mapsto \beta\circ\vartheta$ is a natural homomorphism

$$\operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, \beta) : \operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, U) \to \operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, V) ,$$

of DG modules over $A\langle X\rangle$, which is a chain map if β is one.

Over a commutative ring, the derivation functor is representable by module of Kähler differentials. The next proposition establishes the representability of the functor of Γ-derivations of semi-free Γ-extensions.

Proposition 6.2.3. *There exist a semi-free DG module* $\operatorname{Diff}^{\gamma}_{A} A\langle X\rangle$ *over* $A\langle X\rangle$ *and a degree zero chain* Γ*-derivation* $d\colon A\langle X\rangle \to \operatorname{Diff}^{\gamma}_{A} A\langle X\rangle$ *such that*

(1) $\big(\operatorname{Diff}^{\gamma}_{A} A\langle X\rangle\big)^{\natural}$ *has a basis* $dX = \{dx : |dx| = |x|\}_{x\in X}$ *over* $A\langle X\rangle^{\natural}$.

(2) $d(x) = dx$ *for all* $x \in X$.

(3) $\partial(b(dx)) = \partial(b)(dx) + (-1)^{|b|} b\, d(\partial(x))$ *for all* $b \in A\langle X\rangle$.

(4) *The map* $\beta \mapsto \beta\circ d$ *is a natural in* U *isomorphism*

$$\operatorname{Hom}_{A\langle X\rangle}(\operatorname{Diff}^{\gamma}_{A} A\langle X\rangle, U) \to \operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, U)$$

of DG modules over $A\langle X\rangle$, *with inverse given by*

$$\widetilde{\vartheta}\Big(\sum_{x\in X} a_x dx\Big) = \sum_{x\in X} (-1)^{|\vartheta||a_x|} a_x \vartheta(x) .$$

Remark. We call $\operatorname{Diff}^{\gamma}_{A} A\langle X\rangle$ the *DG module of* Γ*-differentials* of $A\langle X\rangle$ over A, and d the *universal chain* Γ*-derivation* of $A\langle X\rangle$ over A.

Proof. Let D be a module with basis $dX = \{dx : |dx| = |x|\}_{x\in X}$ over $A\langle X\rangle^{\natural}$. By Remark 6.2.2, there is a unique degree zero Γ-derivation $d\colon A\langle X\rangle \to D$, such that $d(x) = dx$ for all $x \in X$. A short computation shows that $\partial\circ d - d\circ\partial\colon A\langle X\rangle \to D$ is an A–linear Γ-derivation. It is trivial on X, hence $\partial\circ d = d\circ\partial$. In particular, $\partial^2(dx)) = d(\partial^2(x)) = 0$ for all $x \in X$, so $\partial^2 = 0$.

The DG module $\operatorname{Diff}^{\gamma}_{A} A\langle X\rangle = (D, \partial)$ has the first three properties by construction. The last one is verified by inspection. □

In combination with Proposition 1.3.2, the preceding result yields:

Corollary 6.2.4. *If* $U \to V$ *is a* (*surjective*) *quasi-isomorphism of DG modules over* $A\langle X\rangle$, *then so is* $\operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, U) \to \operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, V)$. □

Construction 6.2.5. Indecomposables. Let J denote the kernel of the morphism $A \to S = \mathrm{H}_0(A\langle X\rangle)$, and let $X^{(\geqslant 2)}$ be the set of normal Γ-monomials $x_{\mu}^{(i_{\mu})} \cdots x_{\nu}^{(i_{\nu})}$ that are *decomposable*, that is, satisfy $i_{\mu} + \cdots + i_{\nu} \geq 2$. It is clear that $A + JX + AX^{(\geqslant 2)}$ is a DG submodule of $A\langle X\rangle$ over A, hence the projection $\pi\colon A\langle X\rangle \to A\langle X\rangle/\big(A + JX + AX^{(\geqslant 2)}\big)$ defines a complex of free S–modules

$$\operatorname{Ind}^{\gamma}_{A} A\langle X\rangle\colon \quad \ldots \to SX_{n+1} \xrightarrow{\delta_{n+1}} SX_n \to \ldots .$$

We call it the *complex of* Γ*-indecomposables* of the extension $A \hookrightarrow A\langle X\rangle$. It is used to construct DG Γ-derivations, by means of the next lemma.

Lemma 6.2.6. *Let V be a complex of S–modules, let U a DG module over A with $U_i = 0$ for $i < 0$, and let $\beta\colon U \to V$ a surjective quasi-isomorphism.*

For each chain map $\xi\colon \operatorname{Ind}^{\gamma}_{A} A\langle X\rangle \to V$ of degree $-n$ there exists a degree $-n$ chain Γ-derivation $\vartheta\colon A\langle X\rangle \to U$ such that $\beta\vartheta = \xi\pi$; any two such derivations are homotopic by a homotopy that is itself an A–linear Γ-derivation.

Furthermore, for each family $\{u_x \in U_0 \mid \beta(u_x) = \xi(x) \text{ for } x \in X_n\} \subseteq U$ there is a chain Γ-derivation ϑ, satisfying $\vartheta(x) = u_x$ for all $x \in X_n$.

Proof. The canonical projection $A\langle X\rangle \to \operatorname{Ind}^{\gamma}_{A} A\langle X\rangle$ is an A–linear chain Γ-derivation, so Proposition 6.2.3.4 yields a morphism $D \to \operatorname{Ind}^{\gamma}_{A} A\langle X\rangle$ of DG modules over $A\langle X\rangle$, that maps dx to x for each $x \in X$. It induces a morphism of complexes of free S–modules $S \otimes_{A\langle X\rangle} D \to \operatorname{Ind}^{\gamma}_{A} A\langle X\rangle$ that is bijective on the bases, and hence is an isomorphism. Thus, we have isomorphisms

$$\operatorname{Hom}_{A\langle X\rangle}(D, V) \cong \operatorname{Hom}_S\big(S \otimes_{A\langle X\rangle} D, V\big) \cong \operatorname{Hom}_S(\operatorname{Ind}^{\gamma}_{A} A\langle X\rangle, V)$$

On the other hand, for the semi-free module $D = \operatorname{Diff}^{\gamma}_{A} A\langle X\rangle$ over $A\langle X\rangle$, Corollary 6.2.4 gives the surjective quasi-isomorphism below

$$\operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, U) \xrightarrow{\simeq} \operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, V) \cong \operatorname{Hom}_{A\langle X\rangle}(D, V)$$

while the isomorphism comes from Proposition 6.2.3.4.

Concatenating these two sequences of morphisms, we get a surjective quasi-somorphism $\alpha\colon \operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, U) \to \operatorname{Hom}_S(\operatorname{Ind}^{\gamma}_{A} A\langle X\rangle, V)$, so we can choose a cycle $\vartheta \in \operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, U)$ with $\alpha(\vartheta) = \xi$: this is the desired chain Γ-derivation. Any two choices differ by the boundary of some $\upsilon \in \operatorname{Der}^{\gamma}_{A}(A\langle X\rangle, U)$, that is, of a Γ-derivation. Finally, observe that the $\vartheta_i = 0$ for $i < n$, and the choice of ϑ_n is only subject to the condition $\beta_0\vartheta_n = \xi_0\pi_n$, so $\vartheta_n(x) = u_x$ for $x \in X_n$ is a possible choice. □

In these notes, applications of the lemma go through the following

Proposition 6.2.7. *Assume that $A\langle X\rangle \to S$ is a quasi-isomorphism.*

If $x \in X_n \subset \operatorname{Ind}^{\gamma}_{A} A\langle X\rangle$ is such that $\overline{x} = x \in X_n + \delta_{n+1}(X_{n+1})$ generates a free direct summand of $\operatorname{Coker}\delta_{n+1}$, then there is an A-linear chain Γ-derivation $\vartheta\colon A\langle X\rangle \to A\langle X\rangle$ of degree $-n$, with $\vartheta(x) = 1$ and $\vartheta(X_n \smallsetminus \{x\}) = 0$.

Proof. Since $Sx \cap \operatorname{Im}\delta_{n+1} = 0$, the homomorphism of S–modules $\xi_n\colon SX_n \to S$ defined by $\xi_n(x) = 1$ and $\xi_n(X_n \smallsetminus \{x\}) = 0$ extends to a morphisms of complexes $\operatorname{Ind}^{\gamma}_{A} A\langle X\rangle \to S$; apply the lemma with $U = A\langle X\rangle$ and $V = S$. □

6.3. **Acyclic closures.** The notion is introduced by Gulliksen in [83], where the main results below may be found. Our approach is somewhat different, as it is based on techniques from the preceding section.

Construction 6.3.1. Acyclic closures. Let A be a DG algebra, such that A_0 is a local ring $(R, \mathfrak{m}, k)$, and each R–module $\operatorname{H}_n(A)$ is finitely generated, let $A \to S$ be a surjective augmentation, and set $J = \operatorname{Ker}(A \to S)$.

Successively adjoining finite packages of Γ-variables in degrees 1, 2, 3, etc., one arrives at a semi-free Γ-extension $A \hookrightarrow A\langle X\rangle$, such that $\operatorname{H}_0(A\langle X\rangle) = S$, and $\operatorname{H}_n(A\langle X\rangle) = 0$ for $n \neq 0$ (recall the argument for Proposition 2.1.10).

The Third Commandment imposes the following decisions:

(1) $X = X_{\geqslant 1}$;
(2) $\partial(X_1)$ minimally generates $J_0 \mod \partial_1(A_1)$;

(3) $\{\text{cls}(\partial(x)) \mid x \in X_{n+1}\}$ minimally generates $\text{H}_n(A\langle X_{\leqslant n}\rangle)$ for $n \geq 1$.

Extensions obtained in that way are called *acyclic closures* of S over A.

The set X_n is finite for each n, so we number the Γ-variables X by the natural numbers, in such a way that $|x_i| \leq |x_j|$ for $i < j$. The standard basis of Remark 6.2.1 is then indexed by infinite sequences $I = (i_1, \ldots, i_j, \ldots)$, such that i_j is a non-negative integer, $i_j \leq 1$ when $|x_j|$ is odd, and $i_j = 0$ for $j > q = q(I)$; we call such an I an *indexing sequence*, and set $x^{(I)} = x_1^{(i_1)} \cdots x_q^{(i_q)}$.

We set $|I| = \sum_{j=0}^{\infty} i_j|x_j|$. For an indexing sequence $H = (h_1, \ldots, h_j, \ldots)$, we set $I > H$ if $|I| > |H|$; when $|I| = |H|$, we set $I > H$ if there is an $\ell \geq 0$, such that $i_\ell > h_\ell$, and $i_j = h_j$ for $j > \ell$. We now have a linear order on all indexing sequences, and we linearly order the basis accordingly. Since $|x^{(I)}| = |I|$, it refines the order on the variables, and is just (an extension of) the usual *degree-lexicographic order*.

To recognize an acyclic closure when we see one, we prove:

Lemma 6.3.2. *A semi-free Γ-extension $A \hookrightarrow A\langle X\rangle$ is an acyclic closure of S if and only if $X = X_{\geqslant 1}$, $\text{H}(A\langle X\rangle) = S$, and the complex of free S–modules $\text{Ind}_A^\gamma A\langle X\rangle$ of Construction* 6.2.5 *is minimal.*

Proof. As $X_0 = \varnothing$, the complex $\text{Ind}_A^\gamma A\langle X\rangle$ is trivial in degrees ≤ 0.

Assume that $A\langle X\rangle$ is an acyclic closure. For $x' \in X_{n+1}$, write $z = \partial(x')$ as $\sum_x a_x x + w$ with $x \in X_n$, $a_x \in R$, and $w \in A\langle X_{<n}\rangle$. If $n = 1$, then $\partial(z) = 0$ means $\sum_x a_x\partial(x) \in \partial_1(A_1)$, so $a_x \in \mathfrak{m}$ by (2). If $n \geq 2$, then $\sum_x a_x \,\text{cls}(\partial(x)) = 0$, so $a_x \in \mathfrak{m}$ by (3); thus, $\text{Ind}_A^\gamma A\langle X\rangle$ is minimal.

Assume that $\text{Ind}_A^\gamma A\langle X\rangle$ is minimal. If (2) or (3) fails, then there are $x_{i_1}, \ldots, x_{i_s} \in X_{n+1}$; $a_{i_2}, \ldots, a_{i_s} \in R$; $y \in A\langle X_{\leqslant n}\rangle$, such that $x_{i_1} - a_{i_2}x_{i_2} - \cdots - a_{i_s}x_{i_s} - y$ is a cycle. In $A\langle X\rangle$, it is equal to $\partial(u + \sum_{v\in X_{n+1}} c_v v + w)$, with $u \in RX_{n+2}$, $c_v \in A_1 = J_1$, and $w \in A\langle X_{\leqslant n}\rangle$. As $\partial(c_v v) = \partial(c_v)v - c_v\partial(v) \in JX_{n+1} + JX_n + AX^{(\geqslant 2)}$ and $\partial(w) \in AX^{(\geqslant 2)}$, where $X^{(\geqslant 2)}$ is the set of decomposable Γ-monomials from Construction 6.2.5, we get $\partial(u) = x_{i_1} - a_2x_{i_2} - \cdots - a_s x_{i_s} \notin \mathfrak{m}\,\text{Ind}_A^\gamma A\langle X\rangle$. This contradicts the minimality of $\text{Ind}_A^\gamma A\langle X\rangle$. □

We are now ready to prove a key technical fact.

Lemma 6.3.3. *If $A\langle X\rangle$ is an acyclic closure of k over A, then there exist A–linear chain Γ-derivations $\vartheta_i \colon A\langle X\rangle \to A\langle X\rangle$ for $i \geq 1$, such that:*

(1)
$$\vartheta_i(x_h) = \begin{cases} 0 & \text{for } |x_h| \leq |x_i| \text{ and } h \neq i\,; \\ 1 & \text{for } h = i\,. \end{cases}$$

(2) *Each ϑ_i is unique up to an A–linear Γ-derivation homotopy.*

(3) *When I is an indexing sequence, q is such that $i_j = 0$ for $j > q$, $\vartheta^I = \vartheta_q^{i_q} \cdots \vartheta_1^{i_1}$, and H is an indexing sequence, then*

$$\vartheta^I\big(x^{(H)}\big) = \begin{cases} 0 & \text{for } H < I\,; \\ 1 & \text{for } H = I\,. \end{cases}$$

Remark. In the composition ϑ^I, the indices of ϑ_{i_j} appear in *decreasing* order.

Proof. By Lemma 6.3.2, $\text{Ind}_A^\gamma A\langle X\rangle$ is a complex of k–vector spaces with trivial differential, so derivations ϑ_i satisfying (1) and (2) are provided by Proposition 6.2.7. As the ϑ_i are Γ-derivations, (3) follows by induction on $\sum_j i_j$. □

The next result is due to Gulliksen [83].

Theorem 6.3.4. *Let A be a DG algebra, such that A_0 is a local ring $(R, \mathfrak{m}, k)$, and each R–module $\mathrm{H}_n(A)$ is finitely generated. If $A\langle X\rangle$ is an acyclic closure of k over A, then $\partial(A\langle X\rangle) \subseteq JA\langle X\rangle$, where $J_0 = \mathfrak{m}$, and $J_n = A_n$ for $n > 0$.*

Proof. Take an arbitrary $b \in A\langle X\rangle$, and write its boundary in the standard basis: $\partial(b) = \sum_H a_H x^{(H)}$. We have to prove that if $|H| = |b| - 1$, then $a_H \in \mathfrak{m}$. Assuming the contrary, we can find an indexing sequence I with $a_I \notin \mathfrak{m}$ and $a_H \in \mathfrak{m}$ for $H > I$. Using the preceding lemma, we get

$$\pm\partial_1(\vartheta^I(b)) = \vartheta^I(\partial(b)) = a_I + \sum_{H>I} a_H \vartheta^I\big(x^{(H)}\big) \equiv a_I \mod \mathfrak{m}A\langle X\rangle \ .$$

This is a contradiction, because $\partial_1(A\langle X\rangle_1) = \mathfrak{m}$. □

The important special case when $A = A_0$ is proved independently by Gulliksen [77] (using derivations) and Schoeller [140] (using Hopf algebras):

Theorem 6.3.5. *If $(R, \mathfrak{m}, k)$ is a local ring and $R\langle X\rangle$ is an acyclic closure of the R–algebra k, then $R\langle X\rangle$ is a minimal resolution of the R–module k.* □

As a first application we prove a result of Levin [111], where $\mathrm{H}_N^R(t)$ denotes the *Hilbert series* $\sum_{n=0}^{\infty} \mathrm{rank}_k(\mathfrak{m}^n N/\mathfrak{m}^{n+1}N)t^n$ of a finite R–module N.

Theorem 6.3.6. *For each finite R–module M there is an integer s such that*

$$\mathrm{P}^R_{\mathfrak{m}^i M}(t) = \mathrm{H}^R_{\mathfrak{m}^i M}(-t)\,\mathrm{P}^R_k(t) \qquad \textit{for each} \quad i \geq s\,.$$

Proof. By Theorem 6.3.5, there is a minimal resolution $U = R\langle X\rangle$ of k over R that is a semi-free DG module over the Koszul complex $K = R\langle X_1\rangle$. Choose s as in Lemma 4.1.6.3, so that for $i \geq s$ the complexes

$$C^i: \quad 0 \to \mathfrak{m}^{i-e}M \otimes_R K_e \to \dots \to \mathfrak{m}^{i-1}M \otimes_R K_1 \to \mathfrak{m}^i M \otimes_R K_0 \to 0$$

are exact (here $e = \mathrm{edim}\, R$). Fix such an i, set $N = \mathfrak{m}^i M$, and for each $p \geq 0$ set $F^p = \bigoplus_{|e_\lambda| \leqslant p} (\mathfrak{m}N \otimes_R K)e_\lambda$, where $\{e_\lambda\}_{\lambda\in\Lambda}$ is a $K^\natural$–basis of $U^\natural$.

Take $z \in \mathrm{Z}_n(\mathfrak{m}N \otimes_R U) \cap F^p$. When $p = 0$ we have

$$\mathrm{Z}_n(\mathfrak{m}N \otimes_R K) = \mathrm{Z}_n(\mathfrak{m}^{i+1}M \otimes_R K) = \partial(\mathfrak{m}^i M \otimes_R K_{n+1}) = \partial(N \otimes_R K_{n+1})$$

with the second equality due to the exactness of C^{i+1+n}. When $p > 0$, assume by induction that $\mathrm{Z}(\mathfrak{m}N \otimes_R U) \cap F^{p-1} \subseteq \partial(N \otimes_R U)$, and write $z = \sum_{\lambda\in\Lambda_p} a_\lambda e_\lambda + v$ with $v \in F^{p-1}$. Now

$$0 = \partial(z) = \sum_{\lambda\in\Lambda_p} \partial(a_\lambda)e_\lambda \pm \sum_{\lambda\in\Lambda_p} a_\lambda \partial(e_\lambda) + \partial(v)$$

implies $\partial(a_\lambda) = 0$ for $\lambda \in \Lambda_p$, hence $a_\lambda = \partial(b_\lambda)$ with $b_\lambda \in N \otimes_R K$, and

$$z = \sum_{\lambda\in\Lambda_p} \partial(b_\lambda)e_\lambda + v = \partial\bigg(\sum_{\lambda\in\Lambda_p} b_\lambda e_\lambda\bigg) \mp \sum_{\lambda\in\Lambda_p} b_\lambda \partial(e_\lambda) + v\,.$$

Since $u = \sum_{\lambda\in\Lambda_p} b_\lambda\partial(e_\lambda) \in N \otimes_R \mathfrak{m}U = \mathfrak{m}N \otimes_R U$, we see that $u + v$ lies in $\mathrm{Z}(\mathfrak{m}N \otimes_R U) \cap F^{p-1} \subseteq \partial(N \otimes_R U)$, and hence $z \in \partial(N \otimes_R U)$.

We have $\mathrm{Z}(\mathfrak{m}N \otimes_R U) \subseteq \partial(N \otimes_R U)$, so $\mathfrak{m}N \otimes_R U \subseteq N \otimes_R U$ induces the zero map in homology, that is, $\mathrm{Tor}^R\left(\mathfrak{m}^{i+1}M, k\right) \to \mathrm{Tor}^R\left(\mathfrak{m}^i M, k\right)$ is trivial. Thus, for each $n \in \mathbb{Z}$ we get an exact sequence

$$0 \to \mathrm{Tor}_n^R\left(\mathfrak{m}^i M, k\right) \to \mathrm{Tor}_n^R\left(\mathfrak{m}^i M/\mathfrak{m}^{i+1}M, k\right) \to \mathrm{Tor}_{n-1}^R\left(\mathfrak{m}^{i+1}M, k\right) \to 0$$

of k–vector spaces. They yield an equality of Poincaré series

$$\mathrm{P}^R_{\mathfrak{m}^j M}(t) + t\,\mathrm{P}^R_{\mathfrak{m}^{j+1}M}(t) = \mathrm{rank}_k(\mathfrak{m}^j M/\mathfrak{m}^{j+1}M)\,\mathrm{P}^R_k(t) \qquad (*_j)$$

for each $j \geq i$. Multiplying $(*_j)$ by $(-t)^{j-i}$, and summing the resulting equalities in $\mathbb{Z}[[t]]$ over $j \geq i$, we obtain $\mathrm{P}^R_{\mathfrak{m}^i M}(t) = \mathrm{H}^R_{\mathfrak{m}^i M}(-t)\,\mathrm{P}^R_k(t)$, as desired. □

The[25] reader will note that the argument above may be used to yield a new proof of Theorem 4.1.8. By that result, $\mathrm{P}^R_M(t) = \mathrm{P}^R_{M/\mathfrak{m}^i M}(t) - t\,\mathrm{P}^R_{\mathfrak{m}^i M}(t)$; by Hilbert theory we know that $\mathrm{H}^R_{\mathfrak{m}^i M}(t)(1-t)^{\dim M} \in \mathbb{Z}[t]$ (for each i), hence

Corollary 6.3.7. *If all R–modules of finite length have rational Poincaré series, then $\mathrm{P}^R_M(t)$ is rational for all R–modules M.* □

A second application, from Gulliksen [78], treats *partial acyclic closures*.

Proposition 6.3.8. *If $e = \mathrm{edim}\, R$, then $\mathrm{H}_{\geqslant 1}(R\langle X_{\leqslant n}\rangle)^{e+1} = 0$ for each $n \geq 1$.*

Proof. By Theorem 6.3.5, $\mathrm{Z}_i(R\langle X\rangle) = (\partial(R\langle X\rangle))_i \subseteq \mathfrak{m}(R\langle X\rangle)_i$ for $i \geq 1$, so

$$\mathrm{Z}_i(R\langle X_{\leqslant n}\rangle) = \mathrm{Z}(R\langle X\rangle) \cap R\langle X_{\leqslant n}\rangle_i \subseteq \mathfrak{m}(R\langle X\rangle) \cap R\langle X_{\leqslant n}\rangle_i = \mathfrak{m}(R\langle X_{\leqslant n}\rangle_i)\,.$$

Thus, every cycle $z \in \mathrm{Z}_i(R\langle X_{\leqslant n}\rangle)$ can be written in the form

$$z = \sum_{j=1}^{e} t_j v_j = \sum_{j=1}^{e} \partial(x_j) v_j = \sum_{j=1}^{e} x_j \partial(v_j) + \partial\biggl(\sum_{j=1}^{e} x_j v_j\biggr)\,,$$

so each element of $\mathrm{H}_{\geqslant 1}(R\langle X_{\leqslant n}\rangle)$ is represented by a cycle in $X_1 R\langle X_{\leqslant n}\rangle$. As $(X_1 R\langle X_{\leqslant n}\rangle)^{e+1} = (X_1)^{e+1} R\langle X_{\leqslant n}\rangle = 0$, we get $\mathrm{H}_{\geqslant 1}(R\langle X_{\leqslant n}\rangle)^{e+1} = 0$. □

Remark 6.3.9. To study 'uniqueness' of acyclic closures, one needs the category of *DG algebras with divided powers*: these are DG algebras, whose elements of positive even degree are equipped with a family of operations $\{a \mapsto a^{(i)}\}_{i \geqslant 0}$ that satisfy (among other things) the conditions imposed on the Γ-variables in Construction 6.1.1; they are also known as *DG Γ-algebras*[26], due to the use by Eilenberg-MacLane [56] and Cartan [50] of $\gamma_i(a)$ to denote $a^{(i)}$.

It is proved in [83] that $R\langle X\rangle$ has a unique structure of DG Γ-algebra, that extends the natural divided powers of the Γ-variables in X, and if $R\langle X'\rangle$ is an acyclic closure of S, then $R\langle X\rangle \cong R\langle X'\rangle$ as DG *Γ*-algebras over R.

[25] Attentive.

[26] This accounts for the Γ's appearing from Section 6.1 onward.

7. Deviations of a local ring

In this chapter $(R, \mathfrak{m}, k)$ is a local ring.

We describe a sequence of homological invariants of the R–module k, that are 'logarithmically' related to its Betti numbers. They are introduced formalistically in Section 1, and shown to measure the deviation of R from being regular or a complete intersection in Section 3. In Section 2 we develop tools for their study, that reduce some problems over the singular ring R to problems over a regular ring, by means of *minimal models* for R.

7.1. **Deviations and Betti numbers.** We need an elementary observation.

Remark 7.1.1. For each formal power series $P(t) = 1 + \sum_{j=1}^{\infty} b_j t^j$ with $b_j \in \mathbb{Z}$, there exist uniquely defined $e_n \in \mathbb{Z}$, such that

$$P(t) = \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{e_{2i-1}}}{\prod_{i=1}^{\infty}(1-t^{2i})^{e_{2i}}}$$

where the product converges in the (t)-adic topology of the ring $\mathbb{Z}[[t]]$.

Indeed, let $p_j(t) = (1-(-t)^j)^{(-1)^{j+1}}$. Setting $P_0(t) = 1$, assume by induction that $P_{n-1}(t) = \prod_{h=1}^{n-1} p_h(t)^{e_h}$ satisfies $P(t) \equiv P_{n-1}(t) \pmod{t^n}$ with uniquely defined e_h. If $P(t) - P_{n-1}(t) \equiv e_n t^n \pmod{t^{n+1}}$, then set $P_n(t) = P_{n-1}(t) \cdot p_n(t)^{e_n}$. The binomial expansion of $p_n(t)^{e_n}$ shows that $P(t) \equiv P_n(t) \pmod{t^{n+1}}$, and that e_n is the only integer with that property.

