Math 574,   Exam 3,   Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 8 problems. Problems 1 and 2 are worth 7 points each. Each of the other problems is worth 6 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

YOU MUST JUSTIFY YOUR ANSWERS.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. Prove that for each positive integer n, $\sum\limits_{k=1}^{n}(2k-1)$ is a perfect square.

2. Recall that the Fibonacci numbers are: $f_1 = 1$, $f_2 = 1$, and for each integer n with $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$. Prove that f_{4n} is a multiple of 3, whenever n is a positive integer.

In class we proved the following result. You may use this result in your solution to problem 3.

Result from class. *If each pair in a group of six people consists of two friends or two enemies, then there are three mutual friends or three mutual enemies in the group.*

3. Consider a group of ten people in which each pair of individuals consists of two friends or two enemies. Prove that there are either three mutual friends or four mutual enemies in the group of ten people.

4. Express the sum

$$\binom{500}{0} + \binom{501}{1} + \binom{502}{2} + \cdots + \binom{600}{100}$$

as a single binomial coefficient. PROVE your answer. (If you quote a formula we did in class, then the prove the formula.)

5. Express the sum

$$\binom{20}{0}\binom{30}{10}+\binom{20}{1}\binom{30}{9}+\cdots+\binom{20}{9}\binom{30}{1}+\binom{20}{10}\binom{30}{0}$$

as a single binomial coefficient. PROVE your answer. (If you quote a formula we did in class, then the prove the formula.)

6. How many paths are there from $(0,0)$ to $(6,4)$ on the xy-plane if each path consists of either of a series of steps, where each step is a move one unit to the right or a move one unit up. (No moves to the left or downward are allowed.)

7. How many words of length 20 can be made from the alphabet $\{0,1,2,3\}$ if exactly 10 zeros are used?

8. Let S be the set of integers i with $1 \le i \le 33$. Find the smallest integer k with the property that every subset T of S of size k contains one integer which divides another integer from T. In other words, you must:
   (a) tell me what k is,
   (b) prove that every subset T of S of size k contains one integer which divides another integer from T,
   (c) give an example of a $k-1$ element subset U of S in which no integer from U divides any other integer from U.