Math 554, Exam 2, Summer 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. **Leave room on the upper left hand corner of each page for the staple.** Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 8 problems. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website later this afternoon.

1. (6 points) Define "Cauchy sequence". **Use complete sentences.** Include everything that is necessary, but nothing more.

2. (6 points) Define "limit point". (This concept is also known as "accumulation point".) **Use complete sentences.** Include everything that is necessary, but nothing more.

3. (6 points) Consider the sequence $\{a_n\}$ with $a_1 = \sqrt{20}$, and $a_n = \sqrt{20 + a_{n-1}}$ for $n \geq 2$. Prove that the sequence $\{a_n\}$ converges. Find the limit of the sequence $\{a_n\}$. Write in **complete sentences**.

4. (6 points) Let $\{a_k\}$ be a sequence of real numbers. For each natural number n, let
$$s_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$
Suppose that the sequence $\{a_k\}$ converges to the real number a. Prove that the sequence $\{s_n\}$ also converges to a. Give a complete ε style proof. Write in **complete sentences**.

5. (6 points) Prove that every uncountable subset of $\mathbb{R}$ contains a limit point in $\mathbb{R}$.

6. (8 points) For each question: if the answer is yes, then prove the assertion; if the answer is no, then give a counter example.
   (a) Is the union of an arbitrary collection of closed sets always a closed set?
   (b) Is the union of a finite collection of closed sets always a closed set?
   (c) Is the intersection of an arbitrary collection of closed sets always a closed set?
   (d) Is the intersection of a finite collection of closed sets always a closed set?

7. (6 points) Is the intersection of a finite collection of compact sets always a compact set? If the answer is yes, then prove the assertion. If the answer is no, then give a counter example.

8. (6 points) Let E be the set $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. Give a direct proof (using the definition) that E is compact.