The exponent e_n defined by the product decomposition of the Poincaré series $\mathrm{P}_k^R(t)$ is denoted[27] $\varepsilon_n(R)$ and called the n'th *deviation* of R (for reasons to be clarified in Section 3); we set $\varepsilon_n(R) = 0$ for $n \leq 0$. Here are the first few relations between Betti numbers $\beta_n = \beta_n^R(k)$ and deviations $\varepsilon_n = \varepsilon_n(R)$:

$$\beta_1 = \varepsilon_1\,; \qquad \beta_3 = \varepsilon_3 + \varepsilon_2\varepsilon_1 + \binom{\varepsilon_1}{3};$$

$$\beta_2 = \varepsilon_2 + \binom{\varepsilon_1}{2}; \qquad \beta_4 = \varepsilon_4 + \varepsilon_3\varepsilon_1 + \binom{\varepsilon_2}{2} + \varepsilon_2\binom{\varepsilon_1}{2} + \binom{\varepsilon_1}{4}.$$

Remark 7.1.2. The equality $\mathrm{P}_k^R(t) = \mathrm{P}_k^{\widehat{R}}(t)$ and the uniqueness of the product decomposition show that $\varepsilon_n(R) = \varepsilon_n(\widehat{R})$ for all n.

A first algebraic description of the deviations is given by

Theorem 7.1.3. *If $R\langle X\rangle$ is an acyclic closure of k over R, then*

$$\operatorname{card} X_n = \varepsilon_n(R) \qquad \textit{for} \quad n \in \mathbb{Z}\,.$$

Proof. By the minimality of $R\langle X\rangle$ established in Theorem 6.3.5, we have equalities $\mathrm{Tor}^R(k,k) = \mathrm{H}(R\langle X\rangle \otimes_R k) = k\langle X\rangle = \bigotimes_{x\in X} k\langle x\rangle$. Furthermore,

$$\sum_{n=0}^{\infty} \operatorname{rank}_k(k\langle x\rangle_n) t^n = \begin{cases} 1+t^{2i-1} & \text{if} \quad x \in X_{2i-1}\,; \\ 1/(1-t^{2i}) & \text{if} \quad x \in X_{2i}\,, \end{cases}$$

[27] This numbering is at odds with [83], where ε_n stands for $\varepsilon_{n+1}(R)$.

by Constructions 2.1.7 and 6.1.1, so we get

$$\mathrm{P}^R_k(t) = \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{\operatorname{card}(X_{2i-1})}}{\prod_{i=1}^{\infty}(1-t^{2i})^{\operatorname{card}(X_{2i})}}\,.$$

The coefficient of t^n depends only on the first n factors, so the product converges in the t-adic topology, and the desired equalities follow from Remark 7.1.1. □

We record a couple of easy consequences.

Corollary 7.1.4. *$\varepsilon_n(R) \geq 0$ for all $n \in \mathbb{Z}$.* □

Corollary 7.1.5. *If $\widehat{R} = Q/I$ is a minimal regular presentation, then $\varepsilon_1(R) = \nu_Q(\mathfrak{n})$ and $\varepsilon_2(R) = \nu_Q(I)$.*

Proof. By Remark 7.1.2, we may assume that $R = Q/I$. Condition (6.3.1.0) for acyclic closures then yields the first equality, and implies that $R\langle X_1\rangle$ is the Koszul complex K^R. The second equality now comes from Lemma 4.1.3.3. □

A most important property of deviations is their behavior under change of rings: it is additive, as opposed to the multiplicative nature of Betti numbers. The logarithmic nature of the deviations will reappear in Theorem 7.4.2.

Proposition 7.1.6. *If $g \in R$ is regular, then*

$$\varepsilon_1(R/(g)) = \begin{cases} \varepsilon_1(R) - 1 & \text{if } \ g \notin \mathfrak{m}^2\,; \\ \varepsilon_1(R) & \text{if } \ g \in \mathfrak{m}^2\,; \end{cases}$$

$$\varepsilon_2(R/(g)) = \begin{cases} \varepsilon_2(R) & \text{if } \ g \notin \mathfrak{m}^2\,; \\ \varepsilon_2(R) + 1 & \text{if } \ g \in \mathfrak{m}^2\,; \end{cases}$$

$$\varepsilon_n(R/(g)) = \varepsilon_n(R) \qquad \text{for} \quad n \geq 3\,.$$

Proof. For $R' = R/(g)$, Proposition 3.3.5 yields $\mathrm{P}^R_k(t) = (1-t^2)\,\mathrm{P}^{R'}_k(t)$ if $g \in \mathfrak{m}^2$ and $\mathrm{P}^R_k(t) = (1+t)\,\mathrm{P}^{R'}_k(t)$ if $g \notin \mathfrak{m}^2$; now apply Remark 7.1.1. □

7.2. **Minimal models.** In this section $(Q, \mathfrak{n}, k)$ is a local ring.

A *minimal DG algebra* over Q is a semi-free extension $Q \hookrightarrow Q[Y]$ such that $Y = Y_{\geqslant 1}$ and the differential ∂ is *decomposable* in the sense that

$$\partial(Y_1) \subseteq QY_0^2 \qquad \text{and} \qquad \partial(Y_n) \subseteq \sum_{i=1}^{n-1} QY_{i-1}Y_{n-i} \qquad \text{for} \quad n \geq 2$$

where Y_0 denotes a minimal set of generators of $\mathfrak{n}$; the minimality condition can be rewritten in the handy format $\partial(Q[Y]) \subseteq (Y)^2Q[Y]$.

Remark 7.2.1. Along with a minimal DG algebra $Q[Y]$, we consider the residue DG algebras $Q[Y]/(Y_{<n})Q[Y] = k[Y_{\geqslant n}]$ for $n \geq 1$. Their initial homology is easy to compute: for degree reasons, the decomposability of ∂ implies that $\partial(k[Y_{\geqslant n}]_i) = 0$ when $i \leq 2n$, hence

$$\mathrm{H}_i\left(k[Y_{\geqslant n}]\right) = \begin{cases} k & \text{if } \ i = 0\,; \\ 0 & \text{if } \ 0 < i < n\,; \\ kY_i & \text{if } \ n \leq i < 2n\,. \end{cases}$$

Mimicking the proof of Lemma 6.3.2, we get a criterion for minimality:

Lemma 7.2.2. *A semi-free extension* $Q \hookrightarrow Q[Y]$ *with* $Y = Y_{\geqslant 1}$ *is minimal if and only if* $\partial(Y_1)$ *minimally generates* $\operatorname{Ker}\big(Q \to \mathrm{H}_0(Q[Y])\big) \subseteq \mathfrak{n}^2$ *and* $\partial(Y_n)$ *minimally generates* $\mathrm{H}_{n-1}(Q[Y_{\leqslant n-1}])$ *for* $n \geq 2$. □

Minimal DG algebras are 'as unique as' minimal complexes.

Lemma 7.2.3. *Each quasi-isomorphism* $\phi\colon Q[Y] \to Q[Y']$ *of minimal DG algebras over* Q *is an isomorphism.*

Proof. Consider the restrictions $\phi^{\leqslant n}\colon Q[Y_{\leqslant n}] \to Q[Y'_{\leqslant n}]$ and the morphisms $\phi^{> n} = (k \otimes_{Q[Y_{\leqslant n}]} \phi)\colon k[Y_{> n}] \to k[Y'_{> n}]$ induced by ϕ. By the preceding lemma, $\partial(Y_1)$ and $\partial(Y_1')$ are minimal sets of generators of $\operatorname{Ker}(Q \to \mathrm{H}_0(Q[Y]))$, so by Nakayama $\phi_1\colon QY_1 \to QY_1'$ is an isomorphism of Q–modules, and hence $\phi^{\leqslant 1}$ is an isomorphism of DG algebras over Q.

Assume by induction that $\phi^{\leqslant n}$ is bijective for some $n \geq 1$. By Proposition 1.3.3 then $\phi^{> n}$ is a quasi-isomorphism, hence $\mathrm{H}_{n+1}(\phi^{> n})$ is an isomorphism. By Remark 7.2.1, this is simply $\phi^{> n}_{n+1}\colon kY_{n+1} \to kY'_{n+1}$, hence $(k \otimes_{Q[Y_{\leqslant n}]} \phi^{\leqslant n+1})\colon k[Y_{\leqslant n+1}] \to k[Y'_{\leqslant n+1}]$ is bijective; by Nakayama, so is $\phi^{\leqslant n+1}$. □

If $R = Q/I$, then a *minimal model* of R over Q is a quasi-isomorphism $Q[Y] \to R$, where $Q[Y]$ is a minimal DG algebra.

Proposition 7.2.4. *Each residue ring* $R = Q/I$ *has a minimal model over* Q.

Any two minimal models are isomorphic DG algebras over Q.

Proof. Going through the construction in 2.1.10 of a resolvent of R over Q and strictly observing the Third Commandment, one gets a quasi-isomorphism $Q[Y] \to R$; the DG algebra $Q[Y]$ is minimal by Lemma 7.2.2. If $Q[Y'] \to R$ is a quasi-isomorphism from a minimal DG algebra, then by the lifting property of Proposition 2.1.9 there is a quasi-isomorphism $Q[Y] \to Q[Y']$ of DG algebras over Q; it is an isomorphism by Lemma 7.2.3. □

Remark. The proposition is from Wolffhardt [159], where minimal models are called 'special algebra resolutions'. The 'model' terminology is introduced in [23] so as to reflect the similarity with the DG algebras (over $\mathbb{Q}$) used by Sullivan [148] to encode the rational homotopy type of finite CW complexes. This parallel will bear fruits in Section 8.2.

Minimal models are not to be confused with acyclic closures: If $R = Q/I$, then the minimal model and acyclic closure of R over Q coincide in two cases only—when I is generated by a regular sequence, or when $R \supseteq \mathbb{Q}$.

It is proved in [159] when R contains a field (by using bar constructions) and in [18] in general (by using Hopf algebras) that the deviations of R can be read off a minimal model. More generally, we have:

Proposition 7.2.5. *Let* $(Q, \mathfrak{n}, k)$ *be a regular local ring.*

If $Q[Y]$ *is a semi-free extension with* $Y = Y_{\geqslant 1}$ *and* $\mathrm{H}(Q[Y]) = R$, *then for each* $n \in \mathbb{Z}$ *there is an inequality* $\operatorname{card} Y_n \geq \varepsilon_{n+1}(R)$*; equalities hold for all* n *if and only if* $\partial(Y_1) \subseteq \mathfrak{n}^2$ *and* $Q[Y]$ *is a minimal DG algebra.*

Recall that two DG algebras A and A' are *quasi-isomorphic* if there exists a sequence of quasi-isomorphisms of DG algebras $A \simeq A^1 \simeq \cdots \simeq A^m \simeq A'$, pointing in either direction. The next result is from Avramov [23].

Theorem 7.2.6. *Let $R\langle X\rangle$ be an acyclic closure of k over R, let $\widehat{R} \cong Q/I$ be a minimal Cohen presentation, and let $Q[Y]$ be a minimal model of $\widehat{R}$ over Q.*

For each $n \geq 1$ the DG algebras $R\langle X_{\leqslant n}\rangle$ and $k[Y_{\geqslant n}] = Q[Y]/(Y_{<n})$ are quasi-isomorphic, and $\operatorname{card} Y_n = \operatorname{card} X_{n+1} = \varepsilon_{n+1}(R)$ for $n \geq 0$.

Thus, a minimal model of R over Q contains essentially the same information as an acyclic closure of k over R; the model has an advantage: it is defined over a regular ring, where relations are easier to compute than over R.

In view of Proposition 6.3.8, minimal models are *homologically nilpotent*:

Corollary 7.2.7. *If $e = \operatorname{edim} R$ and $Q[Y]$ is a minimal model of $\widehat{R}$, then for each $n \geq 1$ the product of any $(e+1)$ elements of $\mathrm{H}_{\geqslant 1}(k[Y_{\geqslant n}])$ is equal to 0.* □

Wiebe [158] proves the next result through a lengthy computation.

Corollary 7.2.8. *If $\widehat{R} = Q/I$ is a minimal regular presentation and E is the Koszul complex on a minimal generating set of I, then $\varepsilon_3(R) = \nu_Q(\mathrm{H}_1(E))$.*

Proof. Remark 7.1.2, the theorem, and Lemma 7.2.2 yield $\varepsilon_3(R) = \varepsilon_3(\widehat{R}) = \operatorname{card} Y_2 = \nu_Q(\mathrm{H}_1(E))$. □

Part of Theorem 7.2.6 is generalized in

Proposition 7.2.9. *If $Q[Y]$ is a minimal DG algebra over a regular local ring Q, then there exists an acyclic closure $Q[Y]\langle X\rangle$ of k over $Q[Y]$, such that $X = \{x_y : y \in Y,\ |x_y| = |y|+1\}$ and for each $y \in Y$ there is an inclusion*

$$\partial(x_y) - y \in \sum_{j=0}^{n-1} (Y_j) Q[Y_{<n}]\langle X_{\leqslant n}\rangle \quad \text{where } n = |y|\,.$$

We start the proofs of the theorems with a couple of general lemmas.

Lemma 7.2.10. *For a (surjective) morphism of DG algebras $\phi\colon A \to A'$ and a set $\{z_\lambda\}_{\lambda\in\Lambda} \subseteq \mathrm{Z}(A)$ there is a unique (surjective) morphism of DG algebras*

$$\begin{gathered}\phi\langle X\rangle\colon A\big\langle\{x_\lambda\}_{\lambda\in\Lambda} \mid \partial(x_\lambda) = z_\lambda\big\rangle \to A'\big\langle\{x_\lambda\}_{\lambda\in\Lambda} \mid \partial(x_\lambda) = \phi(z_\lambda)\big\rangle \\ \text{with} \quad \phi\langle X\rangle|_A = \phi \quad \text{and} \quad \phi\langle X\rangle(x_\lambda) = x_\lambda \quad \text{for all } \lambda \in \Lambda\,.\end{gathered}$$

If ϕ is a quasi-isomorphism, then so is $\phi\langle X\rangle$.

Proof. The first assertion is clear. Since homology commutes with direct limits, for the second one we may assume that X is finite. By induction, it suffices to treat the case $X = \{x\}$. The result then follows from the homology exact sequences of Remarks 6.1.5 and 6.1.6, and the Five-Lemma. □

Lemma 7.2.11. *Let $A = k[Y']$ be a minimal DG algebra over k, and let $kY' \subset A$ denote the span of the variables. For a linearly independent set $Z = \{z_\lambda\}_{\lambda\in\Lambda} \subset kY' \cap \mathrm{Z}(A)$ the canonical morphism of DG algebras*

$$\begin{gathered}\xi\colon B = A\langle\{x_\lambda\}_{\lambda\in\Lambda} \mid \partial(x_\lambda) = z_\lambda\rangle \to A/(Z) \\ \text{with} \quad \xi(x_\lambda^{(i)}) = 0 \quad \text{for all } \lambda \in \Lambda \text{ and } i > 0\end{gathered}$$

is a surjective quasi-isomorphism.

Proof. Consider the subalgebra $C = k[Z]\langle \{x_\lambda\}_{\lambda\in\Lambda} \mid \partial(x_\lambda) = z_\lambda\rangle \subseteq B$. By (an easy special case of) Proposition 6.1.7, the canonical projection $\epsilon\colon C \to k$ is a quasi-isomorphism. Since B is a semi-free DG module over C, Proposition 1.3.2 shows that $\pi = B \otimes_C \epsilon\colon B \to A/(Z)$ is a quasi-isomorphism. □

Proof of Theorem 7.2.6. The canonical map $R\langle X\rangle \to \widehat{R} \otimes_R R\langle X\rangle = \widehat{R}\langle X\rangle$ is a quasi-isomorphism, and induces quasi-isomorphisms $R\langle X_{\leqslant n}\rangle \to \widehat{R}\langle X_{\leqslant n}\rangle$ for $n \geq 1$, so we assume that $R = Q/I$ and take a minimal model $\rho\colon Q[Y] \to R$.

Choose a minimal generating set $Y_0 = \{t_1, \dots, t_e\}$ of $\mathfrak{m}$ and pick $s_1, \dots, s_e \in Q$ with $\rho(s_i) = t_i$. Lemma 7.2.10 yelds a quasi-isomorphism

$$R\langle X_1 \mid \partial(x_i) = t_i\rangle \xleftarrow{\rho\langle X_1\rangle} Q[Y]\langle X_1 \mid \partial(x_i) = s_i\rangle$$

for a set of Γ-variables $X_1 = \{x_1, \dots, x_e\}$. On the other hand, as $s_1, \dots, s_e$ is a Q–regular sequence, so $Q\langle X_1\rangle \to k$ is a quasi-isomorphism, and then Proposition 1.3.2 yields a quasi-isomorphism $\pi^1\colon Q[Y]\langle X_1\rangle \to k[Y]$.

Let $n \geq 1$, and assume by induction that we have constructed surjective quasi-isomorphisms of DG algebras

$$R\langle X_{\leqslant n}\rangle \xleftarrow{\rho^n} Q[Y]\langle X_{\leqslant n}\rangle \xrightarrow{\pi^n} k[Y_{\geqslant n}]$$

The classes of $\{\partial(x) \mid x \in X_{n+1}\}$ form a basis of $\mathrm{H}_n(R\langle X_{\leqslant n}\rangle)$ by (6.3.1.2). Since ρ^n is a surjective quasi-isomorphism, these cycles are images of cycles in $Q[Y]\langle X_{\leqslant n}\rangle$, so by Lemma 7.2.10 we get a surjective quasi-isomorphism

$$R\langle X_{\leqslant n+1}\rangle = R\langle X_{\leqslant n}\rangle\langle X_{n+1}\rangle \xleftarrow{\rho^n\langle X_{n+1}\rangle} Q[Y]\langle X_{\leqslant n}\rangle\langle X_{n+1}\rangle = Q[Y]\langle X_{\leqslant n+1}\rangle\,.$$

The same lemma yields a surjective quasi-isomorphism

$$Q[Y]\langle X_{\leqslant n+1}\rangle = Q[Y]\langle X_{\leqslant n}\rangle\langle X_{n+1}\rangle \xrightarrow{\pi^n\langle X_{n+1}\rangle} k[Y_{\geqslant n}]\langle X_{n+1}\rangle\ .$$

As $\mathrm{H}_n(k[Y_{\geqslant n}]) = kY_n$ by Remark 7.2.1, the differential ∂_{n+1} induces an isomorphism $kX_{n+1} \to kY_n$. Lemma 7.2.11 yields a surjective quasi-isomorphism $\xi^{n+1}\colon k[Y_{\geqslant n}]\langle X_{n+1}\rangle \to k[Y_{\geqslant n+1}]$. We set $\pi^{n+1} = \xi^{n+1}\circ\pi\langle X_{n+1}\rangle$, and note that $\mathrm{card}(X_{n+1}) = \varepsilon_{n+1}(R)$ by Theorem 7.1.3. To complete the induction step, set $\rho^{n+1} = \rho^n\langle X_{n+1}\rangle$.

To complete the induction step set $\rho^{n+1} = \rho^n\langle X_{n+1}\rangle$. □

Proof of Proposition 7.2.5. Choose $\boldsymbol{g} = g_1, \dots, g_s \in I$ and $\boldsymbol{f} = f_1, \dots, f_r \in I$ such that $\boldsymbol{f} \cup \boldsymbol{g}$ is a minimal set of generators of I, and $\boldsymbol{g}$ maps to a basis of $I/(I \cap \mathfrak{n}^2)$. The sequence $\boldsymbol{g}$ is then Q–regular, and $R = Q'/I'$ is a minimal regular presentation, with $Q' = Q/(\boldsymbol{g})$ and $I' = I/(\boldsymbol{g})$; set $\mathfrak{n}' = \mathfrak{n}/(\boldsymbol{g})$.

After a linear change of the variables of degree 1, we may assume that there is a sequence of variables $\boldsymbol{y} = y_1, \dots, y_s$ in Y_1, such that $\partial(y_i) = g_i$ for $1 \leq i \leq s$ The Koszul complex $Q[\boldsymbol{y} \mid \partial(y_i) = g_i]$ is then a resolvent of Q', so $Q[Y] \to Q[Y]/(\boldsymbol{g}, \boldsymbol{y}) = Q'[Y']$ is a quasi-isomorphism by Proposition 1.3.2. As $\mathrm{card}(Y'_n) = \mathrm{card}(Y_n) - s$ for $n = 0, 1$, and $\mathrm{card}(Y'_n) = \mathrm{card}(Y_n)$ otherwise, we may assume that $R = Q/I$ is a minimal presentation, and $Q[Y]$ is a semi-free extension with $\mathrm{H}(Q[Y]) = R$; we then have $\mathrm{card}(Y_0) = \varepsilon_1(R)$.

Let $Q[Y']$ be a minimal model of R over Q, so that $\mathrm{card}(Y'_n) = \varepsilon_{n+1}(R)$ by Theorem 7.2.6. Proposition 2.1.9 yields morphisms $\gamma\colon Q[Y'] \to Q[Y]$ and $\beta\colon Q[Y] \to Q[Y']$ of DG algebras over Q, such that $\mathrm{H}(\beta) = \mathrm{id}^R$. By Lemma 7.2.3, $\beta\gamma$ is an automorphism of $Q[Y']$, so for each $n \geq 1$, the composition of

$\gamma^{\geqslant n}\colon k[Y'_{\geqslant n}] \to k[Y_{\geqslant n}]$ with $\beta^{\geqslant n}\colon k[Y_{\geqslant n}] \to k[Y'_{\geqslant n}]$ is then bijective. In particular, $\beta^{\geqslant n}$ is onto, so $\operatorname{card}(Y_n) \geq \operatorname{card}(Y'_n) = \varepsilon_{n+1}(R)$ for $n \geq 1$.

Assume that $\operatorname{card}(Y_n) = \varepsilon_{n+1}(R)$ for all n. The equalities for $n \leq 1$ mean that $I \subseteq \mathfrak{n}^2$ and $\partial(Y_1)$ minimally generates I; equality for $n \geq 2$ implies that $\mathrm{H}_n(\beta^{\geqslant n})$ is bijective, hence in $k[Y_{\geqslant n}]$ the differential ∂_{n+1} is trivial; this is equivalent to saying that in $Q[X]$ the differential ∂_{n+1} is decomposable. □

Proof of Proposition 7.2.9. We first construct surjective quasi-isomorphisms

$$\pi^n\colon Q[Y]\langle X_{\leqslant n}\rangle \to k[Y_{\geqslant n}] \quad \text{with } X_n = \{x_y : y \in Y_{n-1}\}\,.$$

Let $Y_0 = \{s_1, \dots, s_e\}$ be a system of generators of $\mathfrak{n}$, set $X_1 = \{x_1, \dots, x_e\}$, and $Q[Y]\langle X_1 \,|\, \partial(x_i) = s_i\rangle$. As in the proof of Theorem 7.2.6 we have a quasi-isomorphism $\pi^1\colon Q[Y]\langle X_1\rangle \to k[Y]$.

Assume that π^n has been constructed for some $n \geq 1$. Now each $y \in Y_n$ is a cycle in $k[Y_{\geqslant n}]$, so $y = \pi^n(z_y)$ for some cycle z_y; write z_y in the form $y + \sum_{y' \in Y'_n} a_{y'}y' + v$, with $a_{y'} \in \mathfrak{n}$ for $y' \in Y'_n = Y_n \smallsetminus \{y\}$, and $v \in Q[Y_{<n}]\langle X_{\leqslant n}\rangle$. Since $a_{y'} = \partial(b_{y'})$ for appropriate $b_{y'} \in QX_1$, after replacing z_y with the homologous cycle $z_y - \partial\big(\sum_{y'\in Y'_n} b_{y'}y'\big)$, we can assume that $z_y - y \in Q[Y_{<n}]\langle X_{\leqslant n}\rangle$. For each $y \in Y_n$ set $\partial(x_y) = z_y$. Lemma 7.2.10 then yields a surjective quasi-isomorphism

$$Q[Y]\langle X_{\leqslant n+1}\rangle = Q[Y]\langle X_{\leqslant n}\rangle\langle X_{n+1}\rangle \xrightarrow{\pi^n\langle X_{n+1}\rangle} k[Y_{\geqslant n}]\langle X_{n+1}\rangle\ .$$

Lemma 7.2.11 yields a surjective quasi-isomorphism $\xi^n\colon k[Y_{\geqslant n}]\langle X_{n+1}\rangle \to k[Y_{\geqslant n+1}]$.

In the limit, we get a quasi-isomorphism $\pi\colon Q[Y]\langle X\rangle \to k$. As $\partial(X_1) = Y_0$ minimally generates $\mathfrak{n}$, and $\partial(X_n)$ minimally generates $\mathrm{H}_n(Q[Y]\langle X_{\leqslant n}\rangle)$ for $n \geq 2$, the DG algebra $Q[Y]\langle X\rangle$ is an acyclic closure of k over $Q[Y]$. By the choice of z_y above, and Theorem 6.3.4, we see that $\partial(x_y) - y$ lies in

$$\big(Q[Y_{<n}]\langle X_{\leqslant n}\rangle\big)_n \cap \big((Y)Q[Y]\langle X\rangle\big)_n = \sum_{j=0}^{n-1} \big((Y_j)Q[Y_{<n}]\langle X_{\leqslant n}\rangle\big)_n\,.$$

This is the desired condition on the differential. □

7.3. **Complete intersections.** Now we can 'explain' the term *deviation*.

Remark 7.3.1. By Corollary 7.1.5 $\varepsilon_1(R) = \operatorname{edim} R$, so the following conditions are equivalent: (i) R is a field; (ii) $\varepsilon_1(R) = 0$; (iii) $\varepsilon_i(R) = 0$ for $i \geq 1$.

More generally, the regularity of a ring is detected by its deviations.

Theorem 7.3.2. *The following conditions are equivalent.*

(i) *R is regular.*
(ii) $\varepsilon_2(R) = 0$.
(iii) $\varepsilon_n(R) = 0$ *for* $n \geq 2$.

Proof. When R is regular, the Koszul complex K^R is exact, yielding $\mathrm{P}^R_k(t) = (1+t)^{\dim R}$; thus, (i) $\implies$ (iii) by the uniqueness of the product decomposition (7.1.1). On the other hand, (ii) $\implies$ (i) by Corollary 7.1.5. □

Recall that R is a *complete intersection* if $\widehat{R}$ has a minimal Cohen presentation Q/I, with I generated by a regular sequence; in that case, $\varepsilon_2(R) = \operatorname{codim} R$ by Corollary 7.2.8. Complete intersections of codimension 0 (respectively, ≤ 1) are precisely the regular (respectively, hypersurface) rings.

The next result gives characterizations of complete intersections in terms of vanishing of deviations, due to Assmus [15] for (iii) and to Gulliksen [78] for (iv) and [82] for (v). The use of minimal models in their proofs is new.

Theorem 7.3.3. *The following conditions are equivalent.*

(i) *R is a complete intersection.*
(ii) $\varepsilon_3(R) = 0$.
(iii) $\varepsilon_n(R) = 0$ *for* $n \geq 3$.
(iv) $\varepsilon_n(R) = 0$ *for* $n \gg 0$.
(v) $\varepsilon_{2i}(R) = 0$ *for* $i \gg 0$.

Proof. We may take $R \cong Q/I$ with $(Q, \mathfrak{n}, k)$ regular, $\boldsymbol{g} = f_1, \dots, f_c$ minimally generating I, and $\boldsymbol{g} \subseteq \mathfrak{n}^2$, cf. Remark 7.1.2; let E be the Koszul complex on $\boldsymbol{g}$.

