Math 554, Final Exam, Summer 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. **Leave room on the upper left hand corner of each page for the staple.** Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your grade from VIP.

There are 11 problems. The exam is worth a total of 100 points.

I will post the solutions on my website later this afternoon.

Record ALL of your answers in complete sentences.

1. **(9 points) Define "continuous". Use complete sentences. Include everything that is necessary, but nothing more.**

The function $f\colon E \to \mathbb{R}$ is continuous at the point p of E, if, for all $\varepsilon > 0$, there exists $\delta > 0$, such that whenever $|x-p| < \delta$ and $x \in E$, then $|f(x) - f(p)| < \varepsilon$.

2. **(9 points) Define "supremum". Use complete sentences. Include everything that is necessary, but nothing more.**

The real number α is the *supremum* of the non-empty set of real numbers E if α is an upper bound of E and whenever d is a real number with $d < \alpha$, then d is not an upper bound of E.

3. **(9 points) PROVE that the continuous image of a compact set is compact.**

Let K be a compact subset of $\mathbb{R}$ and let $f\colon K \to \mathbb{R}$ be a continuous function. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of $f(K)$. For each point $p \in K$, the element $f(p)$ is in $f(K)$. The set $\mathcal{U}$ covers $f(K)$, so there is an index α_p such that $f(p)$ is in U_{α_p}. The function f is continuous at p; so there exists a $\delta_p > 0$ such that $f(N_{\delta_p}(p) \cap K) \subseteq U_{\alpha_p}$. We create such a neighborhood $N_{\delta_p}(p)$ for each $p \in K$. We see that $\mathcal{N} = \{N_{\delta_p}(p) \mid p \in K\}$ is an open cover of K. The set K is compact; consequently, there exist $p_1, \dots, p_n$ in K such that $N_{\delta_{p_1}}(p_1), \dots, N_{\delta_{p_n}}(p_n)$ cover

K. It follows that $f(N_{\delta_{p_1}}(p_1)\cap K),\dots,f(N_{\delta_{p_n}}(p_n)\cap K)$ cover $f(K)$. But $f(N_{\delta_{p_i}}(p_i)\cap K)\subseteq U_{\alpha_{p_i}}$, for all i; therefore, $U_{\alpha_{p_1}},\dots,U_{\alpha_{p_n}}$ covers $f(K)$.

4. **(9 points) STATE and PROVE the Nested Interval Property.**

The Nested Interval Property. *For each natural number n, let I_n be a bounded closed interval. If $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, then the intersection $\bigcap\limits_{n=1}^{\infty} In$ is not empty.*

Proof. Let $I_n = [a_n, b_n]$, with $a_n < b_n$, for each n. The hypothesis that the intervals are nested tells us that

$$a_1 \le a_2 \le \cdots \le b_2 \le b_1.$$

In fact, if n and m are natural numbers, then $a_n < b_m$. We prove this small claim. If $n < m$, then $a_n \le a_m < b_m$. (The first inquality follows from the hypothesis that the intervals are nested. The second inequality follows from the hypothesis that each interval is an interval.) In a similar manner, if $m < n$, then $a_n < b_n \le b_m$. The claim is established.

The set $A = \{a_1, a_2, \dots\}$ is bounded and not empty. The least upper bound axiom tells us that $\sup A$ exists. Let $a = \sup A$.

We finsh our proof by showing that $a \in \bigcap\limits_{n=1}^{\infty} I_n$. Fix a natural number n. We show $a \in I_n$. It is clear that $a_n \le a$ (beacuse $a_n \in A$ and a is an upper bound for A) On the other hand, the first calculation we made shows that b_n is also an upper bound for A; hence b_n is at least as large as the least upper bound for A; namely, a. We conclude that $a \in [a_n, b_n]$, for all n; hence, $a \in \bigcap\limits_{n=1}^{\infty} I_n$.

5. **(10 points) Let A be a set. For each $\alpha \in A$, let U_α be an open subset of $\mathbb{R}$ and F_α be a closed subset of $\mathbb{R}$. For each question: if the answer is yes, then PROVE the assertion; if the answer is no, then give a counter example.**

   (a) **Does $\bigcup\limits_{\alpha\in A} U_\alpha$ have to be open?**

YES. Let p be an element of $\bigcup_{\alpha\in A} U_\alpha$. Thus, p is in U_{α_0} for some $\alpha_0 \in A$. The set U_{α_0} is open, so there exists $\varepsilon > 0$ with $N_\varepsilon(p) \subseteq U_{\alpha_0}$. Thus, $N_\varepsilon(p) \subseteq \bigcup_{\alpha\in A} U_\alpha$.

