Math 574, 1992, Exam 1

There are 4 problems. Each problem is worth 25 points.

1. Prove that
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Consider the sequence of integers $a_1, a_2, a_3, a_4, \ldots$, where $a_1 = 2$, $a_2 = 12$, and for all n , with $3 \leq n$, $a_n = 2a_{n-1} + 4a_{n-2}$. If $1 \leq n$, then prove that
$$a_n = (1+\sqrt{5})^n + (1-\sqrt{5})^n.$$

3. Suppose that

   (*) $\qquad a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq a_n \quad \text{and} \quad b_1 \leq b_2 \leq \cdots \leq b_{m-1} \leq b_m$

   are lists of integers.
   (a) Give an upper bound on the number of comparisons that are needed in order to merge the lists in (*).
   (b) Prove your answer to (a).

4. Three rods are placed in a board and on one rod is a pile of n different sized disks arranged top to bottom from smallest to largest. You are to transfer the pile from rod A to rod C by moving one disk at a time in such a way that a larger disk is never placed on top of a smaller disk. Prove that this can be done in $2^n - 1$ moves.