(i) $\implies$ (iii). If $\boldsymbol{g}$ is a Q–regular sequence, then $c = \operatorname{codim} R$, so the deviations of R are computed by Theorem 7.3.2 and Proposition 7.1.6.

(ii) $\implies$ (i). By Corollary 7.2.8 we have $\mathrm{H}_1(E) = 0$, so $\boldsymbol{g}$ is Q–regular.

(v) $\implies$ (iv) Take n big enough, so that $\varepsilon_{2i}(R) = 0$ for $2i \geq n$. By Theorem 7.2.6, the DG algebra $k[Y_{\geqslant n}]$ is a polynomial ring with variables of even degree. Their boundaries have odd degree, so are trivial, and thus $\mathrm{H}(k[Y_{\geqslant n}]) = k[Y_{\geqslant n}]$ is a polynomial ring. Each element of $\mathrm{H}_{\geqslant 1}(k[Y_{\geqslant n}])$ is nilpotent by Corollary 7.2.7, so we conclude that $Y_{\geqslant n} = \varnothing$.

(iv) $\implies$ (i) Taking, as we may, $\boldsymbol{f} = f_1, \dots, f_c$ to be a maximal regular sequence in I, we set $R' = Q/(\boldsymbol{f})$. By Corollary 6.1.9, the R'–algebra k has a minimal free resolution of the form $C = R'\langle x_1, \dots, x_{e+c}\rangle$. By the choice of $\boldsymbol{f}$, the DG algebra $C^0 = C \otimes_{R'} R$ can be extended to an acyclic closure $R\langle X\rangle$ of k by adjunction of Γ-variables of degree ≥ 2. We order them in such a way that $|x_i| \leq |x_j|$ for $i < j$, set $C^i = C^0\langle x_{e+c+1}, \dots, x_{e+c+i}\rangle$, and $s_i = \sup\{n \mid \mathrm{H}_n(C^i) \neq 0\}$. Assuming that R is not a complete intersection, we prove that X is infinite by showing that $s_i = \infty$ for each $i \geq 0$.

Each element of I is a zero-divisor modulo $(\boldsymbol{f})$, so $\operatorname{pd}_{R'}(R'/IR') = \infty$ by Proposition 1.2.7.2; as $\mathrm{H}(C^0) = \operatorname{Tor}^{R'}(R, k)$, we get $s_0 = \infty$. For the induction step, set $A = C^{i-1}$, $x = x_{e+c+i}$, $z = \partial(x)$, and $u = \operatorname{cls}(z)$, and assume that $s_i = s < \infty$. When $|x|$ is even, the homology exact sequence in Remark 6.1.6 yields $s_{i-1} \leq s + |x|$, contradicting the induction hypothesis. When $|x|$ is odd the sequence in Remark 6.1.5 shows that $\mathrm{H}(A) = \mathrm{H}_{\leqslant s}(A) + u\,\mathrm{H}(A)$. A simple iteration yields $\mathrm{H}(A) = \mathrm{H}_{\leqslant s+e|u|}(A) + u^{e+1}\,\mathrm{H}(A)$. But $u^{e+1} = 0$ by Proposition 6.3.8, hence $s_{i-1} \leq s + e|u|$; this contradiction proves that $s_i = \infty$. □

The last result is vastly generalized in Halperin's [84] rigidity theorem:

Theorem 7.3.4. *If $\varepsilon_n(R) = 0$ and $n > 0$ then R is a complete intersection.* ■

A proof that uses techniques developed in Section 6.2, and extends the theorem to a relative situation, is given in [30].

7.4. **Localization.** The theme of the preceding section may be summarized as follows: The deviations of a local ring reflect the character of its singularity. Thus, one would expect that they do not go up under localization, and in particular that the complete intersection property localizes.

It is instructive to generalize the discussion by considering the number

$$\operatorname{cid}(R) = \varepsilon_2(R) - \varepsilon_1(R) + \dim R$$

that in view of the next lemma we[28] call the *complete intersection defect* of R. The lemma also shows that in the definition of complete intersection the restriction to minimal presentations is spurious.

Lemma 7.4.1. *If $\widehat{R} \cong Q/I$ is a regular presentation, then*

$$\operatorname{cid} R = \nu_Q(I) - \operatorname{height}(I) \geq 0\,.$$

Furthermore, the following conditions are equivalent: (i) $\operatorname{cid} R = 0$ *;* (ii) *I is generated by a regular sequence;* (iii) *R is a complete intersection,*

Proof. Choose a regular sequence $\boldsymbol{g} = g_1, \dots, g_s$ as in the proof of Proposition 7.2.5. With $Q' = Q/(\boldsymbol{g})$ and $I' = I/(\boldsymbol{g})$ we have a minimal Cohen presentation $\widehat{R} \cong Q'/I'$. Corollary 7.1.5 provides the first equality below, the catenarity of Q' yields the last one, and Krull's Principal Ideal Theorem gives the inequality:

$$\begin{aligned}\operatorname{cid} R &= \nu_{Q'}(I') - \nu_{Q'}(\mathfrak{n}') + \dim R = \nu_Q(I) - \nu_Q(\mathfrak{n}) + \dim R\\ &= \nu_Q(I) - (\dim Q - \dim R) = \nu_Q(I) - \operatorname{height}(I) \geq 0\,.\end{aligned}$$

The equivalence (i) $\iff$ (ii) now follows from the Cohen-Macaulay Theorem. Applied to the minimal presentation $\widehat{R} \cong Q'/I'$, it yields (ii) $\iff$ (iii). □

For the study of complete intersection defects and even deviations, the first part of the next theorem[29] suffices; it is due to Avramov [20]. For odd deviations one needs the second part, due to André [9].

Theorem 7.4.2. *If $R \to S$ is a faithfully flat homomorphism of local rings, then for each $i \geq 1$ there is an integer $\delta_i \geq 0$, such that*

$$\begin{aligned}\varepsilon_{2i}(R) \leq \varepsilon_{2i}(S) &= \varepsilon_{2i}(R) + \varepsilon_{2i}(S/\mathfrak{m}S) - \delta_i\,;\\ \varepsilon_{2i-1}(S/\mathfrak{m}S) \leq \varepsilon_{2i-1}(S) &= \varepsilon_{2i-1}(R) + \varepsilon_{2i-1}(S/\mathfrak{m}S) - \delta_i\,.\end{aligned}$$

Furthermore, $\delta_i = 0$ for $i \gg 0$, and $\sum_{i=0}^{\infty} \delta_i \leq \operatorname{codepth}(S/\mathfrak{m}S)$. ■

As in [20], we deduce:

Theorem 7.4.3. *If $R \to S$ is a flat local homomorphism, then*

$$\operatorname{cid} S = \operatorname{cid} R + \operatorname{cid}(S/\mathfrak{m}S)\,.$$

In particular, S is a complete intersection if and only if both R and $S/\mathfrak{m}S$ are.

Proof. The first two equalities of Theorem 7.4.2 yield

$$\varepsilon_2(S) - \varepsilon_1(S) = \varepsilon_2(R) - \varepsilon_1(R) + \varepsilon_2(S/\mathfrak{m}S) - \varepsilon_1(S/\mathfrak{m}S)\,.$$

Classically, $\dim S = \dim R + \dim(S/\mathfrak{m}S)$, so we have the desired result. □

Corollary 7.4.4. *For each prime ideal $\mathfrak{p}$ of R, $\operatorname{cid}(R_{\mathfrak{p}}) \leq \operatorname{cid} R$.*

Proof. As $\dim R = \dim \widehat{R}$, we have $\operatorname{cid} R = \operatorname{cid} \widehat{R}$ by Remark 7.1.2. Let $\widehat{R} \cong Q/I$ be a regular presentation, and pick prime ideals $\mathfrak{p}' \subseteq \widehat{R}$ and $\mathfrak{q} \subseteq Q$, such that $\mathfrak{p}' \cap R = \mathfrak{p}$ and $\mathfrak{p}' = \mathfrak{q}\widehat{R}$. As $\widehat{R}_{\mathfrak{p}'} \cong Q_{\mathfrak{q}}/I_{\mathfrak{q}}$ is a regular presentation,

$$\operatorname{cid} \widehat{R} = \nu_Q(I) - \operatorname{height}(I) \geq \nu_{Q_{\mathfrak{q}}}(I_{\mathfrak{q}}) - \operatorname{height}(I_{\mathfrak{q}}) = \operatorname{cid}(\widehat{R}_{\mathfrak{p}'})\,,$$

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[28] Kiehl and Kunz introduced it in [96] by the expression in Lemma 7.4.1, and called it the *deviation* of R; that was before an infinite supply of deviations appeared on the scene.

[29] For a more natural statement, cf. Remark 10.2.4.

with equalities coming from Lemma 7.4.1, and inequality from the obvious relations $\nu_Q(I) \geq \nu_{Q_\mathfrak{q}}(I_\mathfrak{q})$ and $\operatorname{height}(I) \leq \operatorname{height}(I_\mathfrak{q})$. Finally, the theorem applied to the flat homomorphism $R_\mathfrak{p} \to \widehat{R}_{\mathfrak{p}'}$ yields $\operatorname{cid}(\widehat{R}_{\mathfrak{p}'}) \geq \operatorname{cid}(R_\mathfrak{p})$. □

It is now clear that complete intersections localize, a fact initially proved in [19]. This is sharpened in the corollary of the next theorem from [20], [9], which represents a quantitative extension to arbitrary local rings of the classical localization of regularity, cf. Corollary 4.1.2.

Theorem 7.4.5. *If $\mathfrak{p}$ is a prime ideal of R, then an inequality $\varepsilon_n(R_\mathfrak{p}) \leq \varepsilon_n(R)$ holds for all even n and for almost all odd n. When R is a residue ring of a regular local ring the inequalities hold for all n.*

Proof. If R is a residue ring of a regular local ring Q, let $Q[Y]$ be a minimal model of R. If $\mathfrak{q}$ is the inverse image of $\mathfrak{p}$ in Q, then $Q_\mathfrak{q}[Y]$ is a semi-free extension of $Q_\mathfrak{q}$ with $\mathrm{H}(Q_\mathfrak{q}[Y]) \cong R_\mathfrak{p}$. Applying Proposition 7.2.5 first to $Q[Y]$, then to $Q_\mathfrak{q}[Y]$, we get $\varepsilon_{n+1}(R) = \operatorname{card}(Y_n) \geq \varepsilon_{n+1}(R_\mathfrak{p})$.

In general, pick (by faithful flatness) a prime ideal $\mathfrak{p}'$ in $\widehat{R}$, such that $\mathfrak{p} = \mathfrak{p}' \cap R$. By Remark 7.1.2 and the preceding case we then have $\varepsilon_n(R) = \varepsilon_n(\widehat{R}) \geq \varepsilon_n(\widehat{R}_{\mathfrak{p}'})$ for all n. On the other hand, Theorem 7.4.2 yields an inequality $\varepsilon_n(\widehat{R}_{\mathfrak{p}'}) \geq \varepsilon_n(R_\mathfrak{p})$ for all even n, and almost all odd n. □

Corollary 7.4.6. *If R is a complete intersection, then for each prime ideal $\mathfrak{p}$ of R the ring $R_\mathfrak{p}$ is a complete intersection with* $\operatorname{codim}(R_\mathfrak{p}) \leq \operatorname{codim} R$. □

Proof. In view of Theorem 7.3.3, the inequalities of deviations for $n = 2i \gg 0$ prove that $R_\mathfrak{p}$ is a complete intersection. Since $\operatorname{codim} R = \varepsilon_2(R)$ by Corollary 7.1.5, the inequality for $n = 2$ shows that $\operatorname{codim}(R_\mathfrak{p}) \leq \operatorname{codim} R$. □

For the first deviation, there is a more precise result of Lech [104]; a simpler proof is given by Vasconcelos [154].

Theorem 7.4.7. *For each prime ideal $\mathfrak{p}$ of R, $\varepsilon_1(R_\mathfrak{p}) + \dim(R/\mathfrak{p}) \leq \varepsilon_1(R)$.* ■

We spell out the obvious remaining problems. The first one has a positive solution when $\operatorname{char}(k) = 2$, due to André [8].

Problem 7.4.8. Let $R \to S$ be a faithfully flat homomorphism of local rings. Does an equality $\varepsilon_n(S) = \varepsilon_n(R) + \varepsilon_n(S/\mathfrak{m}S)$ hold for each $n \geq 3$?

Note that by Corollary 7.1.5 we have

$$\varepsilon_1(R) - \varepsilon_1(S) + \varepsilon_1(S/\mathfrak{m}S) = \operatorname{edim}(R) - \operatorname{edim}(S) + \operatorname{edim}(S/\mathfrak{m}S) \geq 0\,.$$

The inequality is strict unless a minimal generating set of $\mathfrak{m}$ extends to one of $\mathfrak{n}$, so additivity may fail for $n = 1$ and hence, by Theorem 7.4.2, also for $n = 2$.

Problem 7.4.9. Does $\varepsilon_n(R_\mathfrak{p}) \leq \varepsilon_n(R)$ hold for all $\mathfrak{p} \in \operatorname{Spec} R$ and odd $n \geq 3$?

It is easily seen from the proof of Theorem 7.4.5 that a positive solution of the first problem implies one for the second. Larfeldt and Lech [103] prove that the two problems are, in fact, equivalent.

8. Test modules

Ring are 'non-linear' objects, so some of their properties are easier to verify after translation into conditions on some canonically defined modules.

The Auslander-Buchsbaum-Serre Theorem provides a model: the regularity of a local ring $(R, \mathfrak{m}, k)$ is tested by checking the finiteness of the projective dimension of k. In terms of asymptotic invariants, this is stated as

$$\operatorname{cx}_R k = 0 \iff R \text{ is regular} \iff \operatorname{curv}_R k = 0\,.$$

The first two sections establish similar descriptions of complete intersections:

$$\operatorname{cx}_R k < \infty \iff R \text{ is a complete intersection} \iff \operatorname{curv}_R k \leq 1\,.$$

For algebras essentially of finite type, another classical test for regularity is given by the Jacobian criterion. Section 3 discusses extensions to complete intersections, in terms of the homology of Kähler differentials. The results there are partly motivated by (still open in general) conjectures of Vasconcelos.

8.1. **Residue field.** In this section $(R, \mathfrak{m}, k)$ is a local ring.

We start with a few general observations on the Betti numbers of k. They show that an extremal property of Poincaré series characterizes complete intersections—and places across the spectrum from Golod rings.

Remarks 8.1.1. Set $e = \operatorname{edim} R$, $r = \operatorname{rank}_k \operatorname{H}_1(K^R)$, and $\varepsilon_n = \varepsilon_n(R)$.

(1) There is an inequality of formal power series

$$\operatorname{P}_k^R(t) \succcurlyeq \frac{(1+t)^e}{(1-t^2)^r}\;.$$

Indeed, Corollary 7.1.5 yields $\operatorname{P}_k^R(t) = (1+t)^e(1-t^2)^{-r}Q(t)$, with $Q(t) = \prod_{i=2}^{\infty}(1+t^{2i-1})^{\varepsilon_{2i-1}} \big/ \prod_{i=2}^{\infty}(1-t^{2i})^{\varepsilon_{2i}}$; also, $Q(t) \succcurlyeq 1$ by Corollary 7.1.4.

(2) Theorem 7.3.3 shows that equality holds in (1) if and only if R is a complete intersection. In that case, $\operatorname{cx}_R k = r = \operatorname{codim} R$, and

$$\beta_n^R(k) = \sum_{i=0}^{e-r} \binom{e-r}{i}\binom{n+r-1-i}{r-1} \qquad \text{for} \quad n \geq 0.$$

(3) If R is not a hypersurface, then $\beta_n^R(k) > \beta_{n-1}^R(k)$ for $n \geq 1$.

Indeed, then $e \geq 1$ and $r \geq 2$, so we have coefficientwise (in)equalities

$$\sum_{n=0}^{\infty}\big(\beta_n^R(k) - \beta_{n-1}^R(k)\big)t^n = (1-t)\operatorname{P}_k^R(t) = \frac{(1+t)^{e-1}}{(1-t^2)^{r-1}}Q(t) \succcurlyeq \frac{1}{(1-t)}\;.$$

Gulliksen [78], [82] extends the Auslander-Buchsbaum-Serre Theorem in

Theorem 8.1.2. *The following conditions are equivalent.*

(i) *R is a complete intersection (respectively, of codimension $\leq c$).*
(ii) *$\operatorname{cx}_R M < \infty$ (respectively, $\operatorname{cx}_R M \leq c$) for each finite R–module M.*
(iii) *$\operatorname{cx}_R k < \infty$ (respectively, $\operatorname{cx}_R k \leq c$).*

Proof. By Proposition 4.2.4 we have $\operatorname{cx}_R M \leq \operatorname{cx}_R k$, and the complexity of k is equal to $\operatorname{codim} R$ by Remark 8.1.1.2, so (i) implies (ii).

When R is not a complete intersection Theorem 7.3.3 gives infinitely many indices i_d with $\varepsilon_{2i_d}(R) > 0$. Remark 7.1.1 then yields an inequality

$$\mathrm{P}_k^R(t) \succcurlyeq \frac{1}{(1-t^{2i_1})\cdots(1-t^{2i_d})} \succcurlyeq \frac{1}{(1-t^{2i})^d} = \sum_{n=0}^{\infty}\binom{n+d-1}{d-1}t^{(2i)n}$$

with $i = i_1 \cdots i_d$. Thus, $\operatorname{cx}_R k \geq d$ for each $d \geq 1$, so (iii) implies (i). □

Remark 8.1.3. We get a new proof of Corollary 7.4.6 by recycling the classical argument for regularity. Proposition 4.2.4.1 and Remark 8.1.1.2 yield

$$\operatorname{cx}_{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}\right) \leq \operatorname{cx}_R(R/\mathfrak{p}) \leq \operatorname{cx}_R k = \operatorname{codim} R\,,$$

so $R_{\mathfrak{p}}$ is a complete intersection of codimension $\leq \operatorname{codim} R$ by the theorem.

We finish this section by a computation of the curvature of k in terms of the deviations of R; the purely analytical argument is from Babenko [38].

Proposition 8.1.4. *If R is not a complete intersection, then*

$$\operatorname{curv}_R k = \limsup_n \sqrt[n]{\varepsilon_n(R)}\,.$$

Proof. Note that $\limsup_n \sqrt[n]{\varepsilon_n(R)} = 1/\eta$, where η is the radius of convergence of the series $E(t) = \sum_{n=1}^{\infty}\varepsilon_n(R)t^n$. By the definition of $\operatorname{curv}_R k$, we have to show that $\eta = \rho$, where ρ is the radius of convergence of the Poincaré series $P(t) = \mathrm{P}_k^R(t)$; note that $\rho > 0$ by Remark 4.2.3.5. By Corollary 7.1.4, we have $\varepsilon_n(R) = \varepsilon_n \geq 0$, so by the product formula of Remark 7.1.1 we get a coefficientwise inequality $P(t) \succcurlyeq E(t)$, hence $\eta \geq \rho > 0$.

To prove that $\rho \geq \eta$, we show that if $0 < \gamma < \eta$, then $P(t)$ converges at $t = \gamma$. We have $\eta < 1$, because $E(t)$ has integer coefficients and $\varepsilon_n > 0$ for infinitely many n by Theorem 7.3.3. For $j \geq 1$ we then get

$$0 < -\ln(1-\gamma^j) = \sum_{h=1}^{\infty}\frac{\gamma^{jh}}{h} < \sum_{h=1}^{\infty}\gamma^{jh} = \frac{\gamma^j}{1-\gamma^j} < \frac{\gamma^j}{1-\eta}\,.$$

By a similar computation, $0 < \ln(1+\gamma^j) < (1-\eta)^{-1}\gamma^j$, so

$$0 \leq L(\gamma) = \sum_{i=1}^{\infty}\left(\varepsilon_{2i-1}\ln(1+\gamma^{2i-1}) - \varepsilon_{2i}\ln(1-\gamma^{2i})\right) \leq \frac{E(\gamma)}{(1-\eta)} < \infty\,.$$

The numerical series with non-negative coefficients $L(\gamma)$ converges, so the product in 7.1.1 converges at $t = \gamma$, as desired. □

8.2. **Residue domains.** In this section $(R, \mathfrak{m}, k)$ is a local ring.

The results that follow are from Avramov [23].

Theorem 8.2.1. *If $\mathfrak{p}$ is a prime ideal of R such that $R_{\mathfrak{p}}$ is not a complete intersection, then there is a real number $\beta > 1$ with the property that*

$$\beta_n^R(R/\mathfrak{p}) \geq \beta^n \qquad \text{for} \quad n \geq 0\,.$$

The converse may fail: the ring $R_{\mathfrak{p}}$ in Example 5.2.6 is a complete intersection, but $\operatorname{curv}_R(R/\mathfrak{p}) > 1$ by Theorem 5.3.3.2. Finite modules over a complete intersection have curvature ≤ 1 by Proposition 4.2.4 and Remark 8.1.1.2, so

Corollary 8.2.2. *The following conditions are equivalent:*

(i) *R is a complete intersection.*
(ii) $\operatorname{curv}_R M \leq 1$ *for each finite R–module M .*
(iii) $\operatorname{curv}_R k \leq 1$ *.* □

The key to the proof of the theorem is to look at deviations.

Theorem 8.2.3. *When R is not a complete intersection there exist a sequence of integers $0 < s_1 < \cdots < s_j < \ldots$ and a real number $\gamma > 1$, such that*

$$\varepsilon_{s_j}(R) \geq \gamma^{s_j} \qquad \text{for} \quad j \geq 1$$

and $s_{j+1} = i_j(s_j - 1) + 2$ with integers $2 \leq i_j \leq \operatorname{edim} R + 1$.

Remark. The last result and its proof are 'looking glass images'—in the sense of [49], [33]—of a theorem of Félix, Halperin, and Thomas [66] on the rational homotopy groups $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ of a finite CW complex X; it relies heavily on Félix and Halperin's [64] theory of rational Ljusternik-Schnirelmann category.

That theorem was used by Félix and Thomas [67] to prove Corollary 8.2.2 for graded rings over fields of characteristic 0, but the L.-S. category arguments do not extend to local rings or to positive characteristic. This is typical of a larger picture: a theorem in rational homotopy or local algebra raises a conjecture in the other field, but a proof usually requires new tools.

The arguments below use the properties of minimal models already established in Section 7.2, and the additional information contained in the next lemma, proved at the end of the section.

Lemma 8.2.4. *Let $k[Y]$ be a minimal DG algebra, such that $\mathrm{H}_n(k[Y]) = 0$ for $n \geq m$. If $\phi\colon k[Y] \to k[U]$ is a surjective morphism of DG algebras, such that U is a set of exterior variables and $\partial(U) = 0$, then $\operatorname{card}(U) < m$.*

As the proof of the theorem for rational homotopy groups, the one of the theorem on deviations proceeds in three steps. The lemma is needed for the first claim, which (now) can be obtained directly from Theorem 7.3.4, or from its precursor in [34]: *If R is not a complete intersection, then $\varepsilon_n(R) \neq 0$ for $n \gg 0$.* We present the original argument in order to keep the notes self-contained, and because of its intrinsic interest. The exposition of the arguments for Claims 2 and 3 follows [30].

Proof of Theorem 8.2.3. As may assume that R is complete, we take a minimal regular presentation $R \cong Q/I$ and a minimal model $Q[Y]$ of R over Q.

Note that the DG algebra $k[Y] = Q[Y]/\mathfrak{n}Q[Y]$ is minimal by Remark 7.2.1, that $\operatorname{card} Y_n = \varepsilon_{n+1}(R)$ for $n \geq 1$ by Theorem 7.2.6, and that

$$\mathrm{H}_n(k[Y]) \cong \operatorname{Tor}_n^Q(k, R) = 0 \qquad \text{for} \quad n \geq m = \operatorname{edim} R + 1\,. \qquad (*)$$

Assuming that R is not a complete intersection, we show that the numbers

$$a(n) = \operatorname{card} Y_n \qquad \text{and} \qquad s(n) = \sum_{j=n}^{2n} a(j)$$

satisfy the following list of increasingly stronger properties:

Claim 1. The sequence $s(n)$ is unbounded.

Claim 2. The sequence $a(n)$ is unbounded.

Claim 3. There exist positive integers $r_1, r_2, \dots$ with $r_{j+1} = i_j r_j + 1$ and $2 \le i_j \le m$, and a real number $\upsilon > 1$, such that $a(n) > \upsilon^{r_j}$ for each $j \ge 0$.

The last claim yields the theorem: with $\gamma = \sqrt{\upsilon}$ and $s_j = r_j + 1$ we have

$$\varepsilon_{s_j}(R) = a(r_j) > \upsilon^{r_j} = \gamma^{2r_j} \ge \gamma^{r_j+1} = \gamma^{s_j} \qquad \text{for} \quad j \ge 1\,.$$

For the rest of the proof, we write $Y_{[n]}$ for the span of $\bigcup_{j=n}^{2n} Y_j$, and abuse notation by letting Y_n stand also for the k–linear span of the variables $y \in Y_n$; thus, $Y_{[n]}^i$ is the k–linear span of all products involving i elements of $Y_{[n]}$.

Proof of Claim 1. Assume that there is a $c \in \mathbb{N}$ such that $s(n) \le c$ for all $n \ge 1$.

We are going to construct for all $r \ge 1$ and $h \ge 0$ surjective morphisms of DG algebras $\phi_h^r \colon k[Y] \to k[U_h^r]$, where each U_h^r is a set $\{u_{hr+1}^r, \dots, u_{hr+r}^r\}$ of exterior variables subject to the restrictions

$$|u_{n+1}^r| > |u_n^r| + 1 \qquad \text{for} \quad n \ge 1\,;$$
$$|u_{hr+i}^r| > |u_{hr+1}^r| + \dots + |u_{hr+i-1}^r| + 1 \qquad \text{for} \quad i = 2, \dots, r\,.$$

The second condition forces $\partial(U_h^r) = 0$, so in view of $(*)$ Lemma 8.2.4 implies that $r < m$. This contradiction establishes the unboundedness of $s(n)$.

By Theorems 7.2.6 and 7.3.3, there is an infinite sequence $y_1, y_2, \dots \in Y_{\mathrm{odd}}$ with $|y_{h+1}| > |y_h| + 1$. Setting $n_h = |y_h|$, note that the compositions $k[Y] \to k[Y_{\geqslant n_h}] \to k[Y_{\geqslant n_h}]/\big(Y_{\geqslant n_h} \setminus \{y_h\}\big)$ have the desired properties for $r = 1$.

Assume by induction that morphisms ϕ_h^r have been constructed for some $r \ge 1$. We fix $n \ge 0$, simplify the notation by setting $u_{ij} = u_{(n+i)r+j}^r$, $U_i = U_{n+i}^r$, and $\phi_i = \phi_{n+i}^r$, for $i = 0, \dots, c$ and $j = 1, \dots, r$, and embark on an auxilliary construction. Choose an index $q > |u_{c1}| + \dots + |u_{cr}| + 1$ such that $Y_q \ne \varnothing$, and pick $y \in Y_q$. For $i = 0, \dots, c$ the intervals

$$I_i = \big[(q + |u_{i1}| + 1)\,,\ (q + |u_{i1}| + \dots + |u_{ir}| + 1)\big]$$

are disjoint and contained in the interval $[q, 2q]$.