(b) **Does $\bigcap_{\alpha\in A} U_\alpha$ have to be open?**

NO. For each natural number n, let $I_n = (0, 1+\frac{1}{n})$. It is clear that each open interval I_n is an open set in $\mathbb{R}$. It is also clear that $\bigcap_{n\in\mathbb{N}} I_n = (0,1]$; which is not an open subset of $\mathbb{R}$. Indeed, $N_\varepsilon(1)$ is not contained in $(0,1]$ for any $\varepsilon > 0$.

(c) **Does $\bigcup_{\alpha\in A} F_\alpha$ have to be closed?**

NO. For each natural number n with $n \geq 2$, the closed interval $[\frac{1}{n}, 1]$ is a closed subset of $\mathbb{R}$. The union of all of these sets is $(0,1]$, which is not a closed set.

(d) **Does $\bigcap_{\alpha\in A} F_\alpha$ have to be closed?**

YES. We will prove that $\bigcap_{\alpha\in A} F_\alpha$ is a closed set by proving that the complement is open. Let $x \in \mathbb{R}$ with $x \notin \bigcap_{\alpha\in A} F_\alpha$. Thus, there is an index $\alpha_0 \in A$ with $x \notin F_{\alpha_0}$. The set F_{α_0} is closed, so the complement of F_{α_0} is open and there exists an $\varepsilon > 0$ such that $N_\varepsilon(x)$ misses F_{α_0}. It follows that $N_\varepsilon(x)$ misses $\bigcap_{\alpha\in A} F_\alpha$; and therefore, $\bigcap_{\alpha\in A} F_\alpha$ is a closed set.

6. **(9 points) Let $E = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$ and let $F = E \cup \{1\}$.**

(a) **Give an example of an open cover of E which does not admit a finite subcover. PROVE all of your assertions.**

For each $n \in \mathbb{N}$, let U_n be the open set $(-\infty, 1-\frac{1}{n})$. It is clear that $\mathcal{U} = \{U_n | n \in \mathbb{N}\}$ is an open cover of E. It is also clear that the union of any finite subset of sets $U_{n_1} \cup \cdots \cup U_{n_\ell}$ from $\mathcal{U}$ is equal to $U_{\max}$ where max is the maximum of the parameters $\{n_1, \ldots, n_\ell\}$. At any rate this union misses most of the set E.

(b) **Prove DIRECTLY (that is, do not quote any Theorems) that every open cover of F does admit a finite subcover.**

Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an arbitrary open cover of F. The number 1 is in one of the sets of $\mathcal{U}$; in other words, there is an element α_0 of A so that $1 \in U_{\alpha_0}$. The set U_{α_0} is open, so there exists ε with $N_\varepsilon(1) \subseteq U_{\alpha_0}$. Of course, if n_0 is large enough, then $\frac{1}{n_0} < \varepsilon$. If $n \ge n_0$, then $\frac{1}{n} \le \frac{1}{n_0} < \varepsilon$ and $1 - \frac{1}{n} \in N_\varepsilon(1) \subseteq U_{\alpha_0}$. For each n with $n < n_0$, there exists a subscript $\alpha_n \in A$ with $1 - \frac{1}{n} \in U_{\alpha_n}$. We have found a finite subcover $U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_{n_0 - 1}}$ of $\mathcal{U}$ which covers F.

7. **(9 points) Consider the sequence $\{a_n\}$ with $a_n = \sum\limits_{k=1}^{n} \frac{1}{k!}$. Prove that $\{a_n\}$ is a Cauchy sequence.**

For each natural number r, we see that $2^r \le 1 \cdot 2 \cdot 3 \cdot \ldots \cdot r \cdot (r+1)$. In other words, $2^r \le (r+1)$ and $\frac{1}{(r+1)!} \le \frac{1}{2^r}$. It follows that

$$|a_{n+k} - a_n| = \frac{1}{(n+1)!} + \cdots + \frac{1}{(n+k)!} \le \frac{1}{2^n} + \cdots + \frac{1}{2^{n+k-1}} = \frac{\frac{1}{2^n} - \frac{1}{2^{n+k}}}{1 - \frac{1}{2}}$$

$$= \frac{1}{2^{n-1}} - \frac{1}{2^{n+k-1}} \le \frac{1}{2^{n-1}}.$$

Fix $\varepsilon > 0$. Pick n_0 large enough for $\frac{1}{2^{n_0 - 1}} < \varepsilon$. We have just shown that if n and m are both at least n_0, then $|a_n - a_m| < \frac{1}{2^{\min\{n,m\}-1}} \le \frac{1}{2^{n_0 - 1}} < \varepsilon$. We have proven that $\{a_n\}$ is a Cauchy sequence.

