Math 574, Final Exam, Spring 2006 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 11 problems. Problems 1 through 10 are worth 9 points each. Problem 11 is worth 10 points. The exam is worth 100 points.

YOU MUST JUSTIFY YOUR ANSWERS. Write in complete sentences. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Express the sum** $\sum\limits_{k=0}^{n}\binom{n}{k}$ **in a closed form.**

The sum is equal to $\boxed{2^n}$. You can see this by using the binomial theorem $(x+y)^n=\sum\limits_{k=0}^{n}\binom{n}{k}x^ny^n$. Let $x=y=1$ to see that $2^n=(1+1)^n=\sum\limits_{k=0}^{n}\binom{n}{k}$. You can also see the answer because there are $\binom{n}{k}$ subsets of size k in an n element set. So, the given sum counts the number of subsets of an n element set. On the other hand, we know that there are 2^n subsets of an n element set.

2.

(a) **Consider the list of numbers**

$$a_1=4,\quad a_2=6,\quad a_3=2,\quad a_4=8,\quad a_5=10,\quad a_6=1,\quad a_7=5,$$

$$a_8=9,\quad a_9=7,\quad a_{10}=3.$$

For each integer i with $1\le i\le 10$, let u_i be the length of the longest increasing sequence from the above list which starts at a_i, and let d_i be the length of the longest decreasing sequence from the above list which starts at a_i. Write down the value of (u_i,d_i) for each i.

i	(u_i, d_i)
1	$(4, 3)$
2	$(3, 3)$
3	$(3, 2)$
4	$(2, 3)$
5	$(1, 4)$
6	$(3, 1)$
7	$(2, 2)$
8	$(1, 3)$
9	$(1, 2)$
10	$(1, 1)$

(b) **Let $a_1, \ldots, a_{10}$ be any list of 10 distinct numbers. Define (u_i, d_i) as in part (a). Prove that if $i < j$, then $(u_i, d_i) \neq (u_j, d_j)$.**

There are two possibilities. Either $a_i < a_j$ or $a_i > a_j$. If $a_i < a_j$, then every increasing list which starts at a_j can be extended to become an increasing list which starts at a_i. Thus, $u_i \geq u_j + 1$. On the other hand, if $a_i > a_j$, then every decreasing list which starts at a_j can be extended to become an decreasing list which starts at a_i. Thus, $d_i \geq d_j + 1$.

(c) **Prove that every list $a_1, \ldots, a_{10}$ of 10 distinct numbers must contain an increasing sublist of length 4 or a decreasing sublist of length 4.**

There are only 9 distinct pairs (u_i, d_i) made with $1 \leq i \leq u_i, d_i \leq 3$; but there are 10 parameters i with $1 \leq i \leq 10$. It follows that some u_i or some d_i must be at least 4.

(d) **Give an example of a list $a_1, \ldots, a_9$ of 9 distinct numbers which does not contain an increasing sublist of length 4 or a decreasing sublist of length 4.**

$$\boxed{3, 2, 1, 6, 5, 4, 9, 8, 7}.$$

It is easy to see that the pairs (u_i, d_i) are

$$(3, 3), (3, 2), (3, 1), (2, 3), (2, 2), (2, 1), (1, 3), (1, 2), (1, 1).$$

3.

(a) **What is the truth table for $p \to q$?**

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

(b) **What is the converse of** $p \to q$ **?**

$\boxed{q \to p}$

(c) **What is the contrapositive of** $p \to q$ **?**

not $q \to$ not p

(d) **Is the converse of** $p \to q$ **logically equivalent to** $p \to q$ **?**

NO.

p	q	$q \to p$
T	T	T
T	F	$\boxed{T}$
F	T	$\boxed{F}$
F	F	T

Observe that the two boxed entries are different than the corresponding entries for $p \to q$.

(e) **Is the contrapositive of** $p \to q$ **logically equivalent to** $p \to q$ **?**

YES.

p	q	not q	not p	not $q \to$ not p
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

The statements not $q \to$ not p and $p \to q$ take exactly the same truth values for all values of p and q .