Since $s(q) \le c$, we can choose an index i such that $Y_s = \varnothing$ for all $s \in I_i$. The restriction of ϕ_i to $B = k[Y_{<q}]$ yields a surjective morphism of DG algebras $B \to k[U_i]$. Tensoring it with $k[Y]$ over B, we get a surjective morphism $k[Y] \to C = k[U_i] \otimes_B k[Y]$. Note that $C^\natural = k[U_i] \otimes_k k[Y_{\geqslant q}]^\natural$ is equal to $k[U_i]$ in degrees $\le q-1$, to kY_q in degree q, and has no algebra generators in degrees from I_i. As the differential ∂^C is decomposable, the ideal of C generated by the variables $Y_{\geqslant q} \smallsetminus \{y\}$ is closed under the differential of C.

We have now constructed a surjective morphism of DG algebras

$$k[Y] \to C \to C/(Y_{\geqslant q} \smallsetminus \{y\})C = k[u_1 \dots, u_{r+1}]$$

where $u_j = u_{ij}$ for $j = 1, \dots, r$ and u_{r+1} is the image of y; clearly, the condition $|u_j| > |u_1| + \dots + |u_{j-1}| + 1$ holds for $j = 2, \dots, r+1$. To end the auxilliary construction, choose an integer n' such that $rn' > (n + c + 1)r$. Setting $n_1 = 1$ and $n_h = n'_{h-1}$ for $h \ge 2$, and applying the construction to n_h for $h = 1, 2, \dots$, we get a sequence of surjective morphisms ϕ_{h+1}^{r+1} with the desired properties.

Proof of Claim 2. We assume that there exists a number c such that $\operatorname{rank}_k Y_n \le c$ for all n, and work out a contradiction.

Fix for the moment an integer $n \geq 1$. For every $y \in Y_{\geqslant n}$ there are uniquely defined $\alpha_i(y) \in Y^i_{[n]} \subseteq k[Y_{\geqslant n}]$, such that

$$\partial(y) \equiv \sum_{i \geqslant 2} \alpha_i(y) \mod \big((Y_{>2n})k[Y_{\geqslant n}]\big)\,.$$

Clearly, the maps $\alpha_i \colon Y_{\geqslant n} \to Y^i_{[n]}$, where $y \mapsto \alpha_i(y)$, are k-linear. The minimality of $k[Y]$ is inherited by $k[Y_{\geqslant n}]$, so there we have $\partial(Y^i_{[n]}) = 0$ for all i. Recalling from Corollary 7.2.7 that

$$\big(\mathrm{H}_{\geqslant 1}(k[Y_{\geqslant n}])\big)^m = 0 \qquad \text{for every } n \geq 1 \tag{†}$$

we see that $Y^m_{[n]}$ consists of boundaries, so we get an inclusion

$$\sum_{i=2}^{m} Y^{m-i}_{[n]}\alpha_i(Y_{\geqslant n}) \supseteq Y^m_{[n]}\,. \tag{‡}$$

For degree reasons, $\alpha_i(Y_j) = 0$ when $j < in+1$ or $j > i(2n)+1$, so

$$s(in+1) = \sum_{j=in+1}^{2in+2} \operatorname{rank}_k Y_j \geq \sum_{j=in+1}^{i(2n)+1} \operatorname{rank}_k \alpha_i(Y_j) = \operatorname{rank}_k \alpha_i(Y_{\geqslant n})\,. \tag{§}$$

Set $d = (2m)^m$ and choose by Claim 1 an integer n_0 such that $s(n_0) > (md)^2$.

Assume by induction on j that we have integers $n_0, n_1, \dots, n_j$ such that

$$m(n_{h-1}+1) \geq n_h + 1 \text{ and } s(n_h) \geq (md)s(n_{h-1}) \quad \text{for } 1 \leq h \leq j\,. \tag{¶}$$

Choose $n_{j+1} = ln_j + 1$ such that $s(n_{j+1}) = \max\{s(in_j+1) \mid 2 \leq i \leq m\}$. It is then clear that $m(n_j+1) \geq n_{j+1}+1$. Using (§) and (‡), we get

$$\begin{aligned}
(m-1)s(n_j)^{m-2}s(n_{j+1}) &\geq \sum_{i=2}^{m} s(n_j)^{m-i}s(in_j+1)\\
&\geq \sum_{i=2}^{m} \big(\operatorname{rank}_k Y_{[n_j]}\big)^{m-i} \operatorname{rank}_k \alpha_i(Y_{\geqslant n_j})\\
&\geq \operatorname{rank}_k \left(\sum_{i=2}^{m} Y^{m-i}_{[n_j]}\alpha_i(Y_{\geqslant n_j})\right) \geq \operatorname{rank}_k Y^m_{[n_j]}\\
&\geq \binom{s(n_j)}{m} \geq \frac{s(n_j)^m}{(2m)^m} = \frac{s(n_j)}{d} s(n_j)^{m-1}\\
&\geq (m^2 d)s(n_j)^{m-1}
\end{aligned}$$

so $s(n_{j+1}) \geq (md)s(n_j)$, completing the induction step. Clearly, (¶) implies that $m^j(n_0+1) \geq n_j+1$ and $s(n_j) \geq m^j s(n_0)d^j$ hold for $j \geq 1$. Thus, we get $c(n_j+1) \geq s(n_j)$, and hence $c(n_0+1) \geq s(n_0)d^j$ for all j. This is absurd.

Proof of Claim 3. Set $b = (2m)^{m+1}$, choose r_1 so that $a(r_1) = a > b$, and assume by induction that $r_1, \dots, r_j$ have been found with the property that

$$r_h = i_{h-1}r_{h-1} + 1 \quad \text{with } 2 \leq i_{h-1} \leq m \qquad \text{and} \qquad a(r_h) \geq \frac{a(r_{h-1})^{i_{h-1}}}{b}$$

for $1 \leq h \leq j$. The condition $\beta(y) \equiv \partial(y) \mod \big((Y_{>r_j})k[Y_{\geqslant r_j}]\big)$ defines a k–linear homomorphism $\beta \colon Y_{\geqslant r_j} \to \sum_{i \geqslant 2} Y^i_{r_j}$. Noting that $\beta(y) = 0$ unless $|y| \equiv 1 \pmod{r_j}$, and using the fact that $k[Y]$ is minimal and satisfies condition (†), we obtain $\sum_{i=2}^{m} Y_{r_j}^{m-i}\beta(Y_{ir_j+1}) \supseteq Y_{r_j}^{m}$ as in the the proof of the preceding claim. It follows that

$$\sum_{i=2}^{m} a(r_j)^{m-i}a(ir_j+1) \geq \binom{a(r_j)}{m} \geq \frac{a(r_j)^m}{(2m)^m} = (2m)\frac{a(r_j)^m}{b} \ .$$

The assumtion that $a(ir_j+1) < a(r_j)^i/b$ for $2 \leq i \leq m$, leads to the impossible inequality $(m-1)a(r_j)^m > (2m)a(r_j)^m$. Thus, $a(ir_j+1) \geq (r_j)^i/b$ for some $i = i_j$, so the induction step is complete with $r_{j+1} = i_j r_j + 1$.

The quantities $P_j = (i_j \cdots i_1)$ and $S_j = \sum_{h=2}^{j}(i_j \cdots i_h)$ satisfy

$$P_j > P_j\left(\frac{1}{2} + \cdots + \frac{1}{2^j}\right) \geq P_j\left(\frac{1}{i_1} + \cdots + \frac{1}{i_1 \cdots i_j}\right) = S_j \ .$$

Thus, $P_j(r_1+1) > P_j r_1 + S_j = r_{j+1}$, and hence $P_j > r_{j+1}/(r_1+1)$. Since $a > b$, we have

$$a(r_{j+1}) \geq \frac{a(r_j)^{i_j}}{b} \geq \frac{a(r_{j-1})^{(i_j i_{j-1})}}{b^{1+i_j}} \geq \cdots \geq \frac{a^{P_j}}{b^{S_j}} > \left(\frac{a}{b}\right)^{P_j} > \left(\frac{a}{b}\right)^{\frac{r_{j+1}}{r_1+1}} \ .$$

To finish the proof of the claim, note that $\upsilon > 1$ and set $\upsilon = \sqrt[r_1+1]{a/b}$. □

Proof of Theorem 8.2.1. Since $\beta_n^R(R/\mathfrak{p}) \geq \beta_n^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ for each n, we may assume that $\mathfrak{p} = \mathfrak{m}$; set $\beta_n = \beta_n^R(R/\mathfrak{m})$ and $e = \operatorname{edim} R$.

As R is not a complete intersection, Theorem 8.2.3 provides an infinite sequence $s_1, s_2, \ldots$ with $(e+1)s_j > s_{j+1}$, such that $\varepsilon_{s_j}(R) \geq \gamma^{s_j}$ for some real number $\gamma > 1$. For $n \geq 2$ we have $\beta_n > \beta_1 = e > 1$ by Remark 8.1.1.3, hence

$$\beta = \min\left\{ \sqrt[e+1]{\gamma}\, ,\, \beta_1\, , \ldots ,\ \sqrt[s_1]{\beta_{s_1}} \right\} > 1$$

and $\beta_n \geq \beta^n$ for $s_1 \geq n \geq 0$. If $s_{j+1} \geq n > s_j$ with $j \geq 1$, then

$$\beta_n > \beta_{s_j} \geq \varepsilon_{s_j}(R) \geq \gamma^{s_j} \geq \beta^{(e+1)s_j} > \beta^{s_{j+1}} > \beta^n$$

so the desired inequality $\beta_n \geq \beta^n$ holds for all $n \geq 0$. □

Proof of Lemma 8.2.4. Since $\phi^\natural \colon k[Y]^\natural \longrightarrow k[U]$ is a surjective homomorphism of graded free k–algebras, the ideal $\operatorname{Ker}\phi^\natural$ has a linearly independent generating set $Y' = \{y'_1, \ldots, y'_j, \ldots\} \subset kY$, which we can assume ordered in such a way that $|y'_{j+1}| \geq |y'_j|$ for $j \geq 1$. As $\operatorname{Ker}\phi$ is a DG ideal, we have $\partial(y'_1) = 0$ and $\partial(y'_{j+1}) \in (y'_1, \ldots, y'_j)$ for $j \geq 1$. Assume that for some $j \geq 1$ the morphism ϕ factors through a quasi-isomorphism

$$\pi^j \colon A^j = k[Y]\langle x'_1, \ldots, x'_j\rangle \xrightarrow{\pi^j} k[Y]/(y'_1, \ldots, y'_j) = B^j$$

that maps $x_i'^{(n)}$ to zero for $1 \leq i \leq j$ and $n \geq 1$. As the $\overline{y}'_{j+1} = \pi^j(y'_{j+1})$ is a cycle in B^j, there is a cycle $z_{j+1} \in A^j$ such that $\pi^j(z_{j+1}) = \pi^j(y'_{j+1})$. By Lemma 7.2.10, π^j extends to a quasi-isomorphism

$$\pi^j\langle x'_{j+1}\rangle \colon A^{j+1} = A^j\langle x'_{j+1} \mid \partial(x'_{j+1}) = z_{j+1}\rangle \to B^j\langle x'_{j+1} \mid \partial(x'_{j+1}) = \overline{y}'_{j+1}\rangle \ .$$

Lemma 7.2.11 shows that the map $\xi^j \colon B^j\langle x'_{j+1}\rangle \ \longrightarrow\ B^j/(y'_{j+1})$ that sends $\sum_i b_i {x'_{j+1}}^{(i)}$ to $b_0 + (y'_{j+1})$ is a quasi-isomorphism; thus, so is

$$\pi^{j+1} = \xi^j \circ \pi^j\langle x'_{j+1}\rangle \colon A^{j+1} = B^j\langle x'_{j+1}\rangle \to B^j/(y'_{j+1}) = B^{j+1} \ .$$

In the limit, we obtain a factorization of ϕ in the form

$$k[Y] \hookrightarrow k[Y]\langle X'\rangle \xrightarrow{\pi} k[U]$$

with a surjective quasi-isomorphism π, such that $\operatorname{Ker}\pi$ is generated by $\operatorname{Ker}\phi$ and ${x'_j}^{(i)}$ with $i, j \geq 1$. By Lemma 7.2.10, π extends to a quasi-isomorphism

$$\pi\langle X''\rangle \colon k[Y]\langle X'\rangle\langle X''\rangle \longrightarrow k[U]\langle X'' \mid \partial(x''_j) = u_j\rangle \ .$$

where $X'' = \{x''_1, \dots, x''_m\}$. The DG algebra on the right is quasi-isomorphic to k, so we get a semi-free resolution $W' = k[Y]\langle X' \cup X''\rangle$ of k over $k[Y]$. Another semi free resolution $W = k[Y]\langle X\rangle$ of k over $k[Y]$, such that $X = \{x_y : |x_y| = |y| + 1 \,,\, y \in Y\}$, and $\partial(k[Y]\langle X\rangle) \subseteq (Y)W$, is given by Proposition 7.2.9 (applied with $Q = k$). By Propositions 1.3.1 and 1.3.2, the vector spaces $V' = k \otimes_{k[Y]} W' = k\langle X' \cup X''\rangle$ and $V = k \otimes_{k[Y]} W = k\langle X\rangle$ are quasi-isomorphic. As $\partial^V = 0$, we get (in)equalities of formal power series

$$\begin{aligned}
\frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{\operatorname{card}(X_{2i-1})}}{\prod_{i=1}^{\infty}(1-t^{2i})^{\operatorname{card}(X_{2i})}} &= \sum_n \operatorname{rank}_k V_n t^n = \sum_n \operatorname{rank}_k \mathrm{H}_n(V) t^n \\
&= \sum_n \operatorname{rank}_k \mathrm{H}_n(V') t^n \preccurlyeq \sum_n \operatorname{rank}_k V'_n t^n \\
&= \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{\operatorname{card}(X'_{2i-1})}}{\prod_{i=1}^{\infty}(1-t^{2i})^{\operatorname{card}(X'_{2i})}} \cdot \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{\operatorname{card}(X''_{2i-1})}}{\prod_{i=1}^{\infty}(1-t^{2i})^{\operatorname{card}(X''_{2i})}} \ .
\end{aligned}$$

On the other hand, by construction we have for each j an equality

$$\operatorname{card} X_{j+1} = \operatorname{card} Y_j = \operatorname{card} Y'_j + \operatorname{card} U_j = \operatorname{card} X'_{j+1} + \operatorname{card} X''_{j+1} \ .$$

It follows that $\mathrm{H}(V') = V'$, that is, that $\partial(W') \subseteq (Y)W'$.

As $W'^{\natural}$ is a free module over $k[Y]\langle X'\rangle^{\natural}$, and $\mathrm{H}_{\geqslant 1}(W) = 0$, we see that

$$\begin{aligned}
\mathrm{Z}_{\geqslant 1}(k[Y]\langle X'\rangle) &= \mathrm{Z}_{\geqslant 1}(W') \cap (k[Y]\langle X'\rangle) = \partial(W') \cap (k[Y]\langle X'\rangle) \\
&\subseteq (Y)W' \cap k[Y]\langle X'\rangle = (Y)k[Y]\langle X'\rangle \ .
\end{aligned}$$

Since $\pi \colon k[Y]\langle X'\rangle \longrightarrow k[U]$ is a surjective quasi-isomorphism, we can find $z_1, \dots, z_m \in \mathrm{Z}(k[Y]\langle X'\rangle)$ with $\pi(z_i) = u_i$. For them we have

$$z_1 \cdots z_m \in \big(\mathrm{Z}_{\geqslant 1}(k[Y]\langle X'\rangle)\big)^m \subseteq \mathrm{Z}\big((Y)^m k[Y]\langle X'\rangle\,\big) \subseteq \mathrm{Z}(Jk[Y]\langle X'\rangle)$$

where $J \subset k[Y]$ is defined by

$$J_n = \begin{cases} 0 & \text{for } n < m\,; \\ \partial(k[Y]_{m+1}) & \text{for } n = m\,; \\ k[Y]_n & \text{for } n > m\,. \end{cases}$$

For degree reasons, J is a DG ideal of $k[Y]$. By hypothesis $\mathrm{H}_n(k[Y]) = 0$ for $n \geq m$, so $\mathrm{H}(J) \cong \mathrm{H}(k[Y]) = 0$. Thus, the projection $\tau \colon k[Y] \longrightarrow k[Y]/J$ is a quasi-isomorphism; Proposition 1.3.2 then shows that the induced map $k[Y]\langle X'\rangle \longrightarrow$

$k[Y]\langle X'\rangle\,/Jk[Y]\langle X'\rangle$ is one, hence

$$\mathrm{Z}(Jk[Y]\langle X'\rangle) = \partial(Jk[Y]\langle X'\rangle) \subseteq \partial(k[Y]\langle X'\rangle)$$

hence $\mathrm{cls}(z) = 0$. The computation $0 = \mathrm{H}(\pi)(\mathrm{cls}(z_1)\cdots\mathrm{cls}(z_m)) = u_1 \dots u_m \neq 0$ now yields the desired contradiction. □

8.3. **Conormal modules.** In this section we fix a presentation $R = Q/I$, where $(Q, \mathfrak{n}, k)$ is a local or graded ring, and $I \subseteq \mathfrak{n}^2$ is minimally generated by $\boldsymbol{f}$. The R–module I/I^2 is called the *conormal module* of the presentation[30].

If $\boldsymbol{f}$ is a regular sequence, then it is well known and easy to see that the image of $\boldsymbol{f}$ modulo I^2 is a basis of the conormal module, and the projective dimension $\mathrm{pd}_Q R$ is finite. The starting point of the present discussion is a well known converse, due to Ferrand [68] and Vasconcelos [154]:

Theorem 8.3.1. *If $\mathrm{pd}_Q R < \infty$ and the R–module I/I^2 is free, then $\boldsymbol{f}$ is a regular sequence.* ■

Later, Vasconcelos [155] conjectured a considerably stronger statement: *If $\mathrm{pd}_Q R < \infty$ and $\mathrm{pd}_R(I/I^2) < \infty$, then $\boldsymbol{f}$ is a regular sequence.* Various known cases of small projective dimension are surveyed in [156]; the one below is proved by Vasconcelos and Gulliksen.

Theorem 8.3.2. *The conjecture holds if $\mathrm{pd}_R(I/I^2) \leq 1$.*

Proof. In view of the preceding theorem, it suffices to assume that $\mathrm{pd}_R(I/I^2) = 1$, and draw a contradiction.

For the Koszul complex $E = Q\langle X_1 \,|\, \partial(X_1) = \boldsymbol{f}\rangle$, set $Z = \mathrm{Z}_1(E)$ and $H = \mathrm{H}_1(E)$. Tensoring the exact sequence $0 \to Z \to Q^r \to I \to 0$ with R over Q, we get an exact sequence of R–modules $Z/IZ \to R^r \to I/I^2 \to 0$. As $\partial(E_2) \subseteq IE_1$, we have an induced exact sequence $H \to R^r \to I/I^2 \to 0$. The assumption $\mathrm{pd}_R(I/I^2) = 1$ then implies that H contains a free direct summand $R\,\mathrm{cls}(z) \cong R$; note that $z \in \mathfrak{n}E_1$, because $\boldsymbol{f}$ minimally generates I.

Let $E\langle X_{\geqslant 2}\rangle = Q\langle X\rangle$ be an acyclic closure of $R = \mathrm{H}_0(E)$ over E, such that $\partial(x) = z$ for some $x \in X_2$. The cokernel of the differential $\delta_2\colon RX_2 \to RX_1$ of the complex of indecomposables $\mathrm{Ind}^{\gamma}_Q\, Q\langle X\rangle$ is equal to H, so Proposition 6.2.7 yields a Q–linear Γ-derivation $\vartheta\colon Q\langle X\rangle \to Q\langle X\rangle$ of degree -2, with $\vartheta(x) = 1$.

The Γ-derivation $\theta = \mathrm{H}(\vartheta \otimes_Q k)$ of $Q\langle X\rangle \otimes_Q k = k\langle X\rangle$ has $\theta(x) = 1$. As $\partial(x) = z \otimes 1 = 0 \in E_1/\mathfrak{n}E_1$, each $x^{(i)}$ is a cycle. Assuming that $x^{(i)} = \partial(v)$, we get $1 = \theta^i(x^{(i)}) = \theta^i\partial(v) = 0$, which is absurd. Thus, $0 \neq \mathrm{H}_{2i}(k\langle X\rangle) = \mathrm{Tor}^Q_{2i}(R,k)$ for all $i \geq 0$, contradicting the hypothesis that $\mathrm{pd}_Q R$ is finite. □

Next we present the results of Avramov and Herzog [35] on graded ring.

Theorem 8.3.3. *Let $Q = k[s_1, \dots, s_e]$ be a graded polynomial ring over a field k of characteristic 0, with variables of positive degree, let I be a homogeneous ideal of Q, and set $R = Q/I$. The following conditions are equivalent.*

(i) *R is a complete intersection.*
(ii) $\mathrm{pd}_R(I/I^2) < \infty$.
(iii) $\mathrm{cx}_R(I/I^2) < \infty$.
(iv) $\mathrm{curv}_R(I/I^2) \leq 1$.

If R is not a complete intersection, then $\mathrm{curv}_R\, I/I^2 = \mathrm{curv}_R\, k$.

[30]Or: of the embedding $\mathrm{Spec}(R) \subseteq \mathrm{Spec}(Q)$.

The result is proved together with the next one:

Theorem 8.3.4. *If R is as in the preceding theorem, and $\Omega_{R|k}$ is its module of Kähler differentials over k, then the following conditions are equivalent.*

(i) *R is a complete intersection.*
(ii) $\operatorname{cx}_R(\Omega_{R|k}) < \infty$.
(iii) $\operatorname{curv}_R(\Omega_{R|k}) \leq 1$.

If R is not a complete intersection, then $\operatorname{curv}_R \Omega_{R|k} = \operatorname{curv}_R k$.

Remark. If $\operatorname{pd}_R \Omega_{R|k} < \infty$, then the theorem implies that R is a complete intersection—another conjecture of Vasconcelos—but there is more.

If $\mathfrak{p}$ is a minimal prime ideal, then $\Omega_{R_\mathfrak{p}|k} \cong \big(\Omega_{R|k}\big)_\mathfrak{p}$ has finite projective dimension over $R_\mathfrak{p}$. Thus, it is free, hence $R_\mathfrak{p}$ is regular by the Jacobian criterion, and so R is *reduced* by Serre's criterion. Conversely, if R is a reduced complete intersection, then Ferrand [68] and Vasconcelos [154] prove that $\operatorname{pd}_R \Omega_{R|k} \leq 1$.

The asymptotic results are easy consequences of more precise inequalities[31] for the graded invariants described in Remark 1.2.10.

Theorem 8.3.5. *In the notation of Theorem* 8.3.3*, for all $n \geq 0$ and $j \in \mathbb{Z}$ there is an inequality between graded Betti numbers and deviations:*

$$\beta^R_{nj}(\Omega_{R|k}) \geq \varepsilon_{n+1,j}(R) \qquad \text{and} \qquad \beta^R_{nj}(I/I^2) \geq \varepsilon_{n+2,j}(R)\,.$$

Our proof proceeds through a structural result on the resolution of I/I^2, that depends on the grading and on the characteristic; the following is open:

Problem 8.3.6. When R is a local ring and $R \cong Q/I$ is a regular presentation, does an inequality $\beta^R_n(I/I^2) \geq \varepsilon_{n+2}(R)$ hold for each $n \geq 0$?

In the arguments, we use graded versions of some basic constructions.

Remark 8.3.7. The first step in the construction of a minimal model of R over Q is a Koszul complex on the set $\boldsymbol{f}$ of minimal generators of I; we choose $\boldsymbol{f}$ to consist of homogeneous elements, so the first Koszul homology is a finite graded Q–module. Assume by induction that $\mathrm{H}_n(Q[Y_{\leqslant n}])$ has the same property for some $n \geq 1$; to kill it we adjoin a minimal set of homogeneous generators, and assign to each variable $y \in Y_{n+1}$ an internal degree, equal to that of $\partial(y)$.

Thus, we get a *graded minimal model* $Q[Y_{\geqslant 1}] = k[Y]$ of R over Q. Similar considerations yield a *graded acyclic closure* $R\langle X\rangle$ of k over R. The arguments in Sections 6.3 and 7.2 are compatible with the internal gradings, so the 'obvious' graded versions of the results proved there are available.

Remark 8.3.8. Proposition 6.2.3 can be repeated for ordinary (that is, not subject to a condition involving divided powers) k–linear derivations of the DG algebra $k[Y]$ over k, to produce a *DG module of differentials* $\operatorname{Diff}_k k[Y]$ over $k[Y]$. It is semi-free with basis $\{dy \,:\, |dy| = |y|\,; \deg(dy) = \deg(y)\}_{y\in Y}$, where $\deg(a)$ is the internal degree of a; the map $y \mapsto dy$ extends to a universal derivation $d\colon k[Y] \to \operatorname{Diff}_k k[Y]$; the differential is determined by $\partial(dy) = d(\partial(y))$; each k–linear derivation of $k[Y]$ into a DG module U over $k[Y]$ factors uniquely as the composition of d with a homomorphism of DG modules $\operatorname{Diff}_k k[Y] \to U$.

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[31] Equalities hold for $n = 0$ by Corollary 7.1.5, but it appears that the other inequalities are strict unless R is a (reduced) complete intersection.

Consider the complex of free R–modules $L = R \otimes_{k[Y]} \operatorname{Diff}_k k[Y]$. (Using Lemma 7.2.3 on the uniqueness of minimal models and a functorial construction of $\operatorname{Diff}_k k[Y]$, it can be shown that this complex is defined uniquely up to isomorphism by the k–algebra R; we do not use that here, and refer to [35] for details.) For $g \in QY_1$, an easy computation shows that the differential

$$\partial_1 \colon L_1 \longrightarrow L_0 \quad \text{acts by} \quad \partial_1(1 \otimes g) = 1 \otimes \sum_{i=1}^{e} \frac{\partial f}{\partial y_i} dy_i \quad \text{where} \quad \partial_1(g) = f \in Q\,.$$

On the other hand, the 'second fundamental exact sequence' for the module $\Omega_{R|k}$ of *Kähler differentials* of the k–algebra R has the form

$$I/I^2 \xrightarrow{\delta} R \otimes_Q \Omega_{Q|k} \longrightarrow \Omega_{R|k} \longrightarrow 0 \quad \text{with} \quad \delta(f + I^2) = 1 \otimes \sum_{i=1}^{e} \frac{\partial f}{\partial y_i} dy_i\,.$$

As $\Omega_{Q|k}$ is free with basis $\{dy_1, \dots, dy_e\}$, we conclude that $\mathrm{H}_0(L) = \Omega_{R|k}$.

Recall from Remark 4.1.7, that an augmentation $\epsilon \colon F \longrightarrow N$ of a complex of free R–modules F is essential, if for some lifting $\alpha \colon F \longrightarrow G$ to a minimal resolution G of N, the map $k \otimes_R \alpha$ is injective. In that case, α maps F isomorphically onto a subcomplex of G, that splits off as a graded R–module.