8. **(9 points) Let $a_1 \ne a_2$ be real numbers. For $n \ge 3$, let $a_n = \frac{3}{4}a_{n-1} + \frac{1}{4}a_{n-2}$. PROVE that the sequence $\{a_n\}$ is a contractive sequence.**

If a_{n-1} and a_{n-2} ever happen to be equal, then a_n will equal this common value and induction or iteration shows that all of the rest of the terms of the sequence take this common value. A constant sequence is automatically contractive (use any b with $0 < b < 1$), but not very interesting. Henceforth, in this problem, we will only think about sequences with $a_{n+1} - a_n \ne 0$. We see that

$$\frac{|a_{n+2} - a_{n+1}|}{|a_{n+1} - a_n|} = \frac{|\frac{3}{4}a_{n+1} + \frac{1}{4}a_n - a_{n+1}|}{|a_{n+1} - a_n|} = \frac{|-\frac{1}{4}a_{n+1} + \frac{1}{4}a_n|}{|a_{n+1} - a_n|} = \frac{|-\frac{1}{4}||a_{n+1} - a_n|}{|a_{n+1} - a_n|} = \frac{1}{4}.$$

Thus, $|a_{n+2} - a_{n+1}| \le \frac{1}{4}|a_{n+1} - a_n|$ for all n and the sequence $\{a_n\}$ is a contractive sequence.

9. **(9 points) Let $\{a_n\}$ be a sequence of positive real numbers. Suppose that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ for some real number L with $L < 1$. Does the sequence $\{a_n\}$ have to converge? If the answer is yes, then PROVE the assertion; if the answer is no, then give a counter example.**

YES. Pick ρ with $L < \rho < 1$. The hypothesis that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ guarantees that there exists n_0 so that whenever $n \ge n_0$, then $\frac{a_{n+1}}{a_n} < \rho$. In particular, $a_{n_0+1} < \rho a_{n_0}$; $a_{n_0+2} < \rho^2 a_{n_0}$; $\dots$. Induct or iterate to see that $a_{n+k} < \rho^k a_{n_0}$ for all natural numbers k. It is clear that $\rho^k a_{n_0}$ goes to 0 as k goes to ∞; hence, the original sequence $\{a_n\}$ converges to 0.

10. **(9 points) Let A and B be non-empty sets, and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Suppose that the function $g \circ f$ is onto. For each question: if the answer is yes, then PROVE the assertion; if the answer is no, then give a counter example.**

(a) **Does f have to be onto?**

NO Let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1\}$, $f(1) = 1$ and $g(1) = g(2) = 1$. It is clear that $g \circ f$ is onto, but f is not onto.

(b) **Does g have to be onto?**

YES. If c is an arbitrary element of C, then the fact that $g \circ f$ is onto tells us that there exists an element $a \in A$ with $(g \circ f)(a) = c$. We now know that $f(a)$ is an element of B with $g(f(a)) = c$.

11. **(9 points) Let A and B be nonempty subsets of positive real numbers that are bounded from above. Let $C = \{ab \mid a \in A \text{ and } b \in B\}$. PROVE that $\sup C = (\sup A)(\sup B)$.**

Let $\alpha = \sup A$, $\beta = \sup B$ and $\gamma = \sup C$.

We show $\gamma \le \alpha\beta$. It suffices to show that $\alpha\beta$ is an upper bound for C. Let c be an arbitrary element of C. It follows that $c = ab$ for some $a \in A$ and some $b \in B$. We know that α is an upper bound for A, so $a \le \alpha$. We know that β is an upper bound for B, so $b \le \beta$. Multiply $a \le \alpha$ by the positive number b to see thar $ab \le \alpha b$. Multiply $b \le \beta$ by the positive number α to see that $\alpha b \le \alpha\beta$. Conclude that

$$c = ab \le \alpha b \le \alpha\beta;$$

and therefore $\alpha\beta$ is an upper bound for C.

We show $\alpha\beta \le \gamma$. We do this part of the argument by contradiction. If $\gamma < \alpha\beta$, then $\frac{\gamma}{\alpha} < \beta$. (We know that A is a non-empty set of positive numbers, and α is an upper bound for A. It follows that α is positive, and so not zero.) But β is $\sup B$; so there exists $b \in B$ with $\frac{\gamma}{\alpha} < b$. The number b is positive; so not zero. We have $\frac{\gamma}{b} < \alpha$. But α is $\sup A$; so there exists $a \in A$ with $\frac{\gamma}{b} < a$. We now have $\gamma < ab \in C$; which contradicts the fact that γ is an upper bound for C. Our original supposition that $\gamma < \alpha\beta$ must be wrong; so, we have established that $\alpha\beta \le \gamma$.

We have shown that $\gamma \le \alpha\beta$ and $\alpha\beta \le \gamma$. It follows that $\alpha\beta = \gamma$ and the proof is complete.