Math 554, Exam 2, Summer 2006 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. **Leave room on the upper left hand corner of each page for the staple.** Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 8 problems. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website later this afternoon.

1. (6 points) **Define "Cauchy sequence". Use complete sentences. Include everything that is necessary, but nothing more.**

The sequence $\{a_n\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists n_0, such that for every $n, m > n_0$, $|a_n - a_m| < \varepsilon$.

2. (6 points)**Define "limit point". (This concept is also known as "accumulation point".) Use complete sentences. Include everything that is necessary, but nothing more.**

The point p in $\mathbb{R}$ is a *limit point* for the subset E of $\mathbb{R}$ if for every $\varepsilon > 0$, there exists $q \in N_\varepsilon(p) \cap E$, with $q \neq p$.

3. (6 points) **Consider the sequence $\{a_n\}$ with $a_1 = \sqrt{20}$, and $a_n = \sqrt{20 + a_{n-1}}$ for $n \geq 2$. Prove that the sequence $\{a_n\}$ converges. Find the limit of the sequence $\{a_n\}$. Write in complete sentences.**

It is clear that every term a_n is at most 5. We see that $a_1 \leq 5$. If $a_{n-1} \leq 5$, then $a_{n-1} + 20 \leq 25$; so $a_n = \sqrt{a_{n-1} + 20} < \sqrt{25} = 5$. It is also clear that the sequence is an increasing sequence. We just saw that $a_n \leq 5$ for all n. Multiply both sides by the positive number $a_n + 4$ to see that $a_n^2 + 4a_n \leq 5a_n + 20$. In other words, we have $a_n^2 \leq a_n + 20$. The numbers a_n and $a_n + 20$ are both positive. It follows that $a_n \le \sqrt{a_n + 20} = a_{n+1}$. The sequence $\{a_n\}$ is an increasing bounded sequence. We proved in class that every monotone bounded sequence converges. It follows that the sequence $\{a_n\}$ converges. We know that $\lim\limits_{n\to\infty} a_n$ exists. Let L be the name of this limit. Take the limit of both sides of $a_n = \sqrt{20 + a_{n-1}}$ to see that $L = \sqrt{20+L}$, or $L^2 = 20 + L$, which is $L^2 - L - 20 = 0$. This equation factors to become $(L-5)(L+4) = 0$; hence $L = 5$ or $L = -4$. Every a_n is positive so $L = -4$ is not possible. We conclude that $\boxed{\lim\limits_{n\to\infty} a_n = 5}$.

4. (6 points) **Let $\{a_k\}$ be a sequence of real numbers. For each natural number n , let**

$$s_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Suppose that the sequence $\{a_k\}$ converges to the real number a . Prove that the sequence $\{s_n\}$ also converges to a . Give a complete ε style proof. Write in complete sentences.

Fix $\varepsilon > 0$. The sequence $\{a_k\}$ converges to a so there exists a natural number k_0 so that if $k > k_0$, then $|a_k - a| < \frac{\varepsilon}{2}$. Let B equal the fixed number $B = |(\sum\limits_{k=1}^{k_0} a_k) - k_0 a|$. Pick a natural number n_1 with $\frac{2B}{\varepsilon} < n_1$. Let $n_0 = \max\{n_1, k_0\}$. If $n_0 < n$, then

$$|s_n - a| = \left|\frac{a_1 + a_2 + \cdots + a_n}{n} - a\right| = \left|\frac{a_1 + a_2 + \cdots + a_n - na}{n}\right| =$$

$$= \left|\frac{(a_1 + a_2 + \cdots + a_{k_0} - k_0 a) + (a_{k_0+1} - a) + (a_{k_0+2} - a) + \cdots + (a_n - a)}{n}\right|.$$

Use the triangle inequality to see that

$$|s_n - a| \le \frac{|a_1 + a_2 + \cdots + a_{k_0} - k_0 a|}{n} + \frac{|a_{k_0+1} - a|}{n} + \frac{|a_{k_0+2} - a|}{n} + \cdots + \frac{|a_n - a|}{n}.$$

The first term on the right of the sign $\le$ is $\frac{B}{n}$. Our choice of n_0 ensures that $\frac{B}{n} < \frac{\varepsilon}{2}$. If ℓ is a positive integer, then our choice of n_0 ensures that $|a_{k_0+\ell} - a| \le \frac{\varepsilon}{2}$. At this point we have

$$|s_n - a| \le \frac{\varepsilon}{2} + (n - k_0)\frac{\frac{\varepsilon}{2}}{n} \le \frac{\varepsilon}{2} + n\frac{\frac{\varepsilon}{2}}{n} = \varepsilon.$$

We have shown that $n > n_0 \implies |s_n - a| < \varepsilon$. We conclude that the sequence $\{s_n\}$ converges to a.