Theorem 8.3.9. *The augmentation $\epsilon^L \colon L \longrightarrow \mathrm{H}_0(L) = \Omega_{R|k}$ is essential.*

A special morphism is at the heart of the arguments to follow.

Construction 8.3.10. Euler morphisms. The graded algebra R has an *Euler derivation* $R \longrightarrow \mathfrak{m}$, that multiplies each homogeneous element $a \in R$ by its (internal) degree. By Proposition 1.3.1, the R–linear map $\gamma \colon \Omega_{R|k} \longrightarrow \mathfrak{m}$ that it defines lifts to a morphism $\omega \colon \operatorname{Diff}_k k[Y] \longrightarrow V$ of DG modules over $k[Y]$, where $\epsilon^V \colon V \longrightarrow \mathfrak{m}$ is a semi-free resolution of $\mathfrak{m}$ over $k[Y]$. We call such a lifting an *Euler morphism*; it is unique up to $k[Y]$–linear homotopy.

Lemma 8.3.11. *Let $k[Y]$ be a graded minimal model of R over k, and let $U = k[Y]\langle X\rangle$ be a graded acyclic closure of k over $k[Y]$, as in Remark 8.3.7.*

The DG module $V = \Sigma^{-1}(U/k[Y])$ is a semi-free resolution of $\mathfrak{m}$ over $k[Y]$, and there is an Euler morphism $\omega \colon \operatorname{Diff}_k k[Y] \longrightarrow V$, such that

$$\omega(dy) \equiv -\deg(y) x_y \mod \mathfrak{n}X_{n+1} + \big(k[Y_{\leqslant n+1}]\langle X_{\leqslant n}\rangle\big)_{n+1} \qquad \text{for} \quad y \in Y_n\,.$$

Proof. Set $D^n = \coprod_{|y| \leqslant n} k[Y] dy \subseteq \operatorname{Diff}_k k[Y] = D$.

The map $a \mapsto \deg(a)a$ is a k–linear chain Γ-derivation $k[Y] \longrightarrow U$. In degree zero homology it induces the zero map $R \longrightarrow k$, so it is homotopic to 0. If $\xi \colon D \longrightarrow U$ is the $k[Y]$–linear morphism that corresponds to it by Proposition 6.2.3, then ξ is homotopic to 0. We set $\xi^n = \xi|_{D^n}$ and by induction on n construct $k[Y]$–linear homotopies $\sigma^n \colon D^n \longrightarrow U$ between ξ^n and 0, such that

$$\begin{aligned} \sigma^n|_{D^{n-1}} &= \sigma^{n-1}\,; \\ \sigma^n(dy) &\equiv \deg(y) x_y \mod \mathfrak{n}X_n + \big(k[Y_{\leqslant n}]\langle X_{< n}\rangle\big)_n\,. \end{aligned} \tag{$*$}$$

If $|y| = 0$, then set $\sigma^0(dy) = \deg(y) x_y$: clearly, the formula above holds. Let $n \geq 1$, and assume by induction that σ^{n-1} has been found. It is easy to check that $\xi(dy) - \deg(y)\partial(x_y) - \sigma^{n-1}\partial(dy)$ is a cycle; as $n \geq 1$, it is a boundary, that we write as $\partial(u_y+v_y)$ with $u_y\in QX_{n+1}$, and $v_y\in k[Y_{\leqslant n+1}]\langle X_{\leqslant n}\rangle_{n+1}$. Because d is a derivation and $\partial(Y)\subseteq(Y)^2k[Y]$, we get

$$d(\partial Y)\subseteq d\big((Y)^2k[Y]\big)\subseteq (Y)d\big(k[Y]\big)=(Y)D$$

Since σ^{n-1} is $k[Y]$–linear, this implies:

$$\sigma^{n-1}\partial(dy)=\sigma^{n-1}d(\partial(y))\in W_n=\mathfrak{n}Y_n+k[Y_{<n}]\langle X_{\leqslant n}\rangle_n\,.$$

By Proposition 7.2.9, we have $\partial(v_y)\in W_n$, hence

$$\partial(u_y)=\xi(dy)-\deg(y)\partial(x_y)-\sigma^{n-1}\partial(dy)-\partial(v_y)\in W_n\,.$$

By the same theorem, we conclude that $u_y\in\mathfrak{n}X_{n+1}$. The map $\sigma^n\colon D^n\to U$, $\sigma^n(dy)=\deg(y)x_y+u_y+v_y$, defines a homomorphism of DG modules over $k[Y]$ that satisfies $(*)$. As for $|y|\le n$ we have

$$\partial\sigma^n(dy)+\sigma^n\partial(dy)=\deg(y)\partial(x_y)+\partial(u_y+v_y)+\sigma^{n-1}\partial(dy)=\xi(dy)\,,$$

the induction step of the construction is complete.

In the limit, the maps σ^n define a homotopy $\sigma\colon D\to U$ between ξ and 0. Let $\omega\colon D\to V$ be the composition of σ with the canonical $k[Y]$–linear, degree -1 homomorphism $U\to U/k[Y]\to\Sigma^{-1}(U/k[Y])=V$. As $\operatorname{Im}\xi\subseteq k[Y]$, the equality $\partial\sigma+\sigma\partial=\xi$ implies $\partial\omega=\omega\partial$, so ω is a chain map $D\to V$.

The homology exact sequence of $0\to k[Y]\to U\to U/k[Y]\to 0$ yields $\mathrm{H}_n(U/k[Y])=0$ for $n\ne 1$ and $\mathrm{H}_1(V)=\mathfrak{m}$, so $V=\Sigma^{-1}(U/k[Y])$ is a semi-free resolution of $\mathfrak{m}$. For $n=0$, formula $(*)$ shows that $\mathrm{H}_0(\omega)\colon\Omega_{R|k}\to\mathfrak{m}$ is the homomorphism induced by the Euler derivation; for $n\ge 1$, the formula yields a congruence $\omega(dy)\equiv-\deg(y)x_y \mod \mathfrak{n}X_{n+1}+\big(k[Y_{\leqslant n+1}]\langle X_{\leqslant n}\rangle\big)_{n+1}$. □

Proof of Theorem 8.3.9. Let $\omega\colon\operatorname{Diff}_k k[Y]\to V$ be the Euler morphism, constructed in the preceding lemma, and consider the induced morphism

$$\varpi\colon L=R\otimes_{k[Y]}\operatorname{Diff}_k k[Y]\xrightarrow{R\otimes\omega}R\otimes_{k[Y]}V=G$$

of complexes of graded R–modules. The lemma yields congruences

$$(k\otimes_R\varpi)(1\otimes dy)\equiv 1\otimes\deg(y)x_y \mod \big(k\langle X_{\leqslant n}\rangle_{n+1}\big)\ \text{for } y\in Y_n \text{ and } n\ge 0\,,$$

which show[32] that $k\otimes_R\varpi$ is injective. Furthermore, $\mathrm{H}_0(\varpi)$ is the homomorphism $\gamma\colon\Omega_{R|k}\to\mathfrak{m}$ defined by the Euler derivation.

By Proposition 1.3.2, the quasi-isomorphism $\rho\colon k[Y]\to R$ induces a quasi-isomorphism $\rho\otimes V\colon V=k[Y]\otimes_{k[Y]}V\to R\otimes_{k[Y]}V=G$, so G is a minimal free resolution of $\mathfrak{m}$ over R. Let F be a minimal free resolution of $\Omega_{R|k}$ over R, let $\alpha\colon L\to F$ be a lifting of the identity map of $\Omega_{R|k}$, and let $\beta\colon F\to G$ be a lifting of γ. Since $\mathrm{H}_0(\beta\alpha)=\gamma$, the morphisms ϖ and $\beta\alpha$ are homotopic. As noted in Remark 4.1.7, this yields

$$k\otimes_R\varpi=k\otimes_R(\beta\alpha)=(k\otimes_R\beta)(k\otimes_R\alpha)\,,$$

so $k\otimes_R\alpha$ is injective. This is the desired assertion. □

Proof of Theorem 8.3.5. By construction, L_n is a free R–module with basis Y_n, and $\operatorname{card}(Y_n)=\varepsilon_{n+1}(R)$ by Theorem 7.2.6. The inequalities for the Betti numbers of $\Omega_{R|k}$ follow from the result that we have just proved.

The morphism ϖ used in its proof induces a morphism $\varpi'\colon L'=\Sigma^{-1}L_{\geqslant 1}\to\Sigma^{-1}G_{\geqslant 1}=G'$, such that $\varpi'^{\natural}$ is a split injection of R–modules. An easy computation

[32]This is the only place where the hypothesis of characteristic 0 is used.

shows that $\mathrm{H}_0(L') = I/I^2$, so replacing in the preceding argument F by a minimal resolution of I/I^2, we conclude that $\epsilon^{L'}\colon L' \to I/I^2$ is essential. That gives the second series of inequalities. □

Proof of Theorem 8.3.3 *and Theorem* 8.3.4. In view of Corollary 8.2.2, in each case it suffices to prove the last assertion. Using Proposition 4.2.4.1, Theorem 8.3.5, and Proposition 8.1.4, we get

$$\operatorname{curv}_R k \geq \operatorname{curv}_R(I/I^2) = \limsup \sqrt[n]{\beta_n^R(I/I^2)} \geq \limsup \sqrt[n]{\varepsilon_n(R)} = \operatorname{curv}_R k\,.$$

We have Theorem 8.3.3. An identical argument yields Theorem 8.3.4. □

9. Modules over complete intersections

Currently, homological algebra over complete intersections is an active area of research on infinite free resolutions. This chapter describes some basic techniques and results. Most proofs depend on a remarkable higher level structure on resolutions, introduced in Section 1 under more general hypotheses. It is then applied to modules over complete intersections, to study Betti numbers in Section 2, and other homological problems in Section 3.

9.1. **Cohomology operators.** In this section $R = Q/(\boldsymbol{f})$, where $\boldsymbol{f} = f_1, \dots, f_r$ is a regular sequence in a (not necessarily regular local) commutative ring Q. We denote $E = Q[y_1, \dots, y_r \,|\, \partial(y_j) = f_j]$ the Koszul complex on $\boldsymbol{f}$, and let $\kappa\colon E \to R$ be its canonical augmentation.

Extending Shamash's [143] construction of resolutions over hypersurface sections, cf. Theorem 3.1.3, Eisenbud [57] produces (in a finite number of steps, if $\operatorname{pd}_Q M$ is finite) a free resolution of an R–module M starting from any free resolution of M over Q. Here we present a version from Avramov and Buchweitz [31]; the result is somewhat weaker, but easier to prove and sufficient for our purposes.

Theorem 9.1.1. *Let M be a finite R–module, let $\epsilon^U : U \to M$ be a DG module resolution of M over E such that U_n is a free Q–module for each n.*

Let $G = Q\langle v_1, \dots, v_r\rangle$ be a Q–module with basis $\{v^{(H)} = {v_1}^{(h_1)} \cdots {v_r}^{(h_r)} : |v^{(H)}| = 2(h_1|v_1| + \dots + h_r|v_r|)\,,\, H = (h_1, \dots, h_r) \in \mathbb{N}^r\}$, and set

$$C_n(E,U) = \bigoplus_{i \geqslant 0} \overline{G}_i \otimes_R \overline{U}_{n-i}, ;$$

$$\partial(v^{(H)} \otimes u) = -\sum_{j=1}^{r} v^{(H_j)} \otimes y_j u + v^{(H)} \otimes \partial(u)$$

where $\overline{G}_i = R \otimes_Q G_i$, $\overline{U}_j = R \otimes_Q U_j$, and $H_j = (h_1, \dots, h_j - 1, \dots, h_r)$.

Then $(\mathrm{C}(E,U), \partial)$ is a free resolution of M over R.

Remark. The Koszul complex K on a regular sequence $\boldsymbol{s}$ with $(\boldsymbol{s}) \supseteq \boldsymbol{f}$ is a DG module over E; by inspection, $C(E,K) = C$, the resolution of Corollary 6.1.9.

Proof. Let $\mu\colon E \otimes_Q E \to E$ be the morphism of DG algebras, given by the multiplication of the exterior algebra. An elementary computation shows that $\operatorname{Ker}\mu$ is generated by $y'_j = y_j \otimes 1 - 1 \otimes y_j$, for $j = 1, \dots, r$. Thus, μ is the composition of $(E \otimes_Q E) \hookrightarrow D = (E \otimes_Q E)\langle v_1, \dots, v_r \,|\, \partial(v_j) = y'_j\rangle$ with $\nu\colon D \to E$, where $\nu(v^{(H)}) = 0$ if $|H| > 0$. By Proposition 1.3.2, the map

$$E \otimes_Q \kappa\colon E \otimes_Q E \to E \otimes_Q R = R\langle y_1, \dots, y_r \,|\, \partial(y_j) = 0\rangle$$

is a quasi-isomorphism. As $(E \otimes_Q \kappa)(y'_j) = y_j$, we see that $\mathrm{H}(E \otimes_Q E)$ is the exterior algebra on $\mathrm{H}_1(E \otimes_Q E)$, itself a free R–module with basis $\operatorname{cls}(y'_1), \dots, \operatorname{cls}(y'_r)$. Thus, Proposition 6.1.7 applied to the Γ-extension $E \otimes_Q E \hookrightarrow D$, shows that ν is a quasi-isomorphism of DG algebras.

Since ν is a morphism of semi-free DG modules over E for the action of E on the right, by Proposition 1.3.3 so is $\nu \otimes_E U\colon D \otimes_E U \to E \otimes_E U = U$, hence $\mathrm{H}(D \otimes_E U) \cong M$. On the other hand, $(D \otimes_E U)^\natural \cong E^\natural \otimes_Q G \otimes_Q U^\natural$ is a semi-free DG module for the action of E on the left. Thus, by Proposition 1.3.2 the morphism

$\kappa \otimes_E U \colon D \otimes_E U \to R \otimes_E D \otimes_E U$ is a quasi-isomorphism. Comparison shows that $R \otimes_E D \otimes_E U = \mathrm{C}(E,U)$ as complexes of R–modules. □

Construction 9.1.2. Cohomology operators. Let $\mathcal{S} = R[\chi_1,\dots,\chi_r]$ be a graded algebra with variables $\chi_1,\dots,\chi_r$ of degree[33] -2. In the notation of the preceding theorem, set $\chi_j \cdot v^{(H)} = v^{(H_j)}$ for $1 \le j \le r$. These are R–linear endomorphisms of degree -2 of $\mathrm{C}(E,U)^\natural$. They clearly commute with each other, and a glance at the formula for the differential ∂ of the complex $\mathrm{C}(E,U)$ shows that they are chain maps: $\chi_j\partial = \partial\chi_j$. Thus, $\mathrm{C}(E,U)$ is a DG module over the graded *algebra*[34] *of cohomology operators* $\mathcal{S}$ of the presentation $R \cong Q/(\boldsymbol{f})$.

The construction above, taken from [31], is a variant of that of Eisenbud [57], cf. Construction 9.1.5. The introduction of operators of degree -2 on (co)homology is due to Gulliksen [80]; other constructions have been given by Mehta [119] and Avramov [25]. For a long time, it had been held that they coincide, but a close reading of the published arguments has revealed serious flaws. In fact, they yield the same result, but only up to sign: this is proved in [37]; ironically, that proof introduces two new constructions.

Proposition 9.1.3. *For each R–module N there are $\mathcal{S}$–linear homomorphisms*

$$\begin{aligned} \chi_j \colon \operatorname{Tor}^R_n(M,N) &\to \operatorname{Tor}^R_{n-2}(M,N) \\ \chi_j \colon \operatorname{Ext}_R^n(M,N) &\to \operatorname{Ext}_R^{n+2}(M,N) \end{aligned} \qquad \text{for} \quad 1 \le j \le r \quad \text{and all} \quad n\,,$$

which turn $\operatorname{Tor}^R(M,N)$ and $\operatorname{Ext}_R(M,N)$ into modules over $\mathcal{S}$.

These structures depend only on $\boldsymbol{f}$, are natural in both module arguments, and commute with the connecting maps induced by short exact sequences.

Proof. For the first statement, observe that for each R–module N, the complexes $\mathrm{C}(E,U)\otimes_R N$ and $\operatorname{Hom}_R(\mathrm{C}(E,U),N)$ have an induced structure of DG $\mathcal{S}$–module. Naturality in N is clear, as is linearity of the connecting homomorphisms induced by an exact sequence $0 \to N' \to N \to N'' \to 0$.

If $\beta \colon M' \to M$ is a homomorphism of R–modules, and U' is a resolution of M' given by Construction 2.2.7, then by the lifting property of Proposition 1.3.1 there is a morphism $\alpha \colon U' \to U$ of DG modules over E such that $\mathrm{H}(\alpha) = \beta$. The expressions for the differential in Theorem 9.1.1, and for the action of χ_j in Construction 9.1.2 show that $v^{(H)} \otimes u' \mapsto v^{(H)} \otimes \alpha(u')$ defines a morphism of DG $\mathcal{S}$–modules $\mathrm{C}(E,\alpha)\colon \mathrm{C}(E,U') \to \mathrm{C}(E,U)$. All choices of α are homotopic, so the degree 0 maps of $\mathcal{S}$–modules $\mathrm{H}(\mathrm{C}(E,\alpha)\otimes_R N)$ and $\mathrm{H}\operatorname{Hom}_R(\mathrm{C}(E,\alpha),N)$ are uniquely defined, and equal respectively to $\operatorname{Tor}^R(\beta,N)$ and $\operatorname{Ext}_R(\beta,N)$. This proves naturality in M, and independence from the choice of U.

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R–modules, and choose a semi-free resolution U'' of M'' over E, such that $U''^\natural$ is a free module over $E^\natural$. By the usual 'Horseshoe Lemma' argument, there exists a differential on $U^\natural = U'^\natural \oplus U''^\natural$, such that U becomes a DG module resolution of M over E, and the canonical exact sequence $0 \to U' \to U \to U'' \to 0$ is one of DG modules over

[33] This will not be surprising, once the χ_j's reveal their cohomological nature.

[34] The algebra $\mathcal{S}$ itself has a trivial differential; this nicely illustrates the fact that DG module structures are to be found in all walks of life.

E. Due to the expression for the differential in Theorem 9.1.1, it gives rise to an exact sequence of DG modules over $\mathcal{S}$:

$$0 \to \mathrm{C}(E, U') \to \mathrm{C}(E, U) \to \mathrm{C}(E, U'') \to 0$$

that splits over R. It induces short exact sequences of DG modules over $\mathcal{S}$

$$0 \to \mathrm{C}(E, U') \otimes_R N \to \mathrm{C}(E, U) \otimes_R N \to \mathrm{C}(E, U'') \otimes_R N \to 0$$

$$0 \to \mathrm{Hom}_R\left(\mathrm{C}(E, U''), N\right) \to \mathrm{Hom}_R\left(\mathrm{C}(E, U), N\right) \to \mathrm{Hom}_R\left(\mathrm{C}(E, U'), N\right) \to 0$$

Their connecting maps commute with the action of the operators χ_j. □

The importance of the algebra of cohomology operators stems from

Theorem 9.1.4. *If M and N are finite modules over a noetherian ring R, such that $R = Q/(\boldsymbol{f})$ for some Q–regular sequence $\boldsymbol{f}$, then the $\mathcal{S}$–module $\mathrm{Ext}_R(M, N)$ is finite if and only if $\mathrm{Ext}_Q^n(M, N) = 0$ for $n \gg 0$.*

Remark. Most of the remaining results in this chapter are based on this theorem. Section 2 uses the 'if' part; different proofs for it are given in each one of the papers quoted in Construction 9.1.2; here we use an elementary argument to establish a special case, that suffices for many applications. Section 3 is based on the converse statement in the special case $N = k$, proved in [25]; the general result is established in [32].

Partial proof of Theorem 9.1.4. Assume that Q is noetherian, finite projective Q–modules are free, and $\mathrm{pd}_R M$ is finite. Proposition 2.2.8 then yields a DG module resolution U of M over E, which is a finite complex of free Q–modules. By the preceding result, we may use U to compute the action of $\mathcal{S}$. As $\mathrm{Hom}_R(C(E, U), R)$ is a semi-free DG module over $\mathcal{S}$ with underlying module $\mathcal{S} \otimes_R \mathrm{Hom}_Q(U, R)$, we see that it suffices to prove the

Claim. If $\mathcal{F}$ is a semi-free $\mathcal{S}$–module of finite rank, then for each finite R–module N the $\mathcal{S}$–module $\mathrm{H}(\mathcal{F} \otimes_R N)$ is noetherian.

The advantage is that now we can induce on $n = \mathrm{rank}_{\mathcal{S}} \mathcal{F}$. If $n = 1$, then $\mathcal{F}$ is a shift of $\mathcal{S}$, so $\mathrm{H}(\mathcal{F} \otimes_R N) \cong \Sigma^r \mathcal{S} \otimes_R N$ is a finite $\mathcal{S}$–module. If $n > 0$, then choose a basis element $u \in \mathcal{F}$ of minimal degree. As $\partial(u) = 0$ for degree reasons, $\mathcal{S}u$ is a DG submodule of $\mathcal{F}$, and $\mathcal{G} = \mathcal{F}/\mathcal{S}u$ is semi-free of rank $n-1$. The homology exact sequence now yields an exact sequence of degree zero homomorphisms of $\mathcal{S}$–modules $\mathcal{S}u \otimes_R N \to \mathrm{H}(\mathcal{F} \otimes_R N) \to \mathrm{H}(\mathcal{G} \otimes_R N)$, where the two outer ones are noetherian by induction. The claim follows. □

Eisenbud [57] shows how to compute the operators from any resolution.

Construction 9.1.5. Eisenbud operators. A *lifting* to Q of a free resolution (F, ∂) of M over R is a pair $(\widetilde{F}, \widetilde{\partial})$ consisting of a free Q–module $\widetilde{F}$ and a degree -1 endomorphism $\widetilde{\partial}$ of $\widetilde{F}$, such that $(F, \partial) = (\widetilde{F} \otimes_Q R, \widetilde{\partial} \otimes_Q R)$.

Liftings always exist—just take arbitrary inverse images in Q of the elements of the matrices of the differentials ∂_n. The relation $\partial^2 = 0$ yields $\widetilde{\partial}^2(\widetilde{F}) \subseteq (\boldsymbol{f})\widetilde{F}$, hence for $j = 1, \dots, r$ there are degree -2 endomorphisms of Q–modules $\widetilde{\tau}^j : \widetilde{F} \to \widetilde{F}$, such that $\widetilde{\partial}^2 = \sum_{j=1}^r f_j \widetilde{\tau}^j$.

Each lifting produces a family of *Eisenbud operators*

$$\boldsymbol{\tau} = \{\tau^j = \widetilde{\tau}^j \otimes_Q R \colon F \to F\}_{1 \leqslant j \leqslant r} .$$

Proposition 9.1.6. *Let $\boldsymbol{\tau}$ be a family of Eisenbud operators defined by $\boldsymbol{f}$.*

For $1 \le j \le r$ the maps τ^j are chain maps of degree -2, that are defined uniquely up to homotopy, commute with each other up to homotopy, commute up to homotopy with any comparison of resolutions $F' \to F$ constructed over a homomorphism of R–modules $\beta\colon M' \to M$, and satisfy

$$\mathrm{H}(\mathrm{Hom}_R\left(\tau^j, N\right)) = -\chi_j\,.$$

Proof. Let $(\widetilde{F}', \widetilde{\partial}')$ be a lifting of a free resolution (F', ∂') of an R–module M', chosee a family of maps $\widetilde{\boldsymbol{\tau}}' = \{\widetilde{\tau}'^j\}\colon F' \to F'$ as above, and set $\boldsymbol{\tau}' = \{\tau'^j = \widetilde{\tau}'^j \otimes_Q R\}$. If $\alpha\colon F' \to F$ is a chain map and $\widetilde{\alpha}\colon \widetilde{F}' \to \widetilde{F}$ is a map of Q–modules such that $\widetilde{\alpha} \otimes_Q R = \alpha$, then the equality $\partial\alpha = (-1)^{|\alpha|}\alpha\partial'$ implies that for $1 \le j \le r$ there exist Q–linear homomorphisms $\sigma^j\colon \widetilde{F}' \to \widetilde{F}$ with $|\sigma^j| = |\alpha| - 1$ and $\widetilde{\partial}\widetilde{\alpha} - (-1)^{|\alpha|}\widetilde{\alpha}\widetilde{\partial}' = \sum_{j=1}^r f_j\sigma^j$. Thus, we have

$$\begin{aligned}\sum_{j=1}^r f_j\left(\widetilde{\tau}^j\widetilde{\alpha} - \widetilde{\alpha}\widetilde{\tau}'^j\right) &= \widetilde{\partial}^2\widetilde{\alpha} - \widetilde{\alpha}\widetilde{\partial}'^2 \\ &= \left(\sum_{j=1}^r \widetilde{\partial} f_j\sigma^j + (-1)^{|\alpha|}\widetilde{\partial}\widetilde{\alpha}\widetilde{\partial}'\right) + (-1)^{|\alpha|}\left(\sum_{j=1}^r f_j\sigma^j\widetilde{\partial}' - \widetilde{\partial}\widetilde{\alpha}\widetilde{\partial}'\right) \\ &= \sum_{j=1}^r f_j\left(\widetilde{\partial}\sigma^j - (-1)^{|\sigma^j|}\sigma^j\widetilde{\partial}'\right).\end{aligned}$$

Since the elements of $\boldsymbol{f}$ are linearly independent modulo I^2, we get

$$\tau^j\alpha - \alpha\tau'^j = \partial(\sigma^j \otimes_Q R) - (-1)^{|\sigma^j|}(\sigma^j \otimes_Q R)\partial' \qquad \text{for} \quad 1 \le j \le r\,,$$

that is, $\sigma^j \otimes_Q R\colon F' \to F$ is a homotopy from $\tau^j\alpha$ to $\alpha\tau'^j$. We can now get most of the desired assertions by suitably specializing the maps chosen above.

First, letting $\alpha = \partial' = \partial$ and $\widetilde{\alpha} = \widetilde{\partial}' = \widetilde{\partial}$, we can set $\sigma^j = 0$ for $1 \le j \le r$, and so conclude that each τ^j is a chain map. Next, taking $\alpha = \mathrm{id}^F$ and varying τ', we see that $\tau^1, \dots, \tau^r$ are defined uniquely up to homotopy. Then, keeping $\alpha = \tau^j$ and $\boldsymbol{\tau}' = \boldsymbol{\tau}$, we see that τ^j commutes up to homotopy with each τ^i. Finally, choosing α to be a lifting of a homomorphism of R–modules $\beta\colon M' \to M$, we obtain that $\tau^j\alpha$ and $\alpha\tau'^j$ are homotopic for each j.

