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Quiz for June 1, 2004

Let A and B be nonempty subsets of positive real numbers that are bounded from above. Let $C = \{ab \mid a \in A \text{ and } b \in B\}$. Prove that $\sup C = (\sup A)(\sup B)$.

ANSWER: Let $\alpha = \sup A$, $\beta = \sup B$ and $\gamma = \sup C$.

We show $\gamma \le \alpha\beta$. It suffices to show that $\alpha\beta$ is an upper bound for C. Let c be an arbitrary element of C. It follows that $c = ab$ for some $a \in A$ and some $b \in B$. We know that α is an upper bound for A, so $a \le \alpha$. We know that β is an upper bound for B, so $b \le \beta$. Multipliy $a \le \alpha$ by the positive number b to see thar $ab \le \alpha b$. Multiply $b \le \beta$ by the positive number α to see that $\alpha b \le \alpha\beta$. Conclude that

$$c = ab \le \alpha b \le \alpha\beta;$$

and therefore $\alpha\beta$ is an upper bound for C.

We show $\alpha\beta \le \gamma$. We do this part of the argument by contradiction. If $\gamma < \alpha\beta$, then $\frac{\gamma}{\alpha} < \beta$. (We know that A is a non-empty set of positive numbers, and α is an upper bound for A. It follows that α is positive, and so not zero.) But β is $\sup B$; so there exists $b \in B$ with $\frac{\gamma}{\alpha} < b$. The number b is positive; so not zero. We have $\frac{\gamma}{b} < \alpha$. But α is $\sup A$; so there exists $a \in A$ with $\frac{\gamma}{b} < a$. We now have $\gamma < ab \in C$; which contradicts the fact that γ is an upper bound for C. Our original supposition that $\gamma < \alpha\beta$ must be wrong; so, we have established that $\alpha\beta \le \gamma$.

We have shown that $\gamma \le \alpha\beta$ and $\alpha\beta \le \gamma$. It follows that $\alpha\beta = \gamma$ and the proof is complete.