Math 554, Final Exam, Summer 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your grade from VIP.

There are 13 problems. Problems 1 through 9 are worth 8 points each. Problems 10 through 13 are worth 7 points each. The exam is worth a total of 100 points.

I will post the solutions on my website shortly after the class is finished.

1. Let $f\colon E \to \mathbb{R}$ be a function which is defined on a subset E of $\mathbb{R}$. Define $\lim\limits_{x\to p} f(x) = L$. Use complete sentences. (Be sure to tell me what kind of a thing p is, and what kind of a thing L is.)

2. STATE either version of the Bolzano-Weierstrass Theorem.

3. PROVE either version of the Bolzano-Weierstrass Theorem.

4. Define *Cauchy sequence*. Use complete sentences.

5. PROVE that every Cauchy sequence converges.

6. Let I be an interval and $f\colon I \to \mathbb{R}$ be a function which is differentiable at the point p of I. PROVE that f is continuous at p.

7. Let f be a continuous function from the closed interval $[a, b]$ to $\mathbb{R}$. Let $\varepsilon > 0$ be fixed. Prove that there exists $\delta > 0$ such that: whenever x and y are in $[a, b]$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. (Notice that you are supposed to prove that one δ works for every x and y.)

8. Give an example of a bounded infinite closed set that does not contain any intervals. Explain thoroughly.

9. Let A be an index set. For each index a in A, let F_a be a closed subset of $\mathbb{R}$. Is the union $\bigcup\limits_{a\in A} F_a$ always closed? If yes, prove the claim. If no, give a counterexample.

10. Let f and g be functions from the subset E of $\mathbb{R}$ to $\mathbb{R}$, and let p be a limit point of $\mathbb{R}$. Suppose that $\lim_{x \to p} f(x)$ exists and equals A. Suppose, also, that $\lim_{x \to p} g(x)$ exists and equals B. Prove $\lim_{x \to p} f(x)g(x)$ exists and equals AB.

11. Let c_1 be an arbitrary element of the open interval $(0,1)$. For each $n \in \mathbb{N}$, let $c_{n+1} = \frac{1}{5}(c_n^2 + 2)$. Prove that the sequence $\{c_n\}$ is contractive.

12. Consider the sequence $\{a_n\}$ with $a_1 = 4$, and for $n \in \mathbb{N}$, $a_{n+1} = \sqrt{2a_n + 3}$. Prove that the sequence converges. Find the limit of the sequence.

13. Let $f(x) = \begin{cases} x^2 + x & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational.} \end{cases}$ Is f differentiable at 0? Prove your answer.