(f) **Express** $p \to q$ **in a logically equivalent manner using only** $\wedge$ **,** $\vee$ **, and "not".**

$p \to q$ is equivalent to $\boxed{q \vee \text{not } p}$ because $p \to q$ and $q \vee$ not p take exactly the same truth values for all values of p and q .

p	q	not p	$q \vee$ not p
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

4. **Let I be the following interval of real numbers: $I = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$. For each real number x in I, let S_x be the following set of real numbers:**

$$S_x = \{y \in \mathbb{R} \mid x - \tfrac{3}{4} < y < x + \tfrac{3}{4}\}.$$

   (a) Find $\bigcup_{x \in I} S_x$.

$\boxed{\{y \in \mathbb{R} \mid -\tfrac{3}{4} < y < 1 + \tfrac{3}{4}\}.}$

   (b) Find $\bigcap_{x \in I} S_x$.

$\boxed{\{y \in \mathbb{R} \mid \tfrac{1}{4} < y < \tfrac{3}{4}\}.}$

5. **How many words of length 20 can be made from the alphabet $\{0, 1, 2, 3\}$ if exactly 10 zeros are used?**

Select the 10 places to put zero. There are $\binom{20}{10}$ ways to do this. Now fill in the rest of the spots. There are 3^{10} ways to do this. The answer is

$$\boxed{3^{10}\binom{20}{10}}.$$

6. **Prove that every integer greater than 11 is the sum of 2 composite numbers.**

If n is even, then $n = 4 + (n-4)$. It is clear that 4 is a composite integer. On the other hand $n - 4$ is an even integer which is more than 2, so $n - 4$ is also a composite integer. If n is odd, then $n = 9 + (n-9)$. It is clear that 9 is a composite integer. On the other hand, $n - 9$ is an even integer which is greater than 2, so $n - 9$ is a composite integer. In any event, n is the sum of two composite integers.

7. **Let S, T, and U be sets, and let $f\colon S \to T$ and $g\colon T \to U$ be functions. Suppose that $g \circ f$ is onto. For each question, prove or give a counterexample.**
   (a) **Does f have to be onto?**

The function f does NOT have to be onto. Consider $S = U = \{1\}$, $T = \{1, 2\}$, $f(1) = 1$, $g(1) = g(2) = 1$. We see that $g \circ f$ is onto, but f is not onto.

   (b) **Does g have to be onto?**

The function g DOES have to be onto. Let u be an arbitrary element of U. The function $g \circ f$ is onto; so, there exists an element $s \in S$ with $g \circ f(s) = u$. Thus $f(s)$ is an element of T and g sends THIS element of T to u.

8. **Recall that the Fibonacci numbers are: $f_1 = 1$, $f_2 = 1$, and for each integer n with $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$. Prove that f_{4n} is a multiple of 3, whenever n is a positive integer.**

The Fibonacci numbers are $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and $f_4 = 3$. We see that $f_{4 \cdot 1}$ is a multiple of 3 and this takes care of the base case. We continue by induction.

INDUCTIVE HYPOTHESIS: Assume that $f_{4n} = 3\ell$ for some fixed positive integers n and ℓ.

WE WILL PROVE: f_{4n+4} is also a multiple of 3.

We see that

$$f_{4n+4} = f_{4n+3} + f_{4n+2} = 2f_{4n+2} + f_{4n+1} = 3f_{4n+1} + 2f_{4n} = 3(f_{4n+1} + 2\ell).$$

We have shown that the inductive hypothesis ensures that f_{4n+4} is a multiple of 3, and our proof is complete.

9. **How many monomials of degree less than or equal to d can be made using the n variables $x_1, \dots, x_n$? (For example, $x_1^2 x_2^3$ is a monomial of degree 5.)**

We count all monomials of the form $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$. We must count the number of solutions of $e_1 + e_2 + \cdots + e_n \leq d$, where each e_i is a non-negative integer. This is the same as the number of solutions of $e_1 + e_2 + \cdots + e_n + e_{n+1} = d$, where each e_i is a non-negative integer. This is the Candy Store Problem with d picks and n switches. So, there are

$$\boxed{\binom{d+n}{n}}$$

monomials of degree less than or equal to d can be made using the n variables $x_1, \dots, x_n$.

