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Quiz for June 6, 2005

Let $\{p_n\}$ be a bounded sequence of real numbers and let $p \in \mathbb{R}$ be such that every convergent subsequence of $\{p_n\}$ converges to p. Prove that the sequence $\{p_n\}$ converges to p.

ANSWER: Suppose that

$$\text{the sequence } \{p_n\} \text{ does NOT converge to } p. \tag{6}$$

In this case, there exists $\varepsilon > 0$ such that

$$\text{for every } n_0 \in \mathbb{N} \text{ there exists } n > n_0 \text{ such that } |p_n - p| \geq \varepsilon. \tag{7}$$

Apply (7) to find $n_1 > 1$, with $|p_{n_1} - p| > \varepsilon$. Apply (7) to find $n_2 > n_1$, with $|p_{n_2} - p| > \varepsilon$. Apply (7) to find $n_3 > n_2$, with $|p_{n_3} - p| > \varepsilon$. Continue in this manner to construct a subsequence

$$p_{n_1}, p_{n_2}, p_{n_3}, \dots \tag{8}$$

of the original sequence $\{p_n\}$ which never gets closer to p than ε. The Bolzano-Weierstrass Theorem (version 2) guarantees that some subsequence of (8) converges. This subsequence of (8) does not converge to p because the subsequence never gets within ε of p. On the other hand, subsequence of (8) is also a subsequence of the original sequence $\{p_n\}$; and therefore, must converge to p by the original hypothesis. This is a contradiction. The original supposition (6) must be false. We conclude that the sequence $\{p_n\}$ does converge to p.