Homework 2

Due Friday January 25, 2008 at the beginning of class.

Let V, V_1, V_2 and W be finite dimensional vector spaces over the field K. (You may make any necessary assumptions on K.)

2. Let G_1 and G_2 be groups. Suppose that V_1 is an irreducible G_1-module and V_2 is an irreducible G_2-module. Prove that $V_1 \otimes_K V_2$ is an irreducible $G_1 \times G_2$ module. (The direct product $G_1 \times G_2$ acts on the tensor product $V_1 \otimes_K V_2$ by (g_1, g_2) of $v_1 \otimes v_2$ is $g_1(v_1) \otimes g_2(v_2)$.)
3. When is the m^{th} exterior power $\bigwedge^m V$ an irreducible $\mathrm{GL}(V)$-module?
4. Decompose $\mathrm{Sym}_2(V \otimes_K W)$ into a direct sum of irreducible $\mathrm{GL}(V) \times \mathrm{GL}(W)$ modules.

"Recall" that if V and W are vector spaces over the field K, then the tensor product of V and W over K (written as $V \otimes_K W$) is the vector space which is spanned by symbols $v \otimes w$ with $v \in V$ and $w \in W$. (That is, every element of $V \otimes W$ looks like $\sum_{i=1}^{s} \alpha_i (v_i \otimes w_i)$, with $\alpha_i \in K$, $v_i \in V$, and $w_i \in W$.) The operation $\otimes$ is linear in each position:

$$(\alpha_1 v_1 + \alpha_2 v_2) \otimes w = \alpha_1 (v_1 \otimes w) + \alpha_2 (v_2 \otimes w) \quad \text{and}$$

$$v \otimes (\alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 (v \otimes w_1) + \alpha_2 (v \otimes w_2),$$

for all $\alpha_i \in K$, $v_i, v \in V$ and $w_i, w \in W$. In particular, if $\{v_i \mid i \in I\}$ is a basis for V and $\{w_j \mid j \in J\}$ is a basis for W, then $\{v_i \otimes w_j \mid i \in I \quad \text{and} \quad j \in J\}$ is a basis for $V \otimes_k W$.

You might find it informative to notice that

$$\mathrm{Sym}_2 V = \frac{V \otimes_K V}{(\text{the subspace generated by the set of all } v \otimes v' - v' \otimes v \text{ with } v, v' \in V)},$$

and

$$\bigwedge^2 V = \frac{V \otimes_K V}{(\text{the subspace generated by the set of all } v \otimes v \text{ with } v \in V)}.$$