Math 574, Exam 4, Solutions, Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 7 problems. Write in complete sentences **No Calculators.**

YOU MUST JUSTIFY YOUR ANSWERS.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. (7 points) **Express the sum** $\sum_{k=0}^{r}\binom{n+k}{k}$ **as a single binomial coefficient. PROVE your answer. (If you quote a formula we did in class, then the prove the formula.)**

Suppose that I own a candy store with $n+1$ flavors of candy. There are $\binom{n+k}{k}$ types of boxes of candy which consist of k pieces chosen from my $n+1$ flavors. (I count this by thinking of work orders with k picks and n switches.) Thus, there are $\sum_{k=0}^{r}\binom{n+k}{k}$ types of boxes of candy which consist of r or fewer pieces chosen from my $n+1$ flavors. On the other hand, I can count the number of types of boxes of candy which consist of r or fewer pieces chosen from my $n+1$ flavors in a different manner. I can pretend that I have one more flavor (this consists of an empty wrapper). To count the number of types of boxes of candy which consist of r or fewer pieces chosen from my $n+1$ flavors, I count the number of types of boxes of candy which consist of exactly r pieces chosen from $n+2$ flavors (the original $n+1$ flavors together with the fake flavor of empty wrapper). I make this count by thinking of work orders with r picks and $n+1$ switches. There are $\binom{r+n+1}{r}$ such work orders. I conclude that

$$\boxed{\sum_{k=0}^{r}\binom{n+k}{k}=\binom{r+n+1}{r}.}$$

2. (7 points) **How many paths are there from $(0,0)$ to $(8,5)$ on the xy-plane if each path consists of a series of steps, where each step is either a move one unit to the right or a move one unit up. (No moves to the left or downward are allowed.)**

We need to count the number of words made with 8 r's and 5 u's. This is the same as the number of 8-member subcommittees of a 13-member committee: $\boxed{\binom{13}{8}}$.

3. (7 points) **What is the coefficient of $x^4y^2zw^3$ in the expansion of $(x+y+z+w)^{10}$?**

The Multilinear Theorem gives $\boxed{\binom{10}{4,2,1,3}}$.

4. (7 points) **How many different terms are there in the expansion of $(x+y+z+w)^{10}$ (after collecting the common terms)?**

Every term looks like $x^{e_1}y^{e_2}z^{e_3}w^{e_4}$, where the e_i are non-negative integer exponents with $\sum_{i=1}^{4} e_i = 10$. We must count the number of solutions

$$e_1+e_2+e_3+e_4=10,$$

where each e_i is a non-negative integer. This is the candy store problem with 4 flavors and 10 pieces of candy in each box. I count work orders with 10 picks and 3 switches: $\boxed{\binom{13}{3}}$.

5. (7 points) **Consider the Tower of Hanoi problem. There are three towers in a row: tower A, tower B, and tower C. There are n disks of different sizes stacked on tower A. One must move all n disks to tower C. One may NEVER place a bigger disk on top of a smaller disk. In the present problem, one may move a disk only to an ADJACENT tower. Let a_n be the minimum number of moves needed to transfer a stack of n disks from tower A to tower C. Find a_1, a_2, a_3. Find a recurrence relation for $a_1, a_2, a_3, \dots$.**

It is clear that $\boxed{a_1 = 2}$. Move the disk to tower B. Move the disk to tower C. For a_2: move the small disk to tower C (2 moves). Move the big disk to tower B (one move). Move the small disk back to tower A (two moves). Move the big disk to tower C (one move). Move the small disk back to tower C (2 moves). So, $\boxed{a_2 = 8}$. For a_3: move the two small disks to Tower C (a_2 moves). Move the big disk to tower B (1 move). Move the two small disks to tower A (a_2 moves). Move the big disk to tower C (1 move). Move the two small disks back to tower C (a_2 moves). So, $a_3 = 3a_2 + 2 = \boxed{26}$. Of course, we now see how to do the general problem. Move the $n-1$ small disks to Tower C (a_{n-1} moves). Move the big disk to tower B (1 move). Move the $n-1$ small disks to tower A (a_{n-1} moves). Move the big disk to tower C (1 move). Move the $n-1$ small disks back to tower C (a_{n-1} moves). So, $\boxed{a_n = 3a_{n-1}+2}$.