The resolution $F = (\mathrm{C}(E, U), \partial)$ of Theorem 9.1.1 has an obvious lifting:

$$\widetilde{F} = U^\natural \otimes_Q G \qquad \text{with} \quad \widetilde{\partial}(v^{(H)} \otimes u) = -\sum_{j=1}^r v^{(H_j)} \otimes y_j u + v^{(H)} \otimes \partial(u)\,.$$

From it we get $\widetilde{\partial}^2(v^{(H)} \otimes u) = -\sum_{j=1}^r v^{(H_j)} \otimes f_j u = -\sum_{j=1}^r f_j\chi_j(v^{(H)} \otimes u)$ and hence $\mathrm{H}(\mathrm{Hom}_R\left(\tau^j, N\right)) = -\chi_j$ for $1 \leqslant j \leqslant r$. □

Remark 9.1.7. For any integer d with $1 \le d \le r$, the operators $\chi_1, \dots, \chi_d$ act on $\mathrm{Ext}_R^*(M, k)$ in two ways: the initial one, from $R = Q/(\boldsymbol{f})$, and a new one, from the presentation $R = P/(f_1, \dots, f_d)$ with $P = Q/(f_{d+1}, \dots, f_r)$.

These actions coincide. Indeed, if $(\widetilde{F}, \widetilde{\partial})$ is a lifting to Q of a free resolution (F, ∂) of M over R, then it is clear that $(\widetilde{F} \otimes_Q P, \widetilde{\partial} \otimes_Q P)$ is a lifting of (F, ∂) to P. In this case we have $(\widetilde{\partial} \otimes_Q P)^2 = \sum_{j=1}^d f_j(\widetilde{\tau}^j \otimes_Q P)$. Thus, we may use $\widetilde{\tau}^j \otimes_Q P$

to compute the operation of χ_j coming from the new presentation. It remains to observe that $(\widetilde{\tau}^j \otimes_Q P) \otimes_P R = \tau^j \otimes_Q R$.

9.2. **Betti numbers.** Our method for studying homology over complete intersections is to use the action of the algebra of cohomology operators, in order to replace 'degree by degree' computations by 'global' considerations.

At that level, we are essentially dealing with finite graded modules over polynomial rings. This converts homological algebra back into commutative algebra, and opens the door to the use geometric methods to study cohomology. Such an approach was pioneered by Quillen [132] for cohomology of groups, and has evolved into a powerful tool of modular representation theory, cf. Benson [41] and Evens [62] for monographic expositions. Geometric methods are used in [25] to study resolutions over commutative rings.

For reference and comparison, the next theorem is presented along the lines of Theorem 5.3.3. It is compiled from four papers: the fact that $\mathrm{P}^R_M(t)$ is rational with denominator $(1-t^2)^{\operatorname{codim} R}$ is from Gulliksen [80]; the comparison of the orders of the poles, and (3), are from Avramov [25]; the first part of (5) comes from Eisenbud [57], the second from Avramov, Gasharov, and Peeva [32].

Theorem 9.2.1. *Let R be a complete intersection with* $\operatorname{edim} R = e$ *and* $\operatorname{codim} R = r$. *For a finite R–module $M \neq 0$ with* $\operatorname{depth} R - \operatorname{depth} M = m$ *and* $\operatorname{pd}_R M = \infty$, *the following hold.*

(1) *There is a polynomial $p(t) \in \mathbb{Z}[t]$ with $p(\pm 1) \neq 0$, such that*

$$\mathrm{P}^R_M(t) = \frac{p(t)}{(1+t)^c(1-t)^d} \qquad \text{with} \quad c < d\,.$$

(2) $\operatorname{cx}_R M = d \leq \operatorname{codim} R$ *and* $\operatorname{curv}_R M = 1$.

(3) $\beta^R_n(M) \sim \dfrac{b}{2^c(d-1)!}\, n^{d-1}$ *where* $b = p(1) > 0$.

(4) $\displaystyle\lim_{n\to\infty} \frac{\beta^R_{n+1}(M)}{\beta^R_n(M)} = 1\,.$

(5.1) $\dfrac{\beta^R_{n+1}(M)}{\beta^R_n(M)} = 1$ *and* $\operatorname{Syz}^R_{n+2}(M) \cong \operatorname{Syz}^R_n(M)$ *for $n > m$ if* $\operatorname{cx}_R M = 1$.

(5.2) $\dfrac{\beta^R_{n+1}(M)}{\beta^R_n(M)} > 1$ *and* $\operatorname{Syz}^R_{n+2}(M) \twoheadrightarrow \operatorname{Syz}^R_n(M)$ *for $n \gg 0$ if* $\operatorname{cx}_R M \geq 2$.

Remark. A more precise version of the last inequality is proved in [32]: there are polynomials $h_\pm(t)$ of degree $d-2$ with leading terms $a_\pm > 0$, such that for $n \gg 0$ the difference $\beta^R_{n+1}(M) - \beta^R_n(M)$ is equal to $h_+(n)$ if n is even, and to $h_-(n)$ if n is odd; however, it is possible that $a_+ \neq a_-$, cf. Example 9.2.4.

Example 9.2.2. By Remark 8.1.1.2, we have $\beta^R_n(k) \sim 2^{e-r} n^{r-1}/(r-1)!$, so $c = -\dim R$, $d = \operatorname{codim} R$, $b = 1$, and $p_k(t) = 1$.

Recall that if R is a complete intersection, then $\operatorname{mult} R \geq 2^{\operatorname{codim} R}$.

Example 9.2.3. If $\operatorname{mult}(R) = 2^r$, then for each M there is an integer valued polynomial $b(t) \in \mathbb{Q}[t]$ such that $\beta^R_n(M) = b(n)$ for $n \gg 0$, cf. [28]. This generalizes a well known property of complete intersections of *quadrics*.

Not all Betti sequences are eventually given by some polynomial in n.

Example 9.2.4. Let $q = \binom{e+1}{2} - \operatorname{rank}_k \mathfrak{m}^2/\mathfrak{m}^3$ be the number of 'quadratic relations' of R. It is proved in [28] that

$$\mathrm{P}^R_{R/\mathfrak{m}^2}(t) = \frac{(1-t)^q + (1+t)^{e-q-1}\cdot(et-1)}{(1-t)^r\cdot(1+t)^{r-q-1}\cdot t}\ .$$

Thus, when $q \le r-2$ the Poincaré series has poles at $t=1$ *and* at $t=-1$, so the even and odd Betti numbers are each given by a different polynomial. For instance, if $R = k[s_1,s_2]/(s_1^{a_1}, s_2^{a_2})$, with $a_i \ge 3$, then $\beta^R_n(R/\mathfrak{m}^2)$ is equal to $\frac{3}{2}n+1$ if n is even, and to $\frac{3}{2}n+\frac{3}{2}$ if n is odd.

As (5.1) shows, if a Betti sequence is bounded, then it stabilizes after at most $\operatorname{depth} R$ steps. However, if $\operatorname{cx}_R M \ge 2$, then there exist modules whose Betti sequence *strictly decreases* over an initial interval of any given length. This shows that no bound on the degree of the polynomial $p(t)$ can be expressed as a function only of invariants of the ring R:

Example 9.2.5. Let R be a complete intersection of codimension $c \ge 2$. Fix $N = \operatorname{Syz}^R_n(k)$, with $n > \dim R$, and let F be its minimal free resolution. The module N is maximal Cohen-Macaulay, cf. 1.2.8, hence the complex

$$0 \longrightarrow N^* \longrightarrow F_0^* \xrightarrow{\partial_0^*} F_1^* \xrightarrow{\partial_1^*} F_2^* \longrightarrow \dots\ ,$$

where $-^* = \operatorname{Hom}_R(-,R)$, is exact and minimal. Splice it to the right of a minimal free resolution of N^*: now you are holding a 'doubly infinite' exact complex of finite free R–modules, that you can truncate at will. The cokernel of ∂_s^* is guaranteed to have $s+1$ strictly decreasing Betti numbers at the beginning of its resolution, cf. Remark 8.1.1.3.

Before starting on the proof, we make a general observation.

Remark 9.2.6. Let Q be a regular local ring, let $\boldsymbol{f}$ be a Q–regular sequence, and set $R = Q/(\boldsymbol{f})$. If M is a finite R–module, then $\operatorname{Ext}_R(M,k)$ is a finite module over $R[\chi_1,\dots,\chi_r]$ by Theorem 9.1.4.

Since $\mathfrak{m}$ annihilates $\operatorname{Ext}_R(M,k)$, we see that $\mathcal{M} = \operatorname{Ext}_R(M,k)$ is a finit module over the graded polynomial ring $\mathcal{P} = k[\chi_1,\dots,\chi_r]$. In particular, the Hilbert-Serre Theorem applies to the graded $\mathcal{P}$–module $\mathcal{M}$, and shows that $\mathrm{P}^R_M(t) = q(t)/(1-t^2)^r$ for some polynomial $q(t) \in \mathbb{Z}[t]$.

Proof of Theorem 9.2.1. The hypotheses of the theorem and its conclusions do not change if one replaces (R,M) by $(R', M\otimes_R R')$, where R' is the completion of the local ring $R[u]_{\mathfrak{m}[u]}$. Thus, we assume that $R = Q/(\boldsymbol{f})$, where Q is regular with infinite residue field k, and $\boldsymbol{f}$ is a regular sequence.

Let F be a minimal free resolution of M over R, and set $\beta_n = \beta^R_n(M)$.

(1) Due to Remark 9.2.6, $\mathrm{P}^R_M(t)$ can be written in the form

$$\mathrm{P}^R_M(t) = \sum_{j=0}^{d-1}\frac{m_j}{(1-t)^{d-j}} + \sum_{i=0}^{c-1}\frac{\ell_i}{(1+t)^{c-i}} + f(t)\,,$$

with $\max\{c,d\} \le r$ and $f(t) \in \mathbb{Q}[t]$. Thus, for $n \gg 0$, there are equalities

$$\beta_n = \begin{cases} \dfrac{m_0}{(d-1)!} \cdot n^{d-1} + \dfrac{\ell_0}{(c-1)!} \cdot n^{c-1} + g_+(n) & \text{for even } n\,; \\ \dfrac{m_0}{(d-1)!} \cdot n^{d-1} - \dfrac{\ell_0}{(c-1)!} \cdot n^{c-1} + g_-(n) & \text{for odd } n\,; \end{cases} \qquad (*)$$

with $m_0 \neq 0$, and polynomials $g_\pm(t)$ of degree $< \max\{c,d\} - 1$. As the Betti numbers of M are positive, we have $d \ge c$, and $d > 0$.

Assume next that $d = c$, so that $\ell_0 \neq 0$. The positivity of Betti numbers implies that $m_0 \pm \ell_0 > 0$, hence $m_0 > 0$.

Set $\gamma(j, 2s) = \sum_{i=-j}^{j} (-1)^i \beta_{2s-i}$. Formula $(*)$ shows that for all $s, h \gg 0$ the function $2s \mapsto \gamma(2h, 2s)$ is given by a polynomial in $2s$ of degree d with leading coefficient $a_0 = \big((4h+1)\ell_0 + m_0\big)/(d-1)!$, and the function $2s \mapsto \gamma(2h+1, 2s)$ by a polynomial of the same degree with leading coefficient $a_1 = \big((4h+3)\ell_0 - m_0\big)/(d-1)!$. Thus: if $\ell_0 < 0$, then $a_0 < 0$ for $h \gg 0$, so $\gamma(2h, 2s) < 0$; if $\ell_0 > 0$, then $a_1 > 0$ for $h \gg 0$, so $\gamma(2h+1, 2s) > 0$.

Localization of F at a minimal prime ideal $\mathfrak{p}$ of R yields an exact sequence

$$0 \to L_{s,j} \to (F_{2s+j})_{\mathfrak{p}} \to \dots \to (F_{2s})_{\mathfrak{p}} \to \dots \to (F_{2s-j})_{\mathfrak{p}} \to N_{s,j} \to 0\,.$$

Counting lengths over $R_{\mathfrak{p}}$, we get an equality

$$\gamma(j, 2s) \cdot \operatorname{length}(R_{\mathfrak{p}}) = (-1)^j \big(\operatorname{length}(L_{s,j}) + \operatorname{length}(N_{s,j})\big)$$

which shows that $\gamma(2h, 2s) > 0$ and $\gamma(2h+1, 2s) < 0$, regardless of the sign of ℓ_0. We have a contradiction, so we conclude that $d > c$.

(3) Since $d > c$, formula $(*)$ yields $\lim_{n\to\infty} \beta_n / n^{d-1} = m_0/(d-1)!$. On the other hand, $m_0 = \lim_{t\to 1} (1-t)^d \operatorname{P}_M^R(t) = p(1)/2^c$.

(2) and (4) are trivial consequences of (1) and (3).

(5) By Theorem 9.1.4, $\operatorname{Ext}_R(M,k)$ is a finite graded module over the polynomial ring $k[\chi_1, \dots, \chi_r]$. Thus, its graded submodule

$$\{\mu \in \operatorname{Ext}_R(M,k) \mid (\chi_1, \dots, \chi_r)^m \mu = 0 \text{ for some } m\}$$

is finite-dimensional, and hence is trivial, say, in degrees $> s$. Since k is infinite, we can find a linear combination χ of $\chi_1, \dots, \chi_r$, that is a non-zero-divisor on $\operatorname{Ext}_R^{>s}(M,k)$. Thus, the operator χ is injective on $\operatorname{Ext}_R^{>s}(M,k)$. Dualizing, we see that $\chi\colon \operatorname{Tor}^R_{n+2}(M,k) \to \operatorname{Tor}^R_n(M,k)$ is surjective when $n > s$.

Changing bases, we may assume that $\chi = \chi_1$, and switch attention to the presentation $R = P/(f)$, where $P = Q/(f_2, \dots, f_r)$ and f is the image of f_1; note that f is P–regular. Let $(\widetilde{F}, \widetilde{\partial})$ be a lifting of the complex (F, ∂) to P, and let $\widetilde{\tau}\colon \widetilde{F} \to \widetilde{F}$ and $\tau = \widetilde{\tau} \otimes_P R\colon F \to F$ be the degree -2 endomorphisms from Construction 9.1.5. By Remark 9.1.7, we have $\chi = \operatorname{Hom}_R(\tau, k)$.

Since χ_n is surjective for $n > s$, so are the maps $\widetilde{\tau}_{n+2}\colon \widetilde{F}_{n+2} \to \widetilde{F}_n$ and $\tau_{n+2}\colon F_{n+2} \to F_n$ by Nakayama. The chain map τ induces surjections $\operatorname{Syz}^R_{n+2}(M) \to \operatorname{Syz}^R_n(M)$ for $n > s$. Localize the defining exact sequence

$$0 \to \operatorname{Syz}^R_{n+2}(M) \to F_{n+1} \xrightarrow{\partial_{n+1}} F_n \to \operatorname{Syz}^R_n(M) \to 0$$

at a minimal prime $\mathfrak{p}$ of R. For $n > s$ a lengths count over $R_{\mathfrak{p}}$ yields

$$\beta_{n+1} - \beta_n = \big(\operatorname{length} \operatorname{Syz}^{R_{\mathfrak{p}}}_{n+2}(M_{\mathfrak{p}}) - \operatorname{length} \operatorname{Syz}^{R_{\mathfrak{p}}}_n(M_{\mathfrak{p}})\big) \big/ \operatorname{length}(R_{\mathfrak{p}}) \ge 0\,,$$

Next we assume that $\beta_n = \beta_{n+1} = b \neq 0$ for some $n > s$, and show that $\beta_{n+2} = b$. Since $\widetilde{\tau}_{n+2}$ is surjective for $n > s$, we have $\widetilde{F}_{n+2} = E \oplus G$ with $E = \operatorname{Ker} \widetilde{\tau}_{n+2}$, and the restriction θ of $\widetilde{\tau}_{n+2}$ to G is an isomorphism with $\widetilde{F}_n$. Let $\zeta \colon G \longrightarrow \widetilde{F}_{n+1}$ be the restriction of $\widetilde{\partial}_{n+2}$. As $\widetilde{\partial}_{n+1}\zeta$ is the restriction to G of $\widetilde{\partial}_{n+1}\widetilde{\partial}_{n+2} = f\widetilde{\tau}_{n+2}$, where $P/(f) = R$, we have $\widetilde{\partial}_{n+1}\zeta = f\theta$, and hence

$$\bigwedge^b \widetilde{\partial}_{n+1} \bigwedge^b \zeta = \bigwedge^b (\widetilde{\partial}_{n+1}\zeta) = \bigwedge^b (f\theta) = f^b \bigwedge^b \theta \,.$$

Note that G, $\widetilde{F}_{n+1}$, and $\widetilde{F}_n$ have rank b and fix isomorphisms of P with $\bigwedge^b(G)$, $\bigwedge^b(\widetilde{F}_{n+1})$, and $\bigwedge^b(\widetilde{F}_n)$. The maps $\bigwedge^b \widetilde{\partial}_{n+1}$, $\bigwedge^b \zeta$, and $\bigwedge^b \theta$ are then given by multiplication with elements of P, say y, z, and u, respectively. The equality above becomes $yz = f^b u$. As θ is bijective so is $\bigwedge^b(\theta)$, hence u is a unit in P. As f is P-regular, so is y, hence $\widetilde{\partial}_{n+1}$ is injective. From $\widetilde{\partial}_{n+1}\widetilde{\partial}_{n+2}(E) = f\widetilde{\tau}_{n+2}(E) = 0$ we now see that $E \subseteq \operatorname{Ker} \widetilde{\partial}_{n+2}$, so $\operatorname{Im} \widetilde{\partial}_{n+2}$ is a homomorphic image of $\widetilde{F}_{n+2}/E \cong G$. Remarking that

$$(\operatorname{Coker} \widetilde{\partial}_{n+3}) \otimes_P R \cong \operatorname{Coker} \partial_{n+3} = \operatorname{Syz}^R_{n+2}(M)$$

we conclude that $\operatorname{Syz}^R_{n+2}(M)$ is a homomorphic image of the free R–module $G \otimes_P R \cong R^b$. It follows that $\beta_{n+2} \leq b = \beta_n$. On the other hand, we already know that $\beta_{n+2} \geq \beta_{n+1} \geq \beta_n$, hence all three are equal to b. Thus, the sequence $\{\beta_n\}_{n>s}$ is either strictly increasing or constant: we have proved (5.2).

If $\beta_{n+2} = \beta_n$, then $\operatorname{rank}_P \widetilde{F}_{n+2} = \operatorname{rank}_P \widetilde{F}_n$, so $E = 0$ and the surjective homomorphism τ_{n+2} is bijective. To finish the proof of (5.1), we show that for $m = \operatorname{depth} R - \operatorname{depth} M$ the complex $F_{>m}$ is periodic of period 2. It is a minimal resolution of $N = \operatorname{Syz}^R_{m+1}(M)$, and N is maximal Cohen-Macaulay by Proposition 1.2.8. Thus, $F^*_{>m} = \operatorname{Hom}_R(F_{>m}, R)$ is acyclic, with $\operatorname{H}_0(F^*_{>m}) = N^*$. Since $F^*_{>m}$ is minimal, N^* is a syzygy of $C_n = \operatorname{Coker} \partial^*_n$ for each $n \geq m$. For $n \gg 0$ the minimal resolution of C_n is periodic of period 2, hence so is $F^*_{>m}$. □

9.3. **Complexity and Tor.** Let $(R, \mathfrak{m}, k)$ be a local ring.

If $R = Q/I$ is a complete intersection and Q is regular, then the finite global dimension of Q implies that *all* R–modules have finite complexity. However, to study a *specific* R–module, it often pays off to use an intermediate (singular) complete intersection P, that retains the crucial property $\operatorname{pd}_P M < \infty$. With this approach, the following *factorization theorem* is proved in [25].

Theorem 9.3.1. *Let $R \cong Q/I$ be a regular presentation with I generated by a regular sequence. If k is infinite, then for each finite R–module M the surjection $Q \to R$ factors as $Q \to P \to R$, with the kernels of both maps generated by regular sequences, $\operatorname{pd}_P M < \infty$, and $\operatorname{cx}_R M = \operatorname{pd}_P R$.*

Proof. As in Remark 9.2.6, consider the finite graded module $\mathcal{M} = \operatorname{Ext}^*_R(M, k)$ over the ring $\mathcal{P}$ defined by the presentation $R = Q/I$. Elementary dimension theory shows that the Krull dimension of $\mathcal{M}$ over $\mathcal{P}$ is equal to $\operatorname{cx}_R M = d$. As k is infinite, we may choose a homogeneous system of parameters $\chi_1, \ldots, \chi_d$ for $\mathcal{M}$, and extend it to a basis $\chi_1, \ldots, \chi_r$ of $\mathcal{P}^2$, the degree 2 component of $\mathcal{P}$.

It is not hard to see that I can be generated by a Q–regular sequence $\boldsymbol{f} = f_1 \ldots, f_r$ that defines the operators $\chi_1, \ldots, \chi_r$. Remark 9.1.7 identifies $k[\chi_1, \ldots, \chi_d] \subseteq \mathcal{P}$ with the ring $\mathcal{P}'$ of cohomology operators of a presentation

$R = P/(f_1, \dots, f_d)$, where $P = Q/(f_{d+1}, \dots, f_r)$. As $\mathcal{M}$ is finite over $\mathcal{P}'$, Theorem 9.1.4 shows that $\operatorname{Ext}_P^n(M,k) = 0$ for $n \gg 0$, that is, $\operatorname{pd}_P M < \infty$. □

To deal with intrinsic properties of the R–module M, a concept of *virtual projective dimension* is introduced in [25] by the formula[35]

$$\operatorname{vpd}_R M = \inf \left\{ \operatorname{pd}_{Q'} \widehat{M} \;\middle|\; \begin{array}{c} Q' \text{ is a local ring such that } \widehat{R} \cong Q'/(\boldsymbol{f}') \\ \text{for some } Q'\text{–regular sequence } \boldsymbol{f}' \end{array} \right\}.$$

Clearly, $\operatorname{vpd}_R M < \infty$ whenever R is a complete intersection.

Recall that $\operatorname{pd}_R M$ is finite if and only if $\operatorname{cx}_R M = 0$. It is easy to see that in that case, $\operatorname{vpd}_R M = \operatorname{pd}_R M$. Thus, the following result extends the Auslander-Buchsbaum Equality.

Theorem 9.3.2. *If M is a finite R–module and $\operatorname{vpd}_R M$ is finite, then*

$$\operatorname{vpd}_R M = \operatorname{depth} R - \operatorname{depth}_R M + \operatorname{cx}_R M\,.$$

Proof. We may assume that R is complete with infinite residue field.

Choosing Q' with $\operatorname{pd}_{Q'} M = \operatorname{vpd}_R M$, we have

$$\begin{aligned} \operatorname{vpd}_R M &= \operatorname{pd}_{Q'} M = \operatorname{depth} Q' - \operatorname{depth} M \\ &= \operatorname{pd}_{Q'} R + \operatorname{depth} R - \operatorname{depth} M \geq \operatorname{cx}_R M + \operatorname{depth} R - \operatorname{depth} M \end{aligned}$$

where the inequality comes from Corollary 4.2.5.4. On the other hand, the preceding theorem provides a ring P from which R is obtained by factoring out a regular sequence, and that satisfies $\operatorname{pd}_P R = \operatorname{cx}_R M$, so we get

$$\begin{aligned} \operatorname{vpd}_R M &\leq \operatorname{pd}_P M = \operatorname{depth} P - \operatorname{depth} M \\ &= \operatorname{pd}_P R + \operatorname{depth} R - \operatorname{depth} M = \operatorname{cx}_R M + \operatorname{depth} R - \operatorname{depth} M \end{aligned}$$

where the inequality holds by definition. □

Next we study the vanishing of Tor functors over a local ring. The subject starts with a famous rigidity theorem of Auslander [16] and Lichtenbaum [112]:

Theorem 9.3.3. *If M and N are finite modules over a regular ring R and*

$$\operatorname{Tor}_i^R(M,N) = 0 \quad \textit{for some} \quad i > 0\,,$$

then $\operatorname{Tor}_n^R(M,N) = 0$ for all $n \geq i$. ■

Heitmann [85] proves that rigidity may fail, even with R Cohen-Macaulay and $\operatorname{pd}_R M$ finite. On the other hand, there are partial extensions of the theorem to complete intersections. The first one is due to Murthy [123].

Theorem 9.3.4. *If M and N are finite modules over a complete intersection R of codimension r, and for some $i > 0$ there are equalities*

$$\operatorname{Tor}_i^R(M,N) = \cdots = \operatorname{Tor}_{i+r}^R(M,N) = 0$$

then $\operatorname{Tor}_n^R(M,N) = 0$ for all $n \geq i$.

In codimension 1, this is complemented by Huneke and Wiegand [91]:

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[35] If k is infinite; otherwise, R and M are replaced by $\widetilde{R} = R[u]_{\mathfrak{m}[u]}$ and $\widetilde{M} = M \otimes_R \widetilde{R}$.

Theorem 9.3.5. *If M and N are finite modules over a hypersurface R, and*

$$\operatorname{Tor}_i^R(M,N) = \operatorname{Tor}_{i+1}^R(M,N) = 0 \quad \text{for some} \quad i>0\,,$$

then either M or N has finite projective dimension.

When the vanishing occurs outside of an initial interval, Jorgensen [95] draws the conclusion from the vanishing of fewer Tor's.

Theorem 9.3.6. *Let M be a finite module over a complete intersection R, such that $\operatorname{cx}_R M = d$ and $\operatorname{depth} R - \operatorname{depth} M = m$. For a finite R–module N, the following are equivalent.*

(i) $\operatorname{Tor}_n^R(M,N) = 0$ *for* $n > m$.

(ii) $\operatorname{Tor}_n^R(M,N) = 0$ *for* $n \gg 0$.

(iii) $\operatorname{Tor}_i^R(M,N) = \cdots = \operatorname{Tor}_{i+d}^R(M,N) = 0$ *for some* $i > m$.

The number of vanishing Tor's in (iii) cannot be reduced further; the next example elaborates on a construction from [95].

Example 9.3.7. For $i \geq 1$, $R = k[[s_1,\dots,s_{2r}]]/(s_1s_{r+1},\dots,s_rs_{2r})$, and $N = R/(\overline{s}_{r+1},\dots,\overline{s}_{2r})$ there is a module M_i, such that $\operatorname{cx}_R M_i = r$, $\operatorname{Tor}_n^R(M_i,N) = 0$ for $i < n \leq i+r$, but $\operatorname{Tor}_n^R(M_i,N) \neq 0$ for infinitely many n.

Corollary 6.1.9 yields a minimal resolution

$$F = R\langle x_1,\dots,x_{2r} \mid \partial(x_j) = \overline{s}_j\,, \partial(x_{r+j}) = \overline{s}_{r+j}x_j \text{ for } 1 \leq j \leq r\rangle$$

of $M = R/(\overline{s}_1,\dots,\overline{s}_r) \cong k[[s_{r+1},\dots,s_{2r}]]$. As M is maximal Cohen-Macaulay, $F^* = \operatorname{Hom}_R(F,R)$ is exact in degrees $\neq 0$. It is easy to see that the sequence

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\sigma} F_0^* \xrightarrow{-\partial_1^*} F_1^* \qquad \text{with} \quad \sigma(x) = s_{r+1}\cdots s_{2r}x$$

is exact. The splice of F with $\Sigma^{-1}F^*$ along σ is a doubly infinite complex (G,∂) of free R–modules. By construction, $\operatorname{H}_n(G\otimes_R N) = \operatorname{Tor}^n_R(M,N)$ for $n \geq 1$ and $\operatorname{H}_n(G\otimes_R N) = \operatorname{Ext}_R^{1-n}(M,N)$ for $n \leq -2$; as $\sigma\otimes_R N = 0$ trivial, these equalities extend to $n=0$ and $n=-1$, respectively.