10. **Find a recurrence relation for the number of strings made from 0's, 1's, and 2's that do not contain two consecutive zeros or two consecutive ones.**

Let a_n equal the number of strings made from 0's, 1's, and 2's that do not contain two consecutive zeros or two consecutive ones. I see that $a_1 = 3$ and $a_2 = 7$. I notice that the number of legal strings with right-most two in position m is

$$\begin{cases} a_{m-1} & \text{if } m = n \\ 2a_{m-1} & \text{if } m < n \end{cases}$$

because once I put 0 or 1 in position $m+1$, then the rest of the string is completely determined. I now see that if $n \geq 2$, then

$$a_n = a_{n-1} + 2a_{n-2} + \cdots + 2a_1 + 2^{\dagger} + 2^{\ddagger}$$

The number $2^{\dagger}$ counts all strings with right most two in position 1. The number $2^{\ddagger}$ counts all strings without any twos. The above formula is not really a recurrence relation, so we clean it up a little. Observe that a_{n-1} is almost the same as a_n; that is,

$$a_{n-1} = a_{n-2} + 2a_{n-3} + \cdots + 2a_1 + 2^{\dagger} + 2^{\ddagger}.$$

In other words, $a_{n-1} + (a_{n-1} + a_{n-2}) = a_n$. Our answer is

$$\boxed{a_n = 2a_{n-1} + a_{n-2}, \quad a_0 = 1, \quad a_1 = 3.}$$

A good way to check this recurrence relation is to notice that a_2 really is 7 and a_3 really is 17.

11. **Solve the recurrence relation** $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ **with** $a_0 = 1$ **and** $a_1 = 7$**. CHECK your answer.**

The characteristic polynomial $x^2 - 4x + 4 = (x-2)^2$; so we know that the general solution of the homogeneous problem is $a_n = c_1 2^n + c_2 n 2^n$. We look for a particular solution of the given nonhomogeneous problem of the form $a_n = An^2 2^n$. We see that $A = \frac{1}{2}$ works. So, the general solution of the given non-homogeneous problem is

$$a_n = c_1 2^n + c_2 n 2^n + n^2 2^{n-1}.$$

We need to find c_1 and c_2 with

$$1 = a_0 = c_1 \quad \text{and} \quad 7 = a_1 = 2c_1 + 2c_2 + 1.$$

So $c_1 = 1$ and $c_2 = 2$. Our solution is

$$\boxed{a_n = 2^n + n 2^{n+1} + n^2 2^{n-1}.}$$

We see that $a_0 = 1$, $a_1 = 2 + 4 + 1 = 7$, and

$$4a_{n-1} - 4a_{n-2} + 2^n = \begin{cases} 4\left(2^{n-1} + (n-1)2^n + (n-1)^2 2^{n-2}\right) \\ -4\left(2^{n-2} + (n-2)2^{n-1} + (n-2)^2 2^{n-3}\right) \\ +2^n \end{cases}$$

$$= \begin{cases} 4\left(2^{n-1} + n2^n - 2^n + n^2 2^{n-2} - 2n2^{n-2} + 2^{n-2}\right) \\ -4\left(2^{n-2} + n2^{n-1} - (2)2^{n-1} + n^2 2^{n-3} - 4n2^{n-3} + (4)2^{n-3}\right) \\ +2^n \end{cases}$$

$$= \begin{cases} (2)2^n + 2n2^{n+1} - (4)2^n + 2n^2 2^{n-1} - n2^{n+1} + 2^n \\ -2^n - n2^{n+1} + (4)2^n - n^2 2^{n-1} + n2^{n+1} - (2)2^n \\ +2^n \end{cases}$$

$$= \begin{cases} (2)2^n - (4)2^n + 2^n - 2^n + (4)2^n - (2)2^n + 2^n \\ +2n2^{n+1} - n2^{n+1} - n2^{n+1} + n2^{n+1} \\ +2n^2 2^{n-1} - n^2 2^{n-1} \end{cases}$$

$$= 2^n + n2^{n+1} + n^2 2^{n-1} = a_n. \checkmark$$