6. (7 points) **Find a recurrence relation for the number of strings made from 0's, 1's, and 2's that contain two consecutive zeros.**

In this problem, I will say that a string of 0's, 1's, and 2's is a "legal string" if it contains two consecutive zeros. Let a_n be the number of legal strings of length n. It is clear that $a_1 = 0$ and $a_2 = 1$. Every legal string of length n ends in exactly one of the following ways: 1, 2, 10, 20, 00.
There are a_{n-1} legal strings that end in 1.
There are a_{n-1} legal strings that end in 2.
There are a_{n-2} legal strings that end in 10.
There are a_{n-2} legal strings that end in 20.
There are 3^{n-2} legal strings that end in 00.
We conclude that $\boxed{a_n = 2a_{n-1}+2a_{n-2}+3^{n-2}}$.

7. (8 points) **Solve the recurrence relation $a_n = -a_{n-1} + 6a_{n-2} - 4n + 23$ with $a_0 = 0$ and $a_1 = -6$. CHECK your answer.**

First we solve the homogeneous problem

$$\text{(homog)} \qquad a_n = -a_{n-1} + 6a_{n-2}.$$

We look for solutions of the form $a_n = r^n$. If $a_n = r^n$ is a solution of the homogeneous problem, then $r^n = -r^{n-1} + 6r^{n-2}$; so, $r^n + r^{n-1} - 6r^{n-2} = 0$; or $r^{n-2}(r^2 + r - 6) = 0$; which is $r^{n-2}(r-2)(r+3) = 0$. Thus, $r = 0$ or $r = 2$, or $r = -3$. It follows that the general solution of (homog) is $a_n = c_1 2^n + c_2(-3)^n$. (The solution $r = 0$ will be picked up when $c_1 = c_2 = 0$.)

Now we find one solution of the original nonhomogeneous problem:

$$\text{(nonhomog)} \qquad a_n = -a_{n-1} + 6a_{n-2} - 4n + 23.$$

We look for a solution of the form $a_n = A + Bn$. If $a_n = A + Bn$ is a solution of (nonhomog), then

$$A + Bn = -(A + B(n-1)) + 6(A + B(n-2)) - 4n + 23.$$

So,

$$A + Bn = -A + B + 6A - 12B + 23 + n(-B + 6B - 4).$$

I hope to find A and B so that the above equation holds for all values of n; so I equate the corresponding coefficients. I try to solve

$$\begin{aligned} A &= 5A - 11B + 23 \\ B &= 5B - 4 \end{aligned}$$

So, $B = 1$ and $-12 = 4A$ (i.e., $-3 = A$). So, we have shown that $a_n = n - 3$ is one solution of (nonhomog). (You can check this if you want. I'll wait and do only one check at the very end.)

We now know that every solution of (nonhomog) is given by

$$a_n = c_1 2^n + c_2(-3)^n + n - 3.$$

We need only choose c_1 and c_2 so that $a_0 = 0$ and $a_1 = -6$. We must solve

$$\begin{aligned} 0 &= c_1 + c_2 - 3 \\ -6 &= 2c_1 - 3c_2 - 2 \end{aligned}$$

We must solve

$$\begin{aligned} 3 &= c_1 + c_2 \\ -4 &= 2c_1 - 3c_2 \end{aligned}$$

Add -2 times the top equation to the bottom equation to get

$$-10 = -5c_2.$$

Thus, $c_2 = 2$ and $c_1 = 1$. The answer is

$$\boxed{a_n = 2^n + 2(-3)^n + n - 3.}$$

We check our answer: $a_0 = 1 + 2 - 3 = 0$, $a_1 = 2 + 2(-3) + 1 - 3 = -6$, and

$$-a_{n-1} + 6a_{n-2} - 4n + 23$$

$$\begin{aligned} &= -(2^{n-1} + 2(-3)^{n-1} + (n-1) - 3) + 6(2^{n-2} + 2(-3)^{n-2} + n - 2 - 3) - 4n + 23 \\ &= -2^{n-1} + 6(2^{n-2}) - 2(-3)^{n-1} + 12(-3)^{n-2} - n + 6n - 4n + 4 - 30 + 23 \\ &= 2^{n-1}(-1+3) + (-3)^{n-1}(-2-4) + n - 3 \\ &= 2^n + 2(-3)^n + n - 3 = a_n. \end{aligned}$$