For $1 \leq j \leq r$, consider $N_j = N/(\overline{s}_{j+1},\dots,\overline{s}_r)N \cong k[[s_1,\dots,s_j]]$, note that $\operatorname{Tor}^R(M,N_j)$ and $\operatorname{Ext}_R(M,N_j)$ are annihilated by $(\overline{s}_1,\dots,\overline{s}_{2r})$, and set

$$T_j(t) = \sum_{n=0}^{\infty} \operatorname{rank}_k \operatorname{Tor}_n^R(M,N_j)\,t^n \quad \text{and} \quad E_j(t) = \sum_{n=0}^{\infty} \operatorname{rank}_k \operatorname{Ext}_R^n(M,N_j)\,t^n\,.$$

The exact sequences $0 \to N_j \xrightarrow{s_j} N_j \to N_{j-1} \to 0$ induce (co)homology sequences in which multiplication by s_j is the zero map, so

$$T_{j-1}(t) = T_j(t) + tT_j(t) \qquad \text{and} \qquad E_{j-1}(t) = E_j(t) + \frac{1}{t}E_j(t)\,.$$

Since and $N_0 \cong k$, we have $T_0(t) = E_0(t) = \operatorname{P}_M^R(t) = 1/(1-t)^r$, and hence

$$T_r(t) = \frac{T_0(t)}{(1+t)^r} = \frac{1}{(1-t^2)^r} \qquad \text{and} \qquad E_r(t) = \frac{t^rE_0(t)}{(1+t)^r} = \frac{t^r}{(1-t^2)^r}\,.$$

Now set $M_i = \operatorname{Im}\partial_{-r-i-1}$; as $\operatorname{Tor}_n^R(M_i,N) \cong \operatorname{H}_{n-r-i}(G\otimes_R N)$ for $n \geq 1$, these equalities establish the desired property.

We start the proofs with a couple of easy lemmas.

Lemma 9.3.8. *Let $f_1, \dots, f_d$ be a regular sequence in a commutative ring Q, and set $R = Q/(f_1, \dots, f_d)$. If*

$$\mathrm{Tor}_s^R(M,N) = \cdots = \mathrm{Tor}_t^R(M,N) = 0$$

for integers s and t with $s + d \le t$, then there are isomorphisms

$$\mathrm{Tor}^Q_{s+d-1}(M,N) \cong \mathrm{Tor}^R_{s-1}(M,N) ;$$
$$\mathrm{Tor}^Q_{s+d}(M,N) = \cdots = \mathrm{Tor}^Q_t(M,N) = 0 ;$$
$$\mathrm{Tor}^Q_{t+1}(M,N) \cong \mathrm{Tor}^R_{t+1}(M,N) .$$

Proof. The Cartan-Eilenberg change of rings spectral sequence 3.2.1 has

$${}^2\mathrm{E}_{p,q} = \mathrm{Tor}^R_p\left(\mathrm{Tor}^Q_q(M,R), N\right) \implies \mathrm{Tor}^Q_{p+q}(M,N) .$$

If E is the Koszul complex resolving R over Q, then

$$\mathrm{Tor}^Q_q(M,R) = \mathrm{H}_q(M \otimes_Q E) = M \otimes_Q E_q = M^{\binom{d}{q}} ,$$

hence ${}^2\mathrm{E}_{p,q} = \mathrm{Tor}^R_p(M,N)^{\binom{d}{q}}$. Thus, ${}^2\mathrm{E}_{p,q} = 0$ for $s \le p \le t$. It follows that the only possibly non-zero module in total degree $s+d-1$ is ${}^2\mathrm{E}_{s-1\,,\,d} = \mathrm{Tor}^R_{s+d-1}(M,N)$, that all modules in total degree n for $s + d \le n \le t$ are trivial, and that the only possibly non-zero module in total degree $t + 1$ is ${}^2\mathrm{E}_{t+1\,,\,0} = \mathrm{Tor}^R_{t+1}(M,N)$. For degree reasons, no non-trivial differential can enter or quit these modules. This gives the desired isomorphisms. □

Proof of Theorem 9.3.4. By the lemma, $\mathrm{Tor}^Q_{i+r}(M,N) = 0$ and $\mathrm{Tor}^Q_{i+r+1}(M,N) \cong \mathrm{Tor}^R_{i+r+1}(M,N)$, so Theorem 9.3.3 yields $\mathrm{Tor}^Q_{i+r+1}(M,N) = 0$, and we conclude that $\mathrm{Tor}^R_{i+r+1}(M,N) = 0$. Iteration yields $\mathrm{Tor}^R_n(M,N) = 0$ for $n > i + r$. □

C. Miller [120] provides a simple proof of Theorem 9.3.5, based on

Lemma 9.3.9. *Let M, N be finite modules over a complete intersection R.*

If $\mathrm{Tor}^R_n(M,N) = 0$ for $n \ge 1$, then $\mathrm{cx}_R M + \mathrm{cx}_R N = \mathrm{cx}_R(M \otimes_R N)$.

If $\mathrm{Tor}^R_n(M,N) = 0$ for $n \gg 0$, then $\mathrm{cx}_R M + \mathrm{cx}_R N \le \mathrm{codim}\, R$.

Proof. As $\mathrm{P}^R_{M\otimes_Q N}(t) = \mathrm{P}^R_M(t) \cdot \mathrm{P}^R_N(t)$, cf. the proof of Proposition 4.2.4.6, comparison of orders of poles at $t = 1$ and Theorem 9.2.1.2 yield the first assertion. For the second one, replace M by a high syzygy M'; then $\mathrm{cx}_R M' + \mathrm{cx}_R N = \mathrm{cx}_R(M' \otimes_R N) \le \mathrm{codim}\, R$, the inequality coming from *loc. cit.* □

Proof of Theorem 9.3.5. By Theorem 9.3.4, $\mathrm{Tor}^R_n(M,N) = 0$ for $n \ge i$; thus, $\mathrm{cx}_R M + \mathrm{cx}_R N \le 1$ by lemma 9.3.9, so $\mathrm{pd}_R M$ or $\mathrm{pd}_R N$ is finite. □

We use the factorization theorem to give a short

Proof of Theorem 9.3.6. Only (iii) $\implies$ (i) needs a proof. We may assume that R is complete with infinite residue field. By hypothesis, there are $s, t \in \mathbb{N}$, such that $m < s < s + d \le t$ and $\mathrm{Tor}^R_j(M,N) = 0$ for $s \le j \le t$. For the smallest such s, choose P as in Theorem 9.3.1; as $\mathrm{pd}_P M$ is finite,

$$\mathrm{pd}_P M = \mathrm{depth}\, P - \mathrm{depth}\, M = \mathrm{depth}\, R + d - \mathrm{depth}\, M = m + d .$$

We see that if $s > m + 1$, then $\mathrm{Tor}^P_{s+d-1}(M,N) = 0$; the first isomorphism in Lemma 9.3.8 yields $\mathrm{Tor}^R_{s-1}(M,N) = 0$, contradicting the minimality of s. Thus,

$s = m + 1$; it follows that $t + 1 > \operatorname{pd}_P M$, and so $\operatorname{Tor}^P_{t+1}(M, N) = 0$. The last isomorphism of the lemma yields $\operatorname{Tor}^R_{t+1}(M, N) = 0$. Iterate... □

10. Homotopy Lie algebra of a local ring

It is a remarkable phenomenon that very sensitive homological information on a local ring is encrypted in a *non-commutative* object—a graded Lie algebra. We construct it, and show that its very existence affects the size of free resolutions, while its structure influences their form.

This chapter provides a short introduction to a huge area of research: the use of non-commutative algebra for the construction and study of free resolutions. We start by providing a self-contained construction of a graded Lie algebra, whose universal enveloping algebra is the Ext-algebra of the local ring.

10.1. **Products in cohomology.** We revert to a commutative ring $\Bbbk$, and consider *graded associative*[36] algebras over $\Bbbk$. The primitive example of an associative algebra is a matrix ring. The graded version is the $\Bbbk$–module of homogeneous homomorphisms $\operatorname{Hom}_{\Bbbk}(C,C)$, with composition as product and the identity map as unit. If C is a complex, then the derivation on $\operatorname{Hom}_{\Bbbk}(C,C)$, used since Section 1.1, turns it into an associative DG algebra. It appears in

Construction 10.1.1. Ext algebras. When $\epsilon\colon F \to L$ is a free resolution of a $\Bbbk$–module L, Proposition 1.3.2 yields an isomorphism

$$\operatorname{H}\operatorname{Hom}_{\Bbbk}(F,\epsilon):\ \operatorname{H}\operatorname{Hom}_{\Bbbk}(F,F) \cong \operatorname{H}\operatorname{Hom}_{\Bbbk}(F,L) = \operatorname{Ext}_{\Bbbk}(L,L)\ .$$

Thus, F defines a structure of graded $\Bbbk$–algebra on $\operatorname{Ext}_{\Bbbk}(L,L)$.

In degree zero, it is the usual product of $\operatorname{Hom}_{\Bbbk}(L,L)$, an invariant of the $\Bbbk$–module L. To see that all of it is invariant, take a resolution F' of L, and choose morphisms $\alpha\colon F \to F'$ and $\alpha'\colon F' \to F$, lifting the identity of L. As $\alpha'\alpha$ also is such a morphism, there is a homotopy σ with $\alpha'\alpha = \operatorname{id}^F + \partial\sigma + \sigma\partial$. Define $\phi\colon \operatorname{Hom}_{\Bbbk}(F,F) \to \operatorname{Hom}_{\Bbbk}(F',F')$ by $\phi(\beta) = \alpha\beta\alpha'$. If β and $\gamma\colon F \to F$ are chain maps, then

$$\begin{aligned}\phi(\beta\gamma) &= \alpha\beta\gamma\alpha' = \alpha\beta(\alpha'\alpha)\gamma\alpha' - \alpha\beta(\partial\sigma)\gamma\alpha' - \alpha\beta(\sigma\partial)\gamma\alpha' \\ &= (\alpha\beta\alpha')(\alpha\gamma\alpha') - (-1)^{|\beta|}\partial(\alpha\beta\sigma\gamma\alpha') - (-1)^{|\gamma|}(\alpha\beta\sigma\gamma\alpha')\partial \\ &= \phi(\beta)\phi(\gamma) + \partial\tau + (-1)^{|\tau|}\tau\partial\end{aligned}$$

with $\tau = -(-1)^{|\beta|}\phi(\beta\sigma\gamma)$. In homology, this shows that $\operatorname{H}(\phi)$ is an homomorphism of algebras. As ϕ is a quasi-isomorphism by Propositions 1.3.2 and 1.3.3, $\operatorname{H}(\phi)$ is an isomorphism. It is also unique: all choices for α and α' are homotopic to the original ones, producing homotopic maps ϕ, and hence the same $\operatorname{H}(\phi)$.

We have finished the construction of the Ext *algebra* of the $\Bbbk$–module L, with the *composition product*[37].

The next structure[38] might at first seem complicated.

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[36] That is, not assumed positively graded or graded commutative.

[37] Another pairing is the *Yoneda product*, that splices exact sequences representing elements of Ext; they differ by a subtle sign, treated with care by Bourbaki [45].

[38] An early appearance is in the form $\mathfrak{g}^{n+1} = \pi_n(X)$, the n'th homotopy group of a topological space X, with bracket given by the *Whitehead product*; the proof by Uehara and Massey [152] of the Jacobi identity for the Whitehead product was the first major application of the (then) newly discovered Massey triple product.

Remark 10.1.2. A *graded Lie algebra* [39] over $\Bbbk$ is a graded $\Bbbk$–module $\mathfrak{g} = \{\mathfrak{g}^n\}_{n\in\mathbb{Z}}$ equipped with a $\Bbbk$–bilinear pairing, called the *Lie bracket*

$$[\ ,\ ]\colon \mathfrak{g}^i \times \mathfrak{g}^j \longrightarrow \mathfrak{g}^{i+j} \qquad \text{for} \quad i,j\in\mathbb{Z}\,, \qquad (\vartheta,\xi)\mapsto[\vartheta,\xi]\,,$$

such that for all ϑ, ξ, $\zeta \in \mathfrak{g}$ signed versions of the classical conditions hold:

(1) $[\vartheta,\xi] = -(-1)^{|\vartheta||\xi|}[\xi,\vartheta]$ (*anti-commutativity*)

(2) $[\vartheta,[\xi,\zeta]] = [[\vartheta,\xi],\zeta] + (-1)^{|\vartheta||\xi|}[\xi,[\vartheta,\zeta]]$ (*Jacobi identity*)

To deal with deviant behavior over rings without $\frac{1}{6}$, we extend the definition by requiring, in addition[40], that

($1\frac{1}{2}$) $[\vartheta,\vartheta] = 0$ for $\vartheta\in\mathfrak{g}^{\text{even}}$.

($2\frac{1}{3}$) $[\upsilon,[\upsilon,\upsilon]] = 0$ for $\upsilon\in\mathfrak{g}^{\text{odd}}$.

and that $\mathfrak{g}$ be endowed with a *reduced square*[41]

$$\mathfrak{g}^{2h+1} \longrightarrow \mathfrak{g}^{4h+2} \qquad \text{for} \quad h\in\mathbb{Z}\,, \qquad \upsilon\mapsto\upsilon^{[2]}\,,$$

such that the following conditions are satisfied:

(3) $(\upsilon+\omega)^{[2]} = \upsilon^{[2]} + \omega^{[2]} + [\upsilon,\omega]$ for $\upsilon,\omega\in\mathfrak{g}^{\text{odd}}$ with $|\upsilon| = |\omega|$;

(4) $(a\upsilon)^{[2]} = a^2\upsilon^{[2]}$ for $a\in\Bbbk$ and $\upsilon\in\mathfrak{g}^{\text{odd}}$;

(5) $[\upsilon^{[2]},\vartheta] = [\upsilon,[\upsilon,\vartheta]]$ for $\upsilon\in\mathfrak{g}^{\text{odd}}$ and $\vartheta\in\mathfrak{g}$.

A *Lie subalgebra* is a subset of $\mathfrak{g}$ closed under brackets and squares; with the induced operations, it is a graded Lie algebra in its own right. A homomorphism $\beta\colon\mathfrak{h}\longrightarrow\mathfrak{g}$ of graded Lie algebras is a degree zero $\Bbbk$–linear map of the underlying graded $\Bbbk$–modules, such that $\beta[\vartheta,\xi] = [\beta(\vartheta),\beta(\xi)]$ and $\beta\big(\vartheta^{[2]}\big) = \beta(\vartheta)^{[2]}$.

One way to get a Lie structure is to partly forget an associative one. Let B be a graded associative algebra over $\Bbbk$. The underlying module of B, with bracket $[x,y] = xy-(-1)^{|x||y|}yx$ (the *graded commutator*) and reduced square $v^{[2]} = v^2$ for $v\in B^{\text{odd}}$, is a graded Lie algebra, denoted $\operatorname{Lie}(B)$: the axioms are readily verified by direct computations. The non-triviality of the operations measures how far the algebra B is from being graded commutative.

There is also a vehicle to go from Lie to associative algebras. A *universal enveloping algebra* of $\mathfrak{g}$ is a graded associative $\Bbbk$–algebra U together with a degree 0 homomorphism of graded Lie algebras $\iota\colon\mathfrak{g}\longrightarrow\operatorname{Lie}(U)$ with the following property: for each associative algebra B and each Lie algebra homomorphism $\beta\colon\mathfrak{g}\longrightarrow\operatorname{Lie}(B)$, there is a unique homomorphism of associative algebras $\beta'\colon U\longrightarrow B$, such that $\beta=\beta'\iota$; we call β' the *universal extension* of β.

Remark 10.1.3. For the first few statements on enveloping algebras, one just needs to exercise plain abstract nonsense:

(1) Any two universal enveloping algebras of $\mathfrak{g}$ are isomorphic by a unique isomorphism, hence a notation $\mathrm{U}_\Bbbk(\mathfrak{g})$ is warranted.

(2) Each homomorphism $\mathfrak{h}\longrightarrow\mathfrak{g}$ of graded Lie algebras induces a natural homomorphism $\mathrm{U}_\Bbbk(\mathfrak{h})\longrightarrow\mathrm{U}_\Bbbk(\mathfrak{g})$ of graded (associative) algebras.

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[39]In postmodern parlance, a *super Lie algebra*.

[40]Anticommutativity implies $2[\vartheta,\vartheta]=0$ for $\vartheta\in\mathfrak{g}^{\text{even}}$, so ($1\frac{1}{2}$) is superfluous when $\Bbbk\ni\frac{1}{2}$. Jacobi yields $3[\vartheta,[\vartheta,\vartheta]]=0$ for all ϑ, so ($2\frac{1}{3}$) is redundant when $\Bbbk\ni\frac{1}{3}$.

[41]Only needed if $\frac{1}{2}\notin\Bbbk$: conditions (3) and (4) imply that $2\upsilon^{[2]}=[\upsilon,\upsilon]$; when 2 is invertible, $\upsilon^{[2]}=\frac{1}{2}[\upsilon,\upsilon]$ satisfies condition (5), by the Jacobi identity.

(3) The graded $\Bbbk$–algebra $\mathrm{U}_{\Bbbk}(\mathfrak{g})$ is isomorphic to the residue of the tensor algebra $\mathrm{T}_k(\mathfrak{g})$ modulo the two-sided ideal generated by

$$\vartheta\otimes\xi-(-1)^{|\vartheta||\xi|}\xi\otimes\vartheta-[\vartheta,\xi]\qquad\text{for all}\quad \vartheta,\xi\in\mathfrak{g}\,;$$
$$\upsilon\otimes\upsilon-\upsilon^{[2]}\qquad\text{for all}\quad \upsilon\in\mathfrak{g}^{\mathrm{odd}}\,;$$

the map ι is the composition of $\mathfrak{g}\subseteq T(\mathfrak{g})$ with the projection $T(\mathfrak{g})\to U$.

(4) Assume that $\mathfrak{g}^n=0$ for $n\le 0$, and that $\boldsymbol{\vartheta}=\{\vartheta_i\}_{i\geqslant 1}$ is a set of generators of $\mathfrak{g}$, linearly ordered so that $|\vartheta_i|\le|\vartheta_j|$ for $i<j$. We consider *indexing sequences* $I=(i_1,i_2,\dots)$ of integers $i_j\ge 0$, such that $i_j\le 1$ if $|\vartheta_j|$ is odd and $i_j=0$ for $j\gg 0$. For each I, pick any q such that $i_j=0$ for $j>q$, and form the (well defined) *normal monomial* $\vartheta^I=\vartheta_q^{i_q}\cdots\vartheta_1^{i_1}\in\mathrm{U}_{\Bbbk}(\mathfrak{g})$.

The normal monomials span[42] $\mathrm{U}_{\Bbbk}(\mathfrak{g})$. Indeed, (3) shows that $\mathrm{U}_{\Bbbk}(\mathfrak{g})$ is spanned by *all* product of elements of $\boldsymbol{\vartheta}$. If such a product contains ϑ_i^2 with $|\vartheta_i|$ odd, then replace ϑ_i^2 by $\vartheta_i^{[2]}$; if it contains $\vartheta_i\vartheta_j$ with $i<j$, then replace it by $\vartheta_j\vartheta_i\pm[\vartheta_i,\vartheta_j]$; express each ϑ_i^2 and $[\vartheta_i,\vartheta_j]$ as a linear combination of generators. Applying the procedure to each of the new monomials, after a finite number of steps one ends up with a linear combination of normal monomials.

Returning to homological algebra, we show how basic constructions of Lie algebras create *co*homological structures. A *DG Lie algebra* over $\Bbbk$ is a graded Lie algebra $\mathfrak{g}$ with a degree -1 $\Bbbk$-linear map $\partial\colon\mathfrak{g}\to\mathfrak{g}$, such that $\partial^2=0$,

$$\partial[\vartheta,\xi]=[\partial(\vartheta),\xi]+(-1)^{|\vartheta|}[\vartheta,\partial(\xi)]\,,\ \text{and}\ \partial\big(\vartheta^{[2]}\big)=[\partial(\vartheta),\vartheta]\quad\text{for }\vartheta\in\mathfrak{g}^{\mathrm{odd}}\,.$$

A morphism of DG Lie algebras is a homomorphism of the underlying graded Lie algebras, that is also a morphism of complexes. Homology is a functor from DG Lie algebras to graded Lie algebras.

Lie algebras of derivations are paradigmatic throughout Lie theory. Here is a DG version, based on the Γ-free extensions of Chapter 6.

Lemma 10.1.4. *Let $\Bbbk\hookrightarrow\Bbbk\langle X\rangle$ be a semi-free Γ-extension. The inclusion $\mathrm{Der}^{\gamma}_{\Bbbk}(\Bbbk\langle X\rangle,\Bbbk\langle X\rangle)\subseteq\mathrm{Lie}(\mathrm{Hom}_{\Bbbk}(\Bbbk\langle X\rangle,\Bbbk\langle X\rangle))$ is one of DG Lie algebras.*

Proof. The proof is a series of exercises on the Sign Rule.

If b, c, are elements of $\Bbbk\langle X\rangle$, and υ is a derivation of odd degree, then

$$\begin{aligned}\upsilon^2(bc)=&\upsilon\Big(\upsilon(b)c+(-1)^{|b|}b\upsilon(c)\Big)\\=&\upsilon^2(b)c+(-1)^{|\vartheta|(|\vartheta|+|b|)}\upsilon(b)\upsilon(c)+(-1)^{|b|}\upsilon(b)\upsilon(c)+(-1)^{|b|+|b|}b\upsilon^2(c)\\=&\upsilon^2(b)c+b\upsilon^2(c)\,.\end{aligned}$$

If x is a Γ-variable of even degree, and ϑ, ξ are Γ-derivations, then

$$\upsilon^2(x^{(i)})=\upsilon(\upsilon(x)x^{(i-1)})=\upsilon^2(x)x^{(i-1)}-(\upsilon(x))^2x^{(i-2)}=\upsilon^2(x)x^{(i-1)}$$

[42]In fact, if $\boldsymbol{\vartheta}$ is a basis of $\mathfrak{g}$, then the normal monomials form a *basis* of $\mathrm{U}_{\Bbbk}(\mathfrak{g})$: this is the contents of the celebrated Poincaré-Birkhoff-Witt Theorem. The original proof(s) provide one of the first applications of 'standard basis' techniques; for an argument in the graded framework, cf. Milnor and Moore [121]; for the case needed here, cf. Theorem 10.2.1.

and

$$\begin{aligned}[\vartheta,\xi](x^{(i)}) =&\vartheta\xi(x^{(i)}) - (-1)^{|\vartheta||\xi|}\xi\vartheta(x^{(i)})\\ =&\vartheta\big(\xi(x)x^{(i-1)}\big) - (-1)^{|\vartheta||\xi|}\xi\big(\vartheta(x)x^{(i-1)}\big)\\ =&\vartheta\xi(x)x^{(i-1)} + (-1)^{|\vartheta||\xi|}\xi(x)\vartheta(x)x^{(i-2)}\\ &- (-1)^{|\vartheta||\xi|}\xi\vartheta(x)x^{(i-1)} - (-1)^{|\vartheta||\xi|+|\xi||\vartheta|}\vartheta(x)\xi(x)x^{(i-2)}\\ =&\left([\vartheta,\xi](x)\right)x^{(i-1)}\end{aligned}$$

A lengthier computation shows that $[\vartheta,\xi]$ is a derivation, completing the verification that $\operatorname{Der}^{\gamma}_{\Bbbk}(\Bbbk\langle X\rangle,\Bbbk\langle X\rangle)$ is a Lie subalgebra of $\operatorname{Lie}(\operatorname{Hom}_{\Bbbk}(\Bbbk\langle X\rangle,\Bbbk\langle X\rangle))$.

It remains to prove that the differential of the derivation complex satisfies the requirements for a graded Lie algebra. This is best done by 'interiorizing' it: as ∂ is a DG Γ-derivation of $\Bbbk\langle X\rangle$, it may be viewed as an element $\delta\in\operatorname{Der}^{\gamma}_{\Bbbk}(\Bbbk\langle X\rangle,\Bbbk\langle X\rangle)$, and then $\partial(\vartheta)=[\delta,\vartheta]$. The conditions on $\partial[\vartheta,\xi]$ and $\partial(\upsilon^{[2]})$ are now seen to be transcriptions of the Jacobi identity and its complement. □

It is tempting to mimic the construction of Ext algebras: Choose a Tate resolution $\Bbbk\langle X\rangle$ of a commutative $\Bbbk$–algebra P, and associate with P the graded Lie algebra $\operatorname{H}\operatorname{Der}^{\gamma}_{\Bbbk}(\Bbbk\langle X\rangle,\Bbbk\langle X\rangle)$. If $\mathbb{Q}\subseteq P$, then it is an invariant of P: this is proved by Quillen [133], as an outgrowth of his investigation of rational homotopy theory [131]. The general case is very different:

Example 10.1.5. The Tate resolution $A=\Bbbk\langle X\rangle$ of $\Bbbk$ over itself, with $X=\varnothing$, yields $\operatorname{H}\operatorname{Der}^{\gamma}_{\Bbbk}(A,A)=0$. If $\mathbb{F}_2\subseteq\Bbbk$, then another Tate resolution of $\Bbbk$ is

$$B=\Bbbk\big\langle u,\,\{x_i,\,x'_i\}_{i\geqslant 0}\,\big|\,\partial(u)=0\,,\partial(x_i)=u^{(2^i)}\,,\partial(x'_i)=u^{(2^i)}x_i\big\rangle$$

with $|u|=2$ (hence $|x_i|=2^{i+1}+1$ and $|x'_i|=2^{i+2}+2$). Using Corollary 6.2.4 and Construction 6.2.5, one gets $\operatorname{H}\operatorname{Der}^{\gamma}_{\Bbbk}(B,B)\cong\operatorname{Hom}_{\Bbbk}(\operatorname{Ind}^{\gamma}_{\Bbbk}B,\Bbbk)$, and $\operatorname{Ind}^{\gamma}_{\Bbbk}B$ is the free $\Bbbk$–module with basis $\{x_i\}_{i\geqslant 1}\cup\{x'_i\}_{i\geqslant 0}$.

10.2. **Homotopy Lie algebra.**

In this section $(R,\mathfrak{m},k)$ is a local ring.

Some of the problems occurring in the last example may be circumvented, by using acyclic closures. We follows the ideas of Sjödin [144], and simplify the exposition by using complexes of derivations from Section 6.2.

Theorem 10.2.1. *Let $R\langle X\rangle$ be an acyclic closure of k over R, where $X=\{x_i\}_{i\geqslant 1}$ and $|x_i|\leq|x_j|$ for $i<j$, and set $\pi(R)=\operatorname{H}\operatorname{Der}^{\gamma}_{R}(R\langle X\rangle,R\langle X\rangle)$.*

(1) *$\pi(R)$ is a graded Lie algebra over k.*
(2) *$\operatorname{rank}_k\pi^n(R)=\varepsilon_n(R)$ for $n\in\mathbb{Z}$.*
(3) *$\pi(R)$ has a k-basis*
$$\Theta=\{\theta_i=\operatorname{cls}(\vartheta_i)\mid\vartheta_i\in\operatorname{Der}^{\gamma}_{R}(R\langle X\rangle,R\langle X\rangle)\,,\,\vartheta_i(x_j)=\delta_{ij}\text{ for }j\leq i\}_{i\geqslant 1}\,.$$
(4) *The normal monomials on Θ form a k–basis of $\operatorname{U}_k(\pi(R))$.*
(5) *$\operatorname{Der}^{\gamma}_{R}(R\langle X\rangle,R\langle X\rangle)\subseteq\operatorname{Hom}_R(R\langle X\rangle,R\langle X\rangle)$ induces an injective homomorphism of graded Lie algebras $\iota\colon\pi(R)\to\operatorname{Lie}(\operatorname{Ext}_R(k,k))$. Its universal extension is an isomorphism of associative algebras*
$$\iota'\colon\operatorname{U}_k(\pi(R))\cong\operatorname{Ext}_R(k,k)\ .$$

Remark. By Remark 6.3.9, different choices of acyclic closures yield the same Lie algebra $\pi(R)$; it is called the *homotopy Lie algebra* of R.

Proof. (1) and (2). Let ϵ denote the quasi-isomorphism $R\langle X\rangle \to k$, and let $k \hookrightarrow k\langle X\rangle$ be the semi-free Γ-extension with trivial differential. Using Lemma 6.2.4, the minimality of $R\langle X\rangle$, and Proposition 6.2.3.4, we get

$$\begin{aligned}\pi(R) = \mathrm{H}\,\mathrm{Der}^{\gamma}_{R}\left(R\langle X\rangle, R\langle X\rangle\right) &\cong \mathrm{H}\,\mathrm{Der}^{\gamma}_{R}\left(R\langle X\rangle, k\right)\\ &\cong \mathrm{H}\,\mathrm{Der}^{\gamma}_{k}\left(k\langle X\rangle, k\right) = \mathrm{Der}^{\gamma}_{k}\left(k\langle X\rangle, k\right) \cong \mathrm{Hom}_{k}\left(kX, k\right)\,.\end{aligned}$$

So $\pi(R)$ is a k–module; by Theorem 7.1.3, $\mathrm{rank}_k\,\pi^n(R) = \mathrm{card}(X_n) = \varepsilon_n(R)$.

(3) Lemma 6.3.3.1 provides a set of R–linear Γ-derivations $\{\vartheta_i\}_{i\geqslant 1}$, with $\vartheta_i(x_j) = \delta_{ij}$ for $j \le i$. As $\{x_i\}_{i\geqslant 1}$ is a basis of kX, the isomorphisms above imply that Θ is linearly independent in $\pi(R)$. Since Θ_n has $\varepsilon_n(R)$ elements for each n, it is a basis by (2).

(4) and (5). In the commutative diagram

$$\begin{array}{ccc}\mathrm{Der}^{\gamma}_{R}\left(R\langle X\rangle, R\langle X\rangle\right) & \longrightarrow & \mathrm{Hom}_{R}\left(R\langle X\rangle, R\langle X\rangle\right)\\ {\scriptstyle \mathrm{Der}^{\gamma}_{R}(R\langle X\rangle,\epsilon)}\Big\downarrow \simeq & & \simeq \Big\downarrow {\scriptstyle \mathrm{Hom}_{R}(R\langle X\rangle,\epsilon)}\\ \mathrm{Der}^{\gamma}_{R}\left(R\langle X\rangle, k\right) & \xrightarrow{\ \iota\ } & \mathrm{Hom}_{R}\left(R\langle X\rangle, k\right)\end{array}$$

the left hand arrow is a quasi-isomorphism, as noted for (1); the right hand one is a quasi-isomorphism by Proposition 1.3.2.

Let $\theta^I \in \mathrm{U}_k(\pi(R))$ be a normal monomial on Θ. For a normal Γ-monomial $x^{(H)}$ on X, by Lemma 6.3.3.3 we see that $\iota'\big(\theta^I\big)\big(x^{(H)}\big) = \mathrm{cls}\,\big(\vartheta^I(x^{(H)})\big)$ is equal to 0 if $H < I$, and to 1 if $H = I$. As the normal Γ-monomials on X form a basis of $k\langle X\rangle$, the triangular form of the matrix implies that the images of the normal monomials on Θ form a basis of $\mathrm{Hom}_R\left(R\langle X\rangle, k\right) = \mathrm{Ext}_R\left(k, k\right)$. Thus, the homomorphism ι' is surjective, and the images of the normal monomials are linearly independent. By (4) and Remark 10.1.3.4, these monomials generate $\mathrm{U}_k(\pi(R))$, so we conclude that ι' is an isomorphism, as desired. □

Sjödin [144] shows how to compute the Lie operations on $\pi^1(R)$.

Example 10.2.2. Let $R = Q/I$, where $(Q, \mathfrak{n}, k)$ is regular, $\mathfrak{n}$ is minimally generated by $s_1, \dots, s_e$, and I is minimally generated by $f_1, \dots, f_r$, with

$$f_j = \sum_{1\leqslant h\leqslant i\leqslant e} a_{hi,j} s_h s_i \qquad \text{with} \quad a_{hi,j} \in Q \quad \text{for } 1 \le j \le r\,.$$

Using overbars to denote images in R, and setting $z_j = \sum_{h\leqslant i} \overline{a}_{hi,j}\overline{s}_h x_i$, we see that the acyclic closure $R\langle X\rangle$ of k over R is then obtained from

$$R\big\langle x_1, \dots, x_{e+r} \,\big|\, \partial(x_i) = \overline{s}_i \text{ for } 1 \le i \le e\,; \partial(x_{e+j}) = z_j \text{ for } 1 \le j \le r\big\rangle$$

by adjunction of Γ-variables of degree ≥ 3. Let $\vartheta_1, \dots \vartheta_e$ be the Γ-derivations of $R\langle x_1, \dots, x_e\rangle$, defined by $\vartheta_i(x_h) = \delta_{ih}$. To extend them to Γ-derivations of $R\langle x_1, \dots, x_{e+r}\rangle$, such that $\partial\vartheta_i = -\vartheta_i\partial$, note that

$$\vartheta_i\partial(x_{e+j}) = \vartheta_i\bigg(\sum_{h\leqslant i} \overline{a}_{hi,j}\overline{s}_h x_i\bigg) = \sum_{h=1}^{i} \overline{a}_{hi,j}\overline{s}_h = \partial\bigg(\sum_{h=1}^{i} \overline{a}_{hi,j} x_h\bigg)\,,$$

and set $\vartheta_i(x_{e+j}) = -\sum_h \overline{a}_{hi,j} x_h$ for $1 \le i \le e$ and $1 \le j \le r$. This yields

$$[\vartheta_h, \vartheta_i](x_{e+j}) = -\overline{a}_{hi,j} \quad \text{for } h < i \qquad \text{and} \qquad \vartheta_i^{[2]}(x_{e+j}) = -\overline{a}_{ii,j}\,.$$

Thus, on the basis elements of Theorem 10.2.1, the Lie bracket $\pi^1(R)\times\pi^1(R)\to\pi^2(R)$ and reduced square $\pi^1(R)\to\pi^2(R)$ are given by

$$[\theta_h,\theta_i]=-\sum_{j=1}^{r}a'_{hi,j}\theta_{e+j}\quad\text{for } h<i\qquad\text{and}\qquad(\theta_i)^{[2]}=-\sum_{j=1}^{r}a'_{ii,j}\theta_{e+j}$$

where a' denotes the image in k of $a\in R$.

Consider the k–subspace of $\pi^2(R)$, spanned by the commutators and squares of all elements of $\pi^1(R)$ (in fact, the squares suffice, cf. 10.1.2). By the preceding computation, its rank q is equal to that of the $\binom{e+1}{2}\times r$ matrix $(a_{hi,j})$ reduced modulo $\mathfrak{n}$, that is $q=\binom{e+1}{2}-\operatorname{rank}_k(I/I\cap\mathfrak{m}^3)$. In particular, $\pi^1(R)$ generates $\pi^2(R)$ if and only if the r quadratic forms $f_j=\sum_{h\leqslant i}a_{hi,j}s_hs_i$ are linearly independent in $\operatorname{gr}_{\mathfrak{n}}(Q)$. At the other extreme, the Lie subalgebra of $\pi(R)$ generated by $\pi^1(R)$ is reduced to $\pi^1(R)$ itself if and only if $I\subseteq\mathfrak{m}^3$.

Example 10.2.3. By Theorems 10.2.1.2 and 7.3.3, if $\pi(R)$ is finite dimensional, then R is a complete intersection and $\pi(R)$ is concentrated in degrees 1 and 2, so the preceding example determines its structure. The Lie subalgebra $\pi^{\geqslant 2}(R)$ is central in $\pi(R)$, and its universal enveloping algebra is the polynomial ring $\mathcal{P}=k[\theta_1,\dots,\theta_r]$. An isomorphism of $\mathcal{P}$–modules $\operatorname{Ext}_R(k,k)\cong\mathcal{P}\otimes_k\mathcal{E}$, where $\mathcal{E}$ is the vector space underlying the exterior algebra on $\pi^1(R)\cong\operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2,k)$, refines the equality $\mathrm{P}^R_k(t)=(1+t)^e/(1-t^2)^r$.

Conversely, each graded Lie algebra $\mathfrak{g}$ with $\operatorname{rank}_k\mathfrak{g}^1\geq\operatorname{rank}_k\mathfrak{g}^2$ and $\mathfrak{g}^n=0$ for $n\neq 1,2$ is of the form $\pi(R)$ for an appropriate complete intersection: one starts by fixing the desired quadratic parts $g_j=\sum_{h\leqslant i}a_{hi,j}s_hs_i$ of the relations, and uses a 'prime avoidance' argument to find elements p_j in a high power of $\mathfrak{m}$, such that the sequence $g_1+p_1,\dots,g_r+p_r$ is regular, cf. [144].

We conclude with some general remarks on the homotopy Lie algebra. A detailed study belongs to a different exposition.

Remark 10.2.4. A local homomorphism of local rings $\varphi\colon R\to S$ induces a homomorphism of graded Lie algebra $\pi(\varphi)\colon\pi(S)\to\pi(R)\otimes_k\ell$, where ℓ is the residue field of S. This yields a contravariant functor with remarkable properties. For example, if φ is flat, then for each i there is an exact sequence

$$0\longrightarrow\pi^{2i-1}(S/\mathfrak{m}S)\longrightarrow\pi^{2i-1}(S)\longrightarrow\pi^{2i-1}(R)\otimes_k\ell$$
$$\xrightarrow{\eth^{2i-1}}\pi^{2i}(S/\mathfrak{m}S)\longrightarrow\pi^{2i}(S)\longrightarrow\pi^{2i}(R)\otimes_k\ell\longrightarrow 0$$

where $\eth^{2i-1}=0$ for almost all i, and $\sum_{i=0}^{\infty}\operatorname{rank}\eth^{2i-1}\leq\operatorname{codepth}(S/\mathfrak{m}S)$: this is proved (in dual form) in [20] and [9]. The Lie algebra $\pi(R)$ is a looking glass version of the Lie algebra of rational homotopy groups in algebraic topology, cf. [23] and [33] for a systematic discussion.

Remark. The original construction of $\pi(R)$ proceeded in two steps.

The first, initiated by Assmus [15], and completed by Levin [108] and Schoeller [140], constructs a homomorphism $\Delta\colon\operatorname{Tor}^R(k,k)\to\operatorname{Tor}^R(k,k)\otimes_k\operatorname{Tor}^R(k,k)$ of Γ-algebras, giving $\operatorname{Tor}^R(k,k)$ a structure of Hopf algebra. The second identifies the composition product as the dual of Δ under the isomorphism of Hopf algebras $\operatorname{Ext}_R(k,k)=\operatorname{Hom}_k\left(\operatorname{Tor}^R(k,k),k\right)$.

At that point, a structure theorem due to Milnor and Moore [121] in characteristic 0 and to André in characteristic $p > 0$ (adjusted by Sjödin [145] for $p = 2$) shows that such a Hopf algebra is the universal enveloping algebra of graded Lie algebra. In fact, these results prove much more: namely, an equivalence (of certain subcategories) of the categories of Γ-Hopf algebras and graded Lie algebras, given in one direction by the universal enveloping algebra functor.

Remark. A graded Lie algebra $\mathrm{H}^*(R,k,k)$ is attached to R by the simplicially defined tangent cohomology of André and Quillen. There is a homomorphism of graded Lie algebras $\mathrm{H}^*(R,k,k) \longrightarrow \pi(R)$, cf. [4]. It is bijective for complete intersections, or when $\mathrm{char}(k) = 0$, cf. [133]; when $\mathrm{char}(k) = p > 0$, this holds in degrees $\leq 2p$, but not always in degree $2p+1$, cf. [7]. The computation of $\mathrm{H}^*(R,k,k)$ is very difficult in positive characteristic: for the small ring $R = \mathbb{F}_2[s_1,s_2]/(s_1^2, s_1s_2, s_2^2)$ it requires the book of Goerss [73]; for comparison, $\pi(R)$ is the free Lie algebra on the 2-dimensional vector space $\pi^1(R)$.

10.3. **Applications.** Once again, $(R,\mathfrak{m},k)$ denotes a local ring.

We relate this chapter to the bulk of the notes by discussing two kinds of applications of $\pi(R)$ to the study of resolutions. The structure of $\pi(R)$ is reflected in the Poincaré series of finite R–modules. To illustrate the point, we characterize Golod rings in terms of their homotopy Lie algebras.

Example 10.3.1. Let V be a vector space over k. A graded Lie algebra $\mathfrak{g}$ is *free* on V, if $V \subseteq \mathfrak{g}$ and each degree zero k–linear map from V to a graded Lie algebra $\mathfrak{h}$ extends uniquely to a homomorphism of Lie algebras $\mathfrak{g} \longrightarrow \mathfrak{h}$.

It is easy to see that free Lie algebras exist on any V: just take $\mathfrak{g}$ to be the subspace of the tensor algebra $\mathrm{T}_k(V)$, spanned by all commutators of elements of V, and all squares of elements of V_{odd} . Using the universal property of $\mathrm{T}_k(V)$ one sees that $\mathfrak{g}$ is a free Lie algebra on V, and comparing it with that of $\mathrm{U}_k(\mathfrak{g})$ one concludes that these two algebras coincide.

Avramov [21] and Löfwall [113] prove that R is Golod if and only if $\pi^{\geqslant 2}(R)$ is free, and then it is the free Lie algebra on $V = \mathrm{Hom}_k\left(\Sigma\,\mathrm{H}_{\geqslant 1}(K^R),k\right)$.

This nicely 'explains' the formula (5.0.1) for the Poincaré series of k over a Golod ring: $f(t) = (1+t)^e$ is the Hilbert series of the vector space $\mathcal{E}$ underlying the exterior algebra on $\pi^1(R) \cong \mathrm{Hom}_k\left(\mathfrak{m}/\mathfrak{m}^2,k\right)$; the expression $g(t) = 1/(1-\sum_i \mathrm{rank}\,\mathrm{H}_i(K^R)t^{i+1})$ is the Hilbert series of $\mathcal{T}$, the tensor algebra on V, and $\mathrm{P}_k^R(t) = f(t)g(t)$ reflects an isomorphism $\mathrm{Ext}_R(k,k) \cong \mathcal{T} \otimes_k \mathcal{E}$ of $\mathcal{T}$–modules.

The bracket and the square in a free Lie algebra are as non-trivial as possible, so the cohomological descriptions of Golod rings above and of complete intersections in Example 10.2.3 put a maximal distance between them (compare Remark 8.1.1.3). However, there exists a level at which these descriptions coalesce: the Lie algebra $\pi^{\geqslant 3}(R)$ is free, because it is trivial in the first case, and because freeness is inherited by Lie subalgebras, cf. [105], in the second.

Remark 10.3.2. It is proved in [28], using results from [24], that the following conditions on a local ring R are equivalent:

(i) $\pi(R)$ contains a free Lie subalgebra of finite codimension;
(ii) for some $r \in \mathbb{Z}$, the Lie algebra $\pi^{\geqslant r}(R)$ is free;
(iii) for some $s \in \mathbb{Z}$, the DG algebra $R\langle X_{<s}\rangle$ admits a trivial Massey operation, cf. Remark 5.2.1.

When they hold, the ring R is called *generalized Golod* (*of level* $\leq s$). Such rings abound in small codepth: this is the case when $\operatorname{edim} R - \operatorname{depth} R \leq 3$ (Avramov, Kustin, and Miller [36]), or when $\operatorname{edim} R - \operatorname{depth} R = 4$ and R is Gorenstein (Jacobsson, Kustin, and Miller [94]) or an almost complete intersection (Kustin and Palmer [102]). Kustin [99], [100] proves that certain determinantal relations define generalized Golod rings.

Theorem 10.3.3. *Let R be a generalized Golod ring of level $\leq s$.*

(1) *There is a polynomial* $\operatorname{den}(t) \in \mathbb{Z}[t]$*, and for each finite R–module M there is a polynomial $q(t) \in \mathbb{Z}[t]$, such that $\mathrm{P}^R_M(t) = q(t)/\operatorname{den}(t)$; the numerator for $M = k$ divides $\prod_{2i+1<s}(1+t^{2i+1})^{\varepsilon_{2i+1}(R)}$.*

(2) *If* $\operatorname{cx}_R M = \infty$*, then* $\operatorname{curv}_R M = \beta > 1$ *and there is a real number α such that $\beta^R_n(M) \sim \alpha\beta^n$.* ■

The first part is proved by Avramov [28], the second by Sun [150]. Both use, among other things, a theorem of Gulliksen [81] that extends Remark 9.2.6.

Theorem 10.3.4. *Let $R\langle X\rangle$ be an acyclic closure of k over R. If M is a finite R–module, then for each $n \geq 1$ there is a polynomial $h_n(t) \in \mathbb{Z}[t]$, such that*

$$\sum_{i=0}^{\infty} \operatorname{rank}_k \mathrm{H}_i(M \otimes_R R\langle X_{\leqslant n}\rangle)t^i = \frac{h_n(t)}{\prod_{2j\leq n}(1-t^{2j})^{\varepsilon_{2j}(R)}}\ . \qquad ■$$

Theorem 10.3.3, or results that it generalizes, has been used in essentially all cases when the Poincaré series is known to be rational for all finite modules. It is difficult to resist asking the next question; a positive answer would be unexpected and very useful; a negative one might be equally interesting, since it would most likely involve unusual constructions.

Problem 10.3.5. If $\mathrm{P}^R_M(t)$ is rational for each M, is then R generalized Golod?

Next we describe applications of $\pi(R)$ that do not make specific assumptions on its form. They use the following easy consequence of Theorem 10.2.1.

Remark 10.3.6. Let $\mathfrak{h}$ be a graded Lie subalgebra of $\pi(R)$. By completing a basis of $\mathfrak{h}$ to one of $\pi(R)$, and considering the corresponding basis of normal monomials of $\mathrm{U}_k(\pi(R)) = \operatorname{Ext}_R(k,k)$, cf. Theorem 10.2.1, one easily sees that there is an isomorphism $\mathrm{U}_k(\mathfrak{g}) \cong \mathrm{U}_k(\mathfrak{h}) \otimes_k V$ of left modules over $\mathrm{U}_k(\mathfrak{h})$, where V is the tensor product of the exterior algebra on $(\mathfrak{g}/\mathfrak{h})^{\mathrm{odd}}$ with the symmetric algebra on $(\mathfrak{g}/\mathfrak{h})^{\mathrm{even}}$. By a simple count of basis elements,

$$\mathrm{H}^k_V(t) = \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{\ell^{2i-1}}}{\prod_{i=1}^{\infty}(1-t^{2i})^{\ell^{2i}}} \qquad \text{with} \quad \ell^n = \operatorname{rank}_k(\mathfrak{g}/\mathfrak{h})^n\ .$$

For each R–module M, the vector space $\operatorname{Ext}_R(M,k)$ is a left module over the universal enveloping algebra $\operatorname{Ext}_R(k,k)$: it suffices to make the obvious changes in the construction of cohomology products in Section 10.1. This module structure is the essential tool in the proof of the next result. In fact, it has already been used throughout Chapter 9, in a different guise: over a complete intersection, the actions on $\operatorname{Ext}_R(M,k)$ of the graded algebras denoted $\mathcal{P}$ in Remark 9.2.6 and in Remark 10.2.3 are the same, cf. [37].

Motivated by Proposition 4.2.4.1, we say that a module L over a local ring $(R,\mathfrak{m},k)$ is *extremal*, if $\operatorname{cx}_R L = \operatorname{cx}_R k$ and $\operatorname{curv}_R L = \operatorname{curv}_R k$. For instance, Theorems 8.3.3 and 8.3.4 show that (in the graded characteristic zero case) the conormal module and the module of differentials are extremal when R is not a complete intersection. Results from [29] add more instances, among them:

Theorem 10.3.7. *If R is a local ring of embedding dimension e, M is a finite R–module and L is a submodule such that $L \supseteq \mathfrak{m}M$, then*

$$\mathrm{P}^R_L(t) \cdot (1+t)^e \succcurlyeq \mathrm{P}^R_k(t) \cdot \operatorname{rank}_k \left(\frac{\mathfrak{m}M}{\mathfrak{m}L}\right) .$$

For $\mathfrak{m}^i M \subseteq \mathfrak{m}^{i-1}M$, we get a quantitative version of Levin's characterization of regularity [108]: If $\mathfrak{m}M \neq 0$ and $\operatorname{pd}_R(\mathfrak{m}M) < \infty$, then R is regular:

Corollary 10.3.8. *If $\mathfrak{m}^i M \neq 0$ for some $i \geq 1$, then $\mathfrak{m}^i M$ is extremal.* □

Corollary 10.3.9. *Each non-zero R–module $M \neq k$ may be obtained as an extension of an extremal R–module by another such module. In particular, the extremal R–modules generate the Grothendieck group of R.* □

Here is another class of extremal modules from [29].

Remark 10.3.10. If $N \neq 0$ is a homomorphic image of a finite direct sum of syzygies of k, then $\operatorname{cx}_R N = \operatorname{cx}_R k$ and $\operatorname{curv}_R N = \operatorname{curv}_R k$. As a consequence, we get a result of Martsinkovsky [116]: $\operatorname{pd}_R N = \infty$, unless R is regular.

Proof of Theorem 10.3.7. The commutative diagram of R–modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & \dfrac{L}{\mathfrak{m}L} & \longrightarrow & \dfrac{M}{\mathfrak{m}L} & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

induces a commutative square of homomorphisms of graded *left* modules

$$\begin{array}{ccc}
\operatorname{Ext}_R(L,k) & \xrightarrow{\eth'} & \operatorname{Ext}_R(N,k) \\
\big\uparrow & & \big\| \\
\operatorname{Ext}_R\left(\dfrac{L}{\mathfrak{m}L},k\right) & \xrightarrow{\eth} & \operatorname{Ext}_R(N,k)
\end{array}$$

over $E = \operatorname{Ext}_R(k,k)$, where $\eth'$ and $\eth$ are connecting maps of degree 1.

As $\mathfrak{m}$ annihilates $L/\mathfrak{m}L$ and N, we have isomorphisms

$$\operatorname{Ext}_R\left(\frac{L}{\mathfrak{m}L},k\right) \cong E \otimes_k \operatorname{Hom}_k\left(\frac{L}{\mathfrak{m}L},k\right)$$
$$\operatorname{Ext}_R(N,k) \cong E \otimes_k \operatorname{Hom}_k(N,k)$$

of graded E–modules. By Remark 10.3.6, $E \cong U \otimes_k \bigwedge(\pi^1(R))$ as graded left modules over the subalgebra $U = \mathrm{U}_k(\pi^{\geqslant 2}R)$. Noting that $\pi^1(R) = E^1$, we can rewrite $\eth$ as the top map of the commutative diagram

$$\begin{array}{ccc}
U \otimes_k \bigwedge(E^1) \otimes_k \operatorname{Hom}_k\left(\dfrac{L}{\mathfrak{m}L}, k\right) & \xrightarrow{\ \eth\ } & U \otimes_k \bigwedge(E^1) \otimes_k \operatorname{Hom}_k(N,k) \\
\cup\!\Big\uparrow & & \Big\uparrow\!\cup \\
U \otimes_k \operatorname{Hom}_k\left(\dfrac{L}{\mathfrak{m}L}, k\right) & \longrightarrow & U \otimes_k E^1 \otimes_k \operatorname{Hom}_k(N,k) \\
\Big\| & & \Big\uparrow\cong \\
U \otimes_k \operatorname{Hom}_k\left(\dfrac{L}{\mathfrak{m}L}, k\right) & \xrightarrow{U\otimes_k \eth^0} & U \otimes_k \operatorname{Ext}^1_R(N,k)\ .
\end{array}$$

of homomorphisms of graded left U–modules.

The preceding information combines to yield a commutative square

$$\begin{array}{ccc}
\operatorname{Ext}_R(L,k) & \xrightarrow{\ \eth'\ } & \operatorname{Ext}_R(N,k) \\
\Big\uparrow & & \Big\uparrow \\
U \otimes_k \operatorname{Hom}_k\left(\dfrac{L}{\mathfrak{m}L}, k\right) & \xrightarrow{U\otimes_k \eth^0} & U \otimes_k \operatorname{Ext}^1_R(N,k)
\end{array}$$

of homomorphisms of graded U–modules, with injective right hand vertical arrow. Thus, $\operatorname{Ext}_R(N,k)$ contains a copy of the free module $\Sigma(U) \otimes_k \operatorname{Im} \eth_0$. By the commutativity of the square, $\operatorname{Ext}_R(L,k)$ contains a copy of $U \otimes_k \operatorname{Im} \eth^0$.

On the other hand, a length count in the cohomology exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(N,k) \longrightarrow \operatorname{Hom}_R\left(\frac{M}{\mathfrak{m}L}, k\right) \longrightarrow \operatorname{Hom}_R\left(\frac{L}{\mathfrak{m}L}, k\right) \xrightarrow{\eth^0} \operatorname{Ext}^1_R(N,k)$$

yields $\operatorname{rank}_k \eth^0 = \operatorname{rank}_k(\mathfrak{m}M/\mathfrak{m}L)$. Thus, $\mathrm{P}^R_L(t) \succcurlyeq \operatorname{rank}_k(\mathfrak{m}M/\mathfrak{m}L) \cdot \mathrm{H}^k_U(t)$. To finish the proof, multiply this inequality by $(1+t)^e$, then simplify the right hand side by using the equality $(1+t)^e \cdot \mathrm{H}^k_U(t) = \mathrm{P}^R_k(t)$ from 10.3.6. $\square$

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