5. (6 points) **Prove that every uncountable subset of $\mathbb{R}$ contains a limit point in $\mathbb{R}$.**

Lemma. *If A is a countable set and for each $\alpha \in A$, S_α is a finite set, then $\bigcup_{\alpha \in A} S_\alpha$ is a countable set*

Proof. The hypothesis that A is countable assures us that there exists a one-to-one and onto function $f : \mathbb{N} \to A$. Make a grid. List the elements of $S_{f(1)}$ in row 1. List the elements of $S_{f(2)}$ in row 2. etc. No count along diagonals that go from Northeast to Southwest:

$$\begin{array}{ccccc} 1 & 2 & 4 & 7 & \dots \\ 3 & 5 & 8 & \dots & \\ 6 & 9 & \dots & & \\ 10 & \dots & & & \\ \vdots & & & & \end{array}$$

□

Now we do the problem. Let E be an uncountable set. We see that E is the disjoint union of $E \cap [n, n+1)$ as n varies over the countable set $\mathbb{Z}$. The Lemma tells us that at least one of the sets $E \cap [n, n+1)$ is infinite. (If all of these sets were finite, then E would be the countable union of finite sets; hence countable.) Let us assume that $E \cap [n, n+1)$ is finite for some fixed n. We may apply version 1 of the Bolzano-Weierstrass Theorem: the infinite bounded set $E \cap [n, n+1)$ has a limit point, say p, in $\mathbb{R}$. It is clear that p is a limit point of E.

6. (8 points) **For each question: if the answer is yes, then prove the assertion; if the answer is no, then give a counter example.**

   (a) **Is the union of an arbitrary collection of closed sets always a closed set?**

NO. For each natural number n with $n \geq 2$, the closed interval $[\frac{1}{n}, 1]$ is a closed subset of $\mathbb{R}$. The union of all of these sets is $(0, 1]$, which is not a closed set.

   (b) **Is the union of a finite collection of closed sets always a closed set?**

YES. Let $C_1, \ldots, C_n$ be closed sets in $\mathbb{R}$. I will show that the complement of $C_1 \cup \cdots \cup C_n$ is an open set. Take $x \in \mathbb{R}$ with $x \notin C_i$ for any i. The complement of C_i is open for each i; so there exists $\varepsilon_i > 0$, so that $N_{\varepsilon_i}(x)$ misses C_i. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. We see that $\varepsilon > 0$ and that $N_\varepsilon(x)$ misses $C_1 \cup \cdots \cup C_n$.

(c) **Is the intersection of an arbitrary collection of closed sets always a closed set?**

YES. Let I be an index set. For each index $i \in I$, let C_i be a closed subset of $\mathbb{R}$. We will prove that $\bigcap\limits_{i \in I} C_i$ is a closed set by proving that the complement is open. Let $x \in \mathbb{R}$ with $x \notin \bigcap\limits_{i \in I} C_i$. Thus, there is an index $i_0 \in I$ with $x \notin C_{i_0}$. The set C_{i_0} is closed, so the complement of C_{i_0} is open and there exists an $\varepsilon > 0$ such that $N_\varepsilon(x)$ misses C_{i_0}. It follows that $N_\varepsilon(x)$ misses $\bigcap\limits_{i \in I} C_i$; and therefore, $\bigcap\limits_{i \in I} C_i$ is a closed set.

(d) **Is the intersection of a finite collection of closed sets always a closed set?**

YES. This is a special case of part (c). There is no need to copy the proof.

7. (6 points) **Is the intersection of a finite collection of compact sets always a compact set? If the answer is yes, then prove the assertion. If the answer is no, then give a counter example.**

YES. A set is compact if and only if it is closed and bounded. If $K_1, \ldots, K_n$ are compact sets, then each set K_i is closed and bounded. It is clear that the intersection $\bigcap\limits_{i=1}^{n} C_i$ is bounded. Problem 6d tells us that $\bigcap\limits_{i=1}^{n} C_i$ is closed. We conclude that $\bigcap\limits_{i=1}^{n} C_i$ is compact.

8. (6 points) **Let E be the set $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. Give a direct proof (using the definition) that E is compact.**

Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an arbitrary open cover of E. The number zero is in one of the sets of $\mathcal{U}$; in other words, there is an element α_0 of A so that $0 \in U_{\alpha_0}$. The set U_{α_0} is open, so there exists ε with $N_\varepsilon(0) \subseteq U_{\alpha_0}$. Of course, if n_0 is large enough, then $\frac{1}{n_0} < \varepsilon$. If $n \geq n_0$, then $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$ and $\frac{1}{n} \in N_\varepsilon(0) \subseteq U_{\alpha_0}$.

For each n with $n < n_0$, there exists a subscript $\alpha_n \in A$ with $\frac{1}{n} \in U_{\alpha_n}$. We have found a finite subcover $U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_{n_0-1}}$ of $\mathcal{U}$